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DEBT STABILIZATION IN THE PRESENCE OF ENDOGENOUS RISK PREMIA: A DYNAMIC GAME APPROACH

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This paper focuses on the possibility that financial markets require risk premia on holding sovereign debt of countries that appear vulnerable from a fiscal sustainability perspective. Both the level of debt as well as the rate of change of debt are assumed to impact on the risk premium. We analyze the impact of such an endogenous risk premium in a simple debt game between a monetary and a fiscal player, as introduced by [Tabellini (1986) *Journal of Economic Dynamics and Control* 10, 427–442]. The risk premium term adds a nonlinearity to the linear model in case risk premia are absent. We analyze outcomes in case of noncooperative open-loop Nash strategies and in case of cooperative strategies and consider the workings of the risk premium as a market-based disciplining device (in case of high debt) and adjustment rewarding device (in case of a declining debt trajectory).

Keywords: Debt Stabilization, Differential Games, Nonlinear Dynamical Systems, Risk Premium, Economic Dynamics

1. INTRODUCTION

One of the most significant consequences of the global financial crisis has been a substantial increase of government debt in most Organisation for Economic Co-operation and Development (OECD) countries. The economic slowdown, the fiscal balance deterioration, as well as the substantial efforts to save and recapitalize the banking sector explain the significant rise of government debt. Thus, in most OECD countries, government debt stabilization has taken a central stage. In addition, financial markets have been adding pressures on countries that appear

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vulnerable when looking at their government debt dynamics, i.e., countries whose debt level is substantial and/or whose debt is on a rapidly increasing trajectory. In the bond market, prices move inversely to yields so when investors perceive that inflation risk or credit risk could rise as a result of government policies they demand higher yields to compensate for the added risk. As a result, bond prices fall and yields rise.

If such endogenous risk premia are present in bond markets, financial markets may also be considered as an actor in the dynamics of government debt. Although markets are obviously made up of a very large number of essentially atomistic agents, there could be a few reasons why an approach in which it acts as a (kind of) single actor nevertheless makes sense. First, the presence of s.c. "bond market vigilantes": investors that trade on the perceptions and sentiments about current and future government solvency, taking into account a broad range of economic factors and political and social stability conditions. Apart from these more general factors, in particular the level and direction of government debt will be relevant for our analysis. Bond market investors may seek to protest against monetary or fiscal policies they consider as inflationary, fiscally imprudent, or unsustainable in other ways by (short)-selling bonds, thus increasing yields. This increases the cost of borrowing. The presence of the "bond market vigilantes" implies that the bond market can constitute a serious restraint on the government's ability to over-spend and over-borrow. In fact, if government programs to maintain/restore fiscal sustainability are not convincing bond markets, financial markets may cut a government entirely from additional bond market financing in that case it needs to turn to alternative financing e.g., in the form of an International Monetary Fund (IMF) rescue package or seeking for debt rescheduling or to seek a form of debt relief.

Second, recent research in behavioral finance considers the presence of "herding" behavior and "contagion" where market participants appear to be involved in distinct, synchronized buying or selling "frenzies" or "panics," implying bandwagon effects that are characteristic for speculative bubbles and forms of financial market stress and volatility.[1] Although at first sight the herding behavior appears to be driven entirely by irrational elements such as panics, copying behavior, there is also a substantial literature that shows that herding behavior can also be the outcome of more rational considerations. In the presence of large informational asymmetries, the more uninformed investors may clearly benefit from emulating more informed traders.[2] Technical and fundamental analysis carried out by "chartists" is another relevant approach in the behavioral finance literature in this respect. Technical analysis provides methodologies for forecasting the direction of prices through the study of past market data.

Endogenous risk-premia in bond markets that are driven by market sentiments viz. speculation can create additional instability in the debt-dynamics and complicate efforts to stabilize debt: "confidence crises" where investors lose confidence in the government's ability to repay its debt, will fuel risk premia and governments risk being unable to roll over existing debt. Speculation in the government bond market is also fueled by the presence of credit default swaps (CDS) on government debt that enable to speculate on and benefit from a government debt default without owning the underlying asset. Rating agencies also contribute further to this speculation by issuing credit ratings on government debt that are based on a rather opaque and ad-hoc evaluation of possible future fiscal developments, resulting in a set of nonuniform ratings and nonuniform timing of rating-changes.

The theoretical focus of this paper is on the formation of endogenous risk premia and the consequences for government debt stabilization. Government debt stabilization is modeled as a dynamic game between the fiscal and monetary authorities in which features also endogenous risk premia on government debt. We consider both noncooperative and cooperative equilibria. We focus on cases where both authorities attach some weight to debt stabilization: in case no authority is concerned about debt stabilization, debt is exploding, if only one authority is concerned about debt stabilization, the entire adjustment burden is automatically on its shoulders. Risk premia will depend both on the level of debt *and* the change in debt.

Including the direction of government debt dynamics in the risk premium is an important contribution of the paper. The presence of the risk premia terms has implications for the dynamic debt stabilization game as they work like "sticks and carrots": authorities will take into account that interest rates are not fixed and higher risk premia and interest burdens result in case debt is at a high level or increasing rapidly. In this interpretation, a decreasing debt level—even if the actual debt level is still high—is interpreted by financial markets as an indication that budgetary conditions are under control so risk premia can be moderated. Conversely, increasing debt—even if actual debt is low—causes budgetary concerns and an increasing risk premium. Authorities internalize the benefits from lower risk premia that result from a lower debt level and a decreasing government debt trajectory.[3]

The outline of this paper is as follows. Section 2 provides an overview of the relevant literature. Section 3 introduces the model. Section 4 analyzes the noncooperative Nash open-loop equilibrium and the cooperative equilibrium. Section 5 presents a detailed numerical example to illustrate the workings of the model and in particular the effects of different assumptions concerning endogenous risk premia. The conclusions section summarizes some main results and points out some directions for future research. Proofs of some theorems follow in the Appendix at the end of the paper.

2. SOME RELEVANT LITERATURE

Government debt (budgetary/fiscal) sustainability is a relatively vague concept and debt problems can be modeled in different ways [see, e.g., Blanchard et al. (1990), Neck and Sturm (2008), Unctad (2009), Lukkezen et al. (2012) for an overview]. The formulation of the essence of the debt sustainability problem goes back at least as far as Domar (1944). Ghosh et al. (2013) develop a framework

TABLE 1. Some estimates of α and β in the literature

Authors	Countries	Period	α	β
von Hagen et al. (2011)	EU-15	1991–2009	0.003	0.04
Bernoth et al. (2013)	EU-15	1993–2009	0.01	0.04
Baldacci and Kumar (2012)	31 adv countries	1980–2008	0.05	0.17
Heinemann et al. (2014)	EU-15	1981–2008	0.003	0.02

for assessing debt sustainability that enables to determine a debt limit beyond which fiscal solvency is in doubt. It defines a fiscal space: the distance between the current debt level and this limit. It also considers the effects of endogenous risk premia and the occurrence of "fiscal fatigue," the primary balance rises as debt increases but the responsiveness eventually begins to weaken and then actually decreases at very high levels of debt.

This section provides an overview of literature on two aspects that are very important to our analysis: (i) government debt sustainability and risk premia; (ii) government debt sustainability and the interaction of monetary and fiscal policies. We cannot review the entire literature on both topics but focus on the most relevant studies (other sources exist that aim at providing such a detailed literature review, see in particular Staehler (2013) and Foresti (2017).

(i) An important literature has resulted on the risk premium on government debt. Among many factors, this literature singles out the level of debt and also the direction of government debt dynamics (is government debt on an increasing or decreasing trajectory?) as important drivers of risk premia on sovereign debt. The question that is of particular importance to our analysis is as follows: What are empirically plausible values of the sensitiveness of risk premia on government debt with respect to the level of debt—which we will refer to as α- and with respect to the change in debt—which we will refer to as β-? The empirical literature on sovereign debt and bond yields/risk premia is quite substantial and provides us with a variety of estimations. Haugh et al. (2009) made an extensive literature overview dealing with the pre-Financial Crisis literature. Some recent studies that deal mostly with the European Union (EU) case provide estimates of α and β are provided in Table 1.

This literature comes with a number of insights that are relevant here: (i) in times of economic and financial stability, estimates of α and β appear to be smaller than in times of high economic uncertainty and financial markets instability (as during the recent Global Financial Crisis and the European Debt Crisis). (ii) α and β may depend on the various country (group) characteristics, e.g., countries with high (foreign) initial debt levels may have risk premia that are more sensitive to further debt accumulation. In other words, the sensitivity of bond yields/risk premium w.r.t. debt levels and the change in debt/deficit varies considerably and is best looked upon as being time-, state- and country-dependent. (iii) Studies such as Allessandrini et al. (2014) find that the Euro Area sovereign debt crisis has been as much a matter of external imbalances as of fiscal irresponsibility. Since imbalances in public finance often come hand-in-hand with external imbalances, this implies additional upward pressure on risk premia in bond markets as both government debt and external debt sustainability is under scrutiny.

De Grauwe and Ji (2012) argue that, since the start of the sovereign debt crisis, markets have been making errors in the direction of overestimating risks, while before crisis they also falsely tended to underestimate risks. They found evidence that a large part of the surge in the spreads of the periphery countries between 2010 and 2011 was disconnected from underlying increases in the debt-to-gross domestic product (GDP) ratios and current-account positions, and was the result of negative market sentiments, even panic, that became very strong starting at the end of 2010. That was interpreted in their empirical estimates as a value for α typically between 0.02 and 0.08, maybe even more as the European Debt Crisis deepened.

(ii) A large literature analyzes the dynamic interaction of monetary and fiscal policies and the consequences for government debt stabilization. Sargent and Wallace (1981) seminal analysis highlights the coordination problem between fiscal and monetary policies in stabilizing debt: which policy is ultimately responsible for satisfying the public sector's consolidated dynamic government budget constraint? Will the treasury adjust in order to attain the requisite surpluses, given the path of money growth set by the central bank, or will the central bank eventually deliver the base money growth (money printing) needed to make up for a shortfall of fiscal surpluses? Postponing fiscal adjustment in the first case, or postponing monetary accommodation in the second case only aggravates the stabilization burden in the long-run: it lead to unpleasant monetarist arithmetic (in case of fiscal leadership) or unpleasant fiscal arithmetic (in case of monetary leadership) because of the short-run piling up of government debt. Leeper (1991) extends the analysis of Sargent and Wallace. Similar to the fiscal or monetary leadership cases of Sargent and Wallace, Leeper defines an active authority who pays no attention to the state of government debt and is free to set its control variable as it sees fit and a passive authority that responds to government debt shocks. Its behavior is constrained by private optimization and the active authority its actions. It leads to the fiscal theory of the price level: The notion that the government's fiscal policy ultimately determines the price level. A necessary condition for the price level to be stable, government finances must be sustainable: the fiscal policy maker must run a balanced budget over the course of the business cycle, meaning they must not run a structural deficit. It therefore contrasts with the usual monetary theory of the price level, where the price level is primarily or exclusively determined by supply of money.

In the analysis of Sargent and Wallace (1981) and Leeper (1991), the interaction between the Central Bank and the fiscal authority is not modeled explicitly as a conflict of players with alternative objectives. Tabellini (1986) analyzes the problem of government debt stabilization as an explicit dynamic game between a monetary and a fiscal authority. The interaction between these two authorities is cast in the form of a game with linear dynamics, which describe the evolution of debt in case of exogenous interest rates, and authorities trying to minimize a quadratic objective function. One of his main findings is the benefit of cooperation, which states that when the two policy makers coordinate their efforts a smaller steady-state debt value is achieved and this is achieved more rapidly than under noncooperation, implying lower losses under cooperation. Another sensible result is that if a player decreases the relative weight assigned to stabilization of debt the adjustment burden for the other player increases.

Several extensions of this model exist. Van Aarle et al. (1997) consider government debt stabilization issues in a two-country monetary union model extension of the Tabellini model.[4] Van den Broek (2002) situates the debt stabilization game in a moving horizon dynamic game that could for example approximate regular elections of new policy makers. It is found that a shorter planning horizon reduces the debt stabilization efforts in the Nash open-loop case. Castren (1998) considers the finite time horizon variant of the debt stabilization model to analyze electoral effects introduced in the form of end-point debt targets. Di Bartolomeo and Di Gioacchino (2008) analyze the Stackelberg open-loop equilibria of the debt stabilization game, showing that these lead to a further reduction in debt stabilization efforts compared to the Nash case. Engwerda et al. (2016a) consider the impact uncertainty has on the outcomes. They find that in case of higher uncertainty, policy makers react with more active policies yielding a smaller equilibrium debt, and a fiscal leadership in policy making may be preferred above other policy regimes if there is much uncertainty about economic developments. Figuieres (2008) considers the Tabellini debt stabilization game in the context of an n-country extension where countries consider joining a monetary union and shows how this changes the strategic incentives.

Engwerda et al. (2013) extended the Tabellini model with an endogenous risk premium term, that represents the endogenous pressure added by financial markets. This risk premium term depends on the level of debt and changes the game into a nonlinear one. Their main results are that, as the risk premium term increases, the steady-state of debt decreases in both noncooperative and cooperative modes of play. Also, they find that equilibrium debt under a cooperative mode of play is smaller than in the noncooperative case only when the strength of the risk premium term is not too large. Furthermore, for larger values of the risk premium parameter, a reduction of the relative weight a player assigns to the debt stabilization does no longer necessarily increase the steady-state of debt, as it does in the absence of an endogenous risk premium.[5]

3. A NONLINEAR DEBT STABILIZATION GAME

Following Tabellini (1986), we consider a country consisting of one monetary authority, responsible for monetary policy, and one fiscal authority, responsible for the country's fiscal policy. The two players are engaged in a dynamic conflict over stabilization of government debt. Financial markets also play a crucial role in this conflict as they require a risk premium when holding government debt. The dynamic government budget constraint depicts the accumulation of the debt and relates government debt, monetary financing, fiscal deficits as well as interest payments:

$$\dot{d}(t) = r(t)d(t) + f(t) - m(t),\ d(0) = d_0. \tag{1}$$

This differential equation expresses that the change of debt equals the total deficit (the sum of interest payments plus the primary fiscal deficit) minus money growth. A dot above a variable, denotes its rate of change, the first derivative w.r.t. time, $\dot{d} \equiv \partial d/\partial t$.[6] The specific variables have the following interpretation: d denotes the government debt as a percentage of output (note that negative value of debt denotes that the government has obtained a claim on private sector assets); f denotes the primary fiscal deficit as a percentage of output (note that f denotes deficit, so negative values of f imply a budgetary surplus for the fiscal authorities); m denotes money growth (including buying of government debt by the Central Bank-CB-) measured as a fraction of output; and r denotes the interest rate adjusted for the rate of output growth.

Fiscal authorities seek to limit the government debt at a certain level, $\bar{d}_F$, using fiscal instruments in such a way that fiscal deficits remain close to a target value $\bar{f}$ as well. We formalize this, as that fiscal authorities intend to minimize:

$$J_F = \frac{1}{2}\int_0^{\infty} e^{-\theta t}\left\{[f(t) - \bar{f}]^2 + \kappa_F[d(t) - \bar{d}_F]^2\right\} dt. \tag{2}$$

Here, κ_F indicates the relative preference of the fiscal authority attached to debt stabilization, θ denotes the rate of time preference.

Similarly, monetary authorities seek to constrain government debt at their target level, $\bar{d}_M$, while trading off debt fluctuations against the desire that money growth remains as close as possible to a target value $\bar{m}$. This is formalized as that monetary authorities intend to minimize:

$$J_M = \frac{1}{2}\int_0^{\infty} e^{-\theta t}\left\{[m(t) - \bar{m}]^2 + \kappa_M[d(t) - \bar{d}_M]^2\right\} dt, \tag{3}$$

in which κ_M indicates the relative preference of the monetary authority attached to debt stabilization.

We will exclude cases of limitless growth of debt where the no-Ponzi game condition is violated. This assumption of a bounded steady-state level of debt implies the no-Ponzi game condition is respected. It prevents over-accumulation of debt and requires that the discounted value of debt tends to zero when time goes to infinity (see also the Appendix).We additionally restrict the set of admissible policies. We assume that players minimize their loss functions using policies that converge to steady states and which stabilizes government debt at some bounded steady state value. More formally, the set of admissible control policies we consider in this paper is the set of locally square integrable functions,

$$\mathcal{U} := \{(f(.), m(.)) \in L_{2,loc} \mid \lim_{t\to\infty} f(t) = f^e,\ \lim_{t\to\infty} m(t) = m^e,\ \lim_{t\to\infty} d(t) = d^e\}. \quad (4)$$

f^e, m^e, and d^e in other words denote the long-run (steady-state) levels of the (primary) fiscal balance, money growth, and government debt.

(4) In a more practical interpretation assumes that in the monetary and fiscal policy framework sufficient "checks and balances" are build in that provide credible enforcement or punishment strategies concerning sustainable monetary and fiscal policy management leading to adequate government debt control/concern even in precarious conditions. In practical term, such a commitment framework would form e.g., the EU's Stability and Growth Pact combined with an independent Central Bank. Clearly, financial markets are continuously evaluating the credibility of such policy frameworks and want assured that government debt remains sustainable.

Note, that in case the deficit target, $\bar{f}$, is smaller than the money growth target, $\bar{m}$, the debt in (1) is, in principle (the other model parameters play a role here too, but for the sake of simplicity ignore these effects) driven toward a negative steady-state debt level (or positive asset position). These cases are considered of less interest, in the sense that we tend to situate the debt stabilization game in a context of high (initial and steady-state) government debt problems.

In the original Tabellini (1986) model, the interest rate on government bond is assumed to be constant, $r(t) = \bar{r}$, thus leading to a linear quadratic differential game, which can be solved analytically. Inspired by the recent government debt crisis, we include an endogenous risk premium. This risk premium is assumed to depend on the level of debt and on the change in the level of debt. The level of debt will act as a form "market-based" discipline [Bayoumi et al. (1995)] as prudent policy makers would seek to avoid high indebtedness. We consider an additional risk premium term, namely a term associated with the rate of change of debt. This term expresses the idea that financial markets will not only scrutiny the level of government debt but also the rate of change of government debt, and would reward debt dynamics that are on a declining path (even if the level could be high) and punish debt dynamics that are rising (even if the level could be low). Thus we assume that

$$r(t) = \bar{r} + \alpha d(t) + \beta \dot{d}(t). \quad (5)$$

Therefore, the interest rate (corrected for growth)[7] is governed by three parameters: $\bar{r}$, the risk free interest rate (assumed to be constant); α, which denotes the risk premium parameter that indicates how financial markets value the level of debt in setting the interest rate [as in Engwerda et al. (2013)]; and β which denotes the risk premium parameter expressing how financial markets value the direction of the evolution of the debt in setting the interest rate. This gives rise to a nonlinear quadratic differential game, that can still partly be solved analytically. Both α and β are positive real numbers, the larger the values of these parameters the more nonlinear the debt dynamics in (1) will become.

As indicated in the introduction, adding the rate of change of debt to the risk premium (5) is motivated by the notion that the level of debt alone is not always sufficient to judge government debt sustainability. A high debt level (as a percent of national output), but which is decreasing rapidly should not be necessarily connected with high interest rate because it could indicate that public finances are sustainable and that the situation is under control. Thus, the government could be "rewarded" with lower interest rates. Alternatively, low debt could still be accompanied by a sizeable risk premium in cases where debt is increasing swiftly, as it could signal that the budgetary situation is precarious.

The α and β parameters of the risk premium definition (4) play an important role in the debt stabilization game. Together, they determine the degree of nonlinearity of the debt dynamics (1). If α and β are zero, the debt dynamics reduce to the standard linear debt dynamics case and the debt stabilization game reduces to the linear-quadratic debt stabilization model of Tabellini (1986). In case, $\beta = 0$, the debt dynamics adjust according to the quadratic debt dynamics in Engwerda et al. (2013) and the debt stabilization game that results is considerably more complex than in the standard linear case. In case both α and β are non-zero the dynamics can be even more complex as this paper finds. While α and β have similar workings, namely increasing the sensitiveness of risk premia to debt and the change of debt, their workings are not identical in the model: the value of α enters the model in nonlinear relation to the state variable, government debt, whereas β is related to the change of the state variable.

An interesting interpretation of α and β in the context of our study is that these parameters are determined by financial markets, reflecting the perceived stability of financial markets, economic stability and credibility/ reputation of honoring debt by governments. For example, in case such credibility is declining, financial markets will become increasingly concerned about current debt levels and the change in debt, that is, the value of α and β would increase as financial markets require higher risk premia.[8] Conversely, in times of high financial and economic stability, financial markets would lower concerns on fiscal authorities not being able to service their debt obligations.

It is clear that in this small model, five factors determine the strategies of the two players: the relative weights for debt stabilization; the target values for debt; the interest rate charged for debt (and in particular the values of $\bar{r}$, α, and β); the time preference for current w.r.t. future losses (measured by θ); and initial debt. We will focus our attention on these factors in the numerical analysis in Section 5.

4. ANALYSIS OF THE NONLINEAR DEBT STABILIZATION GAME

In this section, we consider the game (1)–(5) in case of (i) noncooperative policies and (ii) cooperative policies.

4.1. Noncooperative Debt Stabilization: The Nash Open-loop Case

Assume that the monetary and fiscal players act noncooperatively and play Nash strategies. That is, they look for actions that have the property that a unilateral deviation from these actions makes them worse off. Moreover, it is assumed that the players have an open-loop information structure (see, e.g., Basar (1999) or Engwerda et al. (2013) for additional discussion on this assumption) about the game. In the case of Nash strategies, players move simultaneously: the case where either the fiscal or the monetary policy maker would act as a Stackelberg leader are equally interesting from a policy perspective but not analyzed here given the complexity [see e.g., Di Bartolomeo and Di Gioacchino (2008) on Stackelberg equilibria in the linear debt stabilization game].

Before presenting our first result we first note that, as we will assume that $\beta \in [0, 0.5]$, within the current context we may assume throughout that $1 - \beta d^e > 0$. In the Appendix, we supply the proofs of the following results.

THEOREM 1. *If $(f^*(.), m^*(.)) \in \mathcal{U}$ is a set of open-loop Nash strategies for (1)–(5), there exist a trajectory for debt $d^*(.)$ and an associated co-state variable $\mu^*(.)$ that satisfy the set of nonlinear differential equations:*

$$\dot{d}(t) = \frac{\bar{r}d(t) + \alpha d^2(t) + \bar{f} - \bar{m} - \mu(t)}{1 - \beta d(t)}, \; d(0) = d_0, \tag{6}$$

$$\dot{\mu}(t) = \left(\theta - \frac{\bar{r} + 2\alpha d(t)}{1 - \beta d(t)}\right)\mu(t) + \frac{\kappa_F[\bar{d}_F - d(t)] + \kappa_M[\bar{d}_M - d(t)]}{1 - \beta d(t)}. \tag{7}$$

with $d^*(0) = d_0$ and where both $\lim_{t\to\infty} d^*(t) = d^e$ and $\lim_{t\to\infty} \mu^*(t) = \mu^e$ exist. Furthermore, the steady-state values (d^e, μ^e) satisfy

$$\mu^e := \bar{r}d^e + \alpha d^{e^2} + \bar{f} - \bar{m}, \tag{8}$$

where d^e is a solution of the third-order polynomial equation

$$g(d) := \gamma_3 d^3 + \gamma_2 d^2 + \gamma_1 d + \gamma_0 = 0. \tag{9}$$

Here, $\gamma_3 := -\alpha(2\alpha + \beta\theta)$, $\gamma_2 := \alpha(\theta - \bar{r}) - \bar{r}(2\alpha + \beta\theta)$, $\gamma_1 := \bar{r}(\theta - \bar{r}) - \kappa_F - \kappa_M - (2\alpha + \beta\theta)(\bar{f} - \bar{m})$, and $\gamma_0 := \kappa_F\bar{d}_F + \kappa_M\bar{d}_M + (\bar{f} - \bar{m})(\theta - \bar{r})$. The values with d^e corresponding steady-state equilibrium strategies are

$$f^e := \bar{f} - \frac{\kappa_F(d^e - \bar{d}_F)}{\theta(1 - \beta d^e) - (\bar{r} + 2\alpha d^e)}, \text{ and } m^e$$

$$:= \bar{m} + \frac{\kappa_M(d^e - \bar{d}_M)}{\theta(1 - \beta d^e) - (\bar{r} + 2\alpha d^e)}. \tag{10}$$

Remark 2. Note that compared to the case studied by Engwerda et al. (2013) where no risk premium associated with the change of debt was considered, $\beta = 0$, the parameters $\gamma_i,\ i = 1, 2, 3$, involved in the debt equation (9) are all reduced by a fraction $\beta\theta$, whereas γ_0 remains the same.

Next consider the discriminant

$$\Delta := 18\gamma_3\gamma_2\gamma_1\gamma_0 - 4\gamma_2^3\gamma_0 + \gamma_2^2\gamma_1^2 - 4\gamma_3\gamma_1^3 - 27\gamma_3^2\gamma_0^2. \qquad (\mathbf{11})$$

Then, see the Appendix, (9) has one real root if $\Delta < 0$; a multiple real root if $\Delta = 0$; and three distinct real roots if $\Delta > 0$.
Since $g(d)$ is a third-order polynomial, which leading coefficient is negative, it follows that $\frac{\partial g(d)}{\partial d} < 0$ at $d = d^e$ if $g(d)$ has one root. If $g(d)$ has three roots $d_l^e < d_m^e < d_r^e$, it follows that $\frac{\partial g(d)}{\partial d} < 0$ at $d = d_i^e,\ i = l, r$ and $\frac{\partial g(d)}{\partial d} > 0$ at $d = d_m^e$. As $\frac{\partial d^e(\alpha)}{\partial \alpha} = -\frac{\frac{\partial g(\alpha)}{\partial \alpha}}{\frac{\partial g(d)}{\partial d}}$ it follows that the sign of $\frac{\partial d^e(\alpha)}{\partial \alpha}$ coincides with that of the sign of $\frac{\partial g(\alpha)}{\partial \alpha}$ in case there is a unique steady-state value of debt. In a similar way, it follows that the sign of $\frac{\partial d^e(\beta)}{\partial \beta}$ coincides with that of the sign of $\frac{\partial g(\beta)}{\partial \beta}$. Evaluation of these partial derivatives leads then to the next conclusion.

THEOREM 3. *Let $\bar{f} \geq \bar{m}$. Assume either there is a unique value of equilibrium debt, d^e, or that there exist three equilibria, in which case d^e equals either the largest or the smallest equilibrium. Then,*
1. $\text{sign}(\frac{\partial d^e(\alpha)}{\partial \alpha}) = \text{sign}(-(4\alpha + \beta\theta)d^{e^3} - (3\bar{r} - \theta)d^{e^2} - 2(\bar{f} - \bar{m})d^e)$*, and*
2. $\text{sign}(\frac{\partial d^e(\beta)}{\partial \beta}) = \text{sign}(-\alpha\theta d^{e^3} - \bar{r}\theta d^{e^2} - (\bar{f} - \bar{m})\theta d^e)$.

From this follows immediately next result.

COROLLARY 4. *Let $\bar{f} \geq \bar{m}$ and $d^e > 0$ be the equilibrium debt introduced in Theorem 3. Then, d^e decreases if either*
1. β increases, or
2. α increases and $\theta < 3\bar{r}$.

To facilitate a comparison with the original Tabellini model, where no risk-premia are involved, we recall from Engwerda et al.(2013) next result.

PROPOSITION 5. *Consider the game (1)–(5) with $\alpha = \beta = 0$.*
In case of Nash open-loop strategies, the game has a unique steady-state value of debt $d^e = \frac{\gamma_0}{b}$, for every initial state d_0, if $b := \bar{r}(\bar{r} - \theta) + \kappa_F + \kappa_M > 0$. If $b < 0$, the game has no open-loop Nash equilibrium unless $d_0 = d^e = \frac{\gamma_0}{b}$.

So, if $b < 0$, the problem has no solution and debt will grow without bound, except if the initial debt level coincides with d^e. Such cases without solution arise if $\theta > r$ and κ_F and κ_M are relatively small. That is, in cases where the authorities do not care too much about the future development of debt. Under those conditions fiscal and monetary authorities are primarily interested in fixing their policies at their target values. This yields a system where debt diverges from its unstable equilibrium.

From the above analysis it follows that by including a risk premium two additional equilibria will occur to the left and right of this unstable equilibrium. As financial markets will not allow debt to grow without bound, endogenous risk premia are imposed. Policy makers, on their turn, take the endogenous risk premium into account when deciding their policies. The extended model provides therefore a more realistic framework for modeling debt stabilization conflicts in the sense of taking into account financial markets reactions to government debt dynamics.

4.2. Debt Stabilization: The Cooperative Case

In this section, we consider the cooperative case. From Engwerda (2005), we have that the Pareto efficient solutions are obtained[9] by solving for all $\omega \in (0, 1)$ the optimal control problem,

$$\min_{f,m} \omega J_F + (1-\omega) J_M \tag{12}$$

subject to,

$$\dot{d}(t) = [\bar{r} + \alpha d(t) + \beta \dot{d}(t)]d(t) + f(t) - m(t), \; d(0) = d_0. \tag{13}$$

By varying ω between zero and one, one obtains a curve of Pareto efficient solutions. In case this coordination parameter is not explicitly agreed upon a priori by the players the question arises which solution on the Pareto curve will be selected by the players. Answers to this question may be found in the literature on bargaining theory. In our examples further on, we will focus on the case that players have equal bargaining strengths, i.e., $\omega = 0.5$ and call the cooperative equilibrium in that case the "social optimum/equilibrium." Along the lines of Appendix B in Engwerda et al. (2013) one easily establishes then next result, which mimics Theorem 4.1 in Engwerda et al. (2013):

THEOREM 6. *If $(f^*(.), m^*(.)) \in \mathcal{U}$ is a set of Pareto efficient strategies for (1)–(5), there exist a $\omega \in (0, 1)$, $\omega_1^2 := \omega(1-\omega)$, a trajectory for debt $d^*(.)$, and an associated co-state variable $\mu^*(.)$ that satisfy the set of nonlinear differential equations:*

$$\dot{d}(t) = \frac{1}{1-\beta d(t)} \left\{ [\bar{r} + \alpha d(t)]d(t) + \bar{f} - \bar{m} - \frac{\mu(t)}{\omega_1^2} \right\} \tag{14}$$

$$\dot{\mu}(t) = \left(\theta - \frac{\bar{r} + 2\alpha d(t)}{1-\beta d(t)}\right)\mu(t) + \frac{\omega\kappa_F[\bar{d}_F - d(t)] + (1-\omega)\kappa_M[\bar{d}_M - d(t)]}{1-\beta d(t)}. \tag{15}$$

with $d^*(0) = d_0$ *and where both* $\lim_{t\to\infty} d^*(t) = d_c^e$ *and* $\lim_{t\to\infty} \mu^*(t) = \mu_c^e$ *exist. Moreover,*

$$f^*(t) = \bar{f} - \frac{\mu(t)}{\omega} \text{ and } m^*(t) = \bar{m} + \frac{\mu(t)}{1-\omega}.$$

Furthermore, the steady state values (d_c^e, μ_c^e) *satisfy*

$$\mu_c^e := \omega_1^2\{\bar{r}d_c^e + \alpha d_c^{e^2} + \bar{f} - \bar{m}\}, \tag{16}$$

where d_c^e *is a solution of the third-order polynomial equation*

$$g_c(d) := \gamma_3^c d^3 + \gamma_2^c d^2 + \gamma_1^c d + \gamma_0^c = 0. \tag{17}$$

Here $,\gamma_3^c := \omega_1^2\gamma_3, \gamma_2^c := \omega_1^2\gamma_2, \gamma_1^c := \omega_1^2(\bar{r}(\theta-\bar{r}) - \frac{\kappa_F}{1-\omega} - \frac{\kappa_M}{\omega} - (2\alpha+\beta\theta)(\bar{f}-\bar{m}))$ *and* $\gamma_0^c := \omega_1^2(\frac{\kappa_F}{1-\omega}\bar{d}_F + \frac{\kappa_M}{\omega}\bar{d}_M + (\bar{f}-\bar{m})(\theta-\bar{r}))$.

Denoting the right-hand side of (14) and (15) by $f_{i,c}(d,\mu)$, $i = 1, 2$, respectively, straightforward differentiation shows that $\frac{\partial f_{1,c}}{\partial d} + \frac{\partial f_{2,c}}{\partial \mu} = \theta$. So, if $\theta \neq 0$, by Bendixson's theorem, this system of differential equations (14), (15) has no periodic solutions.

Following the lines of the noncooperative case in Section 4.1, one can show next that results obtained there apply for the cooperative case as well. Below we summarize some important observations.

COROLLARY 7.
A. *Consider the discriminant*

$$\Delta_c := 18\gamma_3^c\gamma_2^c\gamma_1^c\gamma_0^c - 4\gamma_2^{c^3}\gamma_0^c + \gamma_2^{c^2}\gamma_1^{c^2} - 4\gamma_3^c\gamma_1^{c^3} - 27\gamma_3^{c^2}\gamma_0^{c^2}. \tag{18}$$

Then, (17) has one real root if $\Delta_c < 0$*; a multiple real root if* $\Delta_c = 0$*; and three distinct real roots if* $\Delta_c > 0$.
B. *Let* $\bar{f} \geq \bar{m}$ *and* $d^e > 0$ *be the equilibrium debt introduced in Theorem 6. Then, independent of the choice of the coordination parameter* ω, d_c^e *decreases if either: 1.* β *increases, or 2.* α *increases and* $\theta < 3\bar{r}$.

Furthermore, a similar remark as in Engwerda et al. (2003) applies here concerning the connection between the noncooperative and cooperative equilibrium solutions.

Remark 8. **1.** Comparing the steady state values (d^e, μ^e) corresponding to the noncooperative open-loop equilibrium for the game (1–5) satisfying (8) and (9), we observe they solve these equations if (and only if) we replace κ_F by $(1-\omega)\kappa_F$ and κ_M by $\omega\kappa_M$, $(d_c^e, \mu_c^e) := (d^e, \omega_1^2\mu^e)$ are the steady-state values satisfying (16) and (17) for the cooperative game (with weight ω).
This implies, for instance, that the steady state value of debt for the social outcome (i.e., $\omega = 0.5$) is also obtained as the steady state debt of the noncooperative open-loop game with parameters $\alpha, \theta, \bar{f}, \bar{m}, \bar{r}, 2\kappa_F, 2\kappa_M, \bar{d}_F$, and $\bar{d}_M$. That is, the steady-state

debt under a cooperative strategy coincides with that under a noncooperative equilibrium, where the weight attached to the deviation from its target values is doubled by both players.[10]

2. In case the preferences of the policy makers are symmetric, i.e., $\bar{d}_F = \bar{d}_M =: \bar{d}$ and $\kappa_F = \kappa_M =: \kappa$, the steady state values (d^e, μ^e) corresponding to the open-loop equilibrium for the game (1)–(5) satisfy (8) and (9) if and only if with κ replaced by $(1-\omega)\omega\kappa$ and $\bar{d}$ replaced by $\frac{1}{\omega}\bar{d}$, $(d_c^e, \mu_c^e) := (d^e, \omega_1^2\mu^e)$ are the steady state values satisfying (16) and (17) for the cooperative game (with weight ω). So, in a symmetric setting, the social outcome is also obtained as the steady state debt of the noncooperative open-loop game with parameters α, β, θ, $\bar{f}$, $\bar{m}$, $\bar{r}$, 4κ, and $\frac{1}{2}\bar{d}$. That is, the steady state debt under a cooperative strategy coincides with that under a noncooperative equilibrium, where the weight attached to the deviation from its target values is quadrupled by both players and the target value itself is halved. Or, putting it into another perspective, in the cost functions (2), (12) the term $\kappa(d(t) - \bar{d})^2$ is replaced by $\kappa(2d(t) - \bar{d})^2$ (and in general: $\frac{\omega}{1-\omega}\kappa(\frac{1}{\omega}d(t) - \bar{d})^2$). Which can be interpreted as that the same level of equilibrium debt as in the social outcome is attained in a noncooperative setting if players consider the gap between twice the current debt and its target value. Hence, to attain the same equilibrium debt, players must be much more keen on keeping debt close to the target value.

5. A CASE STUDY

In this section, we use numerical simulations of the model outlined in the previous section, to illustrate the main aspects of the dynamic debt stabilization conflict between the monetary and fiscal authorities in the presence of endogenous risk premia. Centerpoint is a baseline set of model parameters that represent a country where monetary and fiscal players have symmetric priorities about the realization of targets and, moreover, are concerned about the future development of debt. We focus on cases with initial debt substantially above the optimal debt target. Special attention is paid to the effects of variation in the risk premium parameters. Finally, we also consider two additional scenarios: a case where both policymakers are less concerned in controlling debt accumulation and a case where the fiscal authorities are more short-sighted than monetary authorities in the sense of focusing mostly on short-term outcomes while discounting more heavily long-run outcomes than the monetary authorities. Both cases therefore concern the complications that arise in case one or both players become less concerned about debt stabilization. In a broad view, the consequences in the current framework resemble the monetary and fiscal policy coordination problems in Sargent and Wallace (1981) "unpleasant monetarist/fiscal arithmetic" and Leeper's (1991) analysis of "active" and "passive" monetary and fiscal policies, with the endogenous risk premium as an additional transmission channel which complicates the outcomes.

5.1. The Baseline Case

We center the analysis on the following set of baseline parameters: $\theta = 0.1$, $\bar{f} = 0.01$, $\bar{m} = 0$, $\bar{d}_F = 0.6$, $\bar{d}_M = 0.6$, $\kappa_F = 0.04$, $\kappa_M = 0.04$, and $\bar{r} = 0.03$.

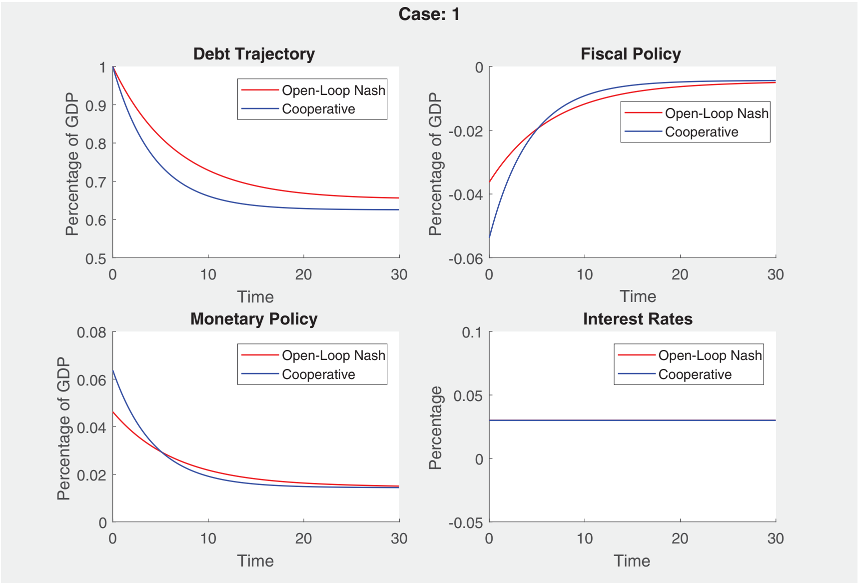

FIGURE 1. Debt, deficits, money growth, and risk premia under cooperative (blue lines) and noncooperative (red lines) policy making: linear model case, $\alpha = 0.0$ and $\beta = 0.0$.

Similar baseline parameter values are used in other extensions of the Tabellini model by Castren (1998) and Figuieres (2008). Clearly, preference parameters and target values remain arbitrary in nature and hard, if not impossible, to estimate in reality and are therefore selected more for their illustrative character. For example, the debt target of 0.6 is of course inspired by the debt criterion in the Maastricht Treaty, the zero primary balance target resembles the close-to-zero medium term objective for the structural budget balance. The initial debt level equals 1.0. In the cooperative case, we assume $\omega = 0.5$.

We want to analyze in particular how outcomes depend on the effects of different values of α and β, i.e., different curvatures of the risk premium formation function. The literature overview provided some discussion of the relevant parameter ranges of α and β. For our purposes, we note that probably a value of α of 0.05 (and lower) holds in case of low sensitivity of risk premia w.r.t. government debt. A value of α of 0.1 (and higher) holds in case of high sensitivity of risk premia w.r.t. government debt. Concerning the sensitivity of the risk premia w.r.t. the debt change, we consider a value of β of 0.2 (and lower) to imply a low sensitivity of risk premia w.r.t. the change in government debt. A value of β of 0.2 (and higher) holds in case of high sensitivity of risk premia w.r.t. the change in government debt.[11]

Starting point is the adjustment in Figure 1 that is produced in case financial markets do not impose a risk premium (the linear model) $\alpha = 0.0$ and $\beta = 0.0$.

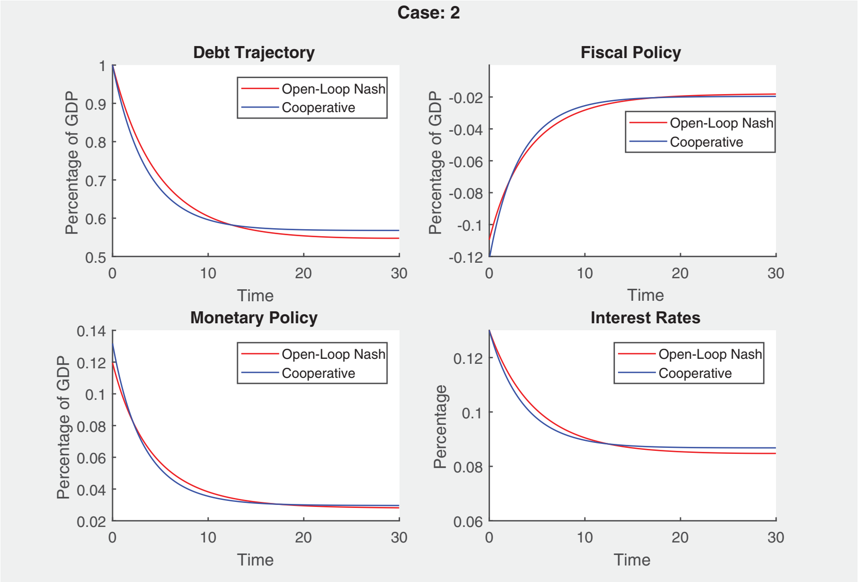

FIGURE 2. Debt, deficits, money growth, and risk premia under cooperative (blue lines) and noncooperative (red lines) policy making: nonlinear model case, $\alpha = 0.1$ and $\beta = 0.0$.

In this case, there is of course no endogenous risk-premium on government debt as in the original model of Tabellini (1986). Clearly, visible is one of the important insights that in the linear model, long-run government debt is lower in case of cooperative policy making than in the noncooperative case. Also visible is that the adjustment speed is higher in the cooperative case.[12]

Next, we consider the effects of introducing nonlinearities in the risk premium. Figure 2 gives the dynamic adjustment of Case 2 where there is an endogenous risk premium associated with the level of debt: $\alpha = 0.1$ and $\beta = 0.0$. This implies nonlinearity in the debt dynamics.

Initially, the risk premium is substantial since the level of debt is high. Over time debt is reduced and in the new steady state the risk premium reaches over 8% . A number of interesting issues result from this case. First, the endogenous risk premium w.r.t. the level of debt leads to a case where debt in the long run is higher in the cooperative than in the noncooperative case. A result that therefore contradicts/extends the finding of Tabellini (1986) for the case without endogenous risk premia in which case long-run debt is always lower in the cooperative case. Second, the difference between the cooperative and noncooperative case is narrowed in case of the endogenous risk premium w.r.t. the level of debt. In other words, if policy coordination is not feasible for some reason, the presence of endogenous risk premia, essentially still forces noncooperative policy makers to follow policies that are approaching outcomes under policy coordination. An important final insight of this example is that the endogenous risk premia induces

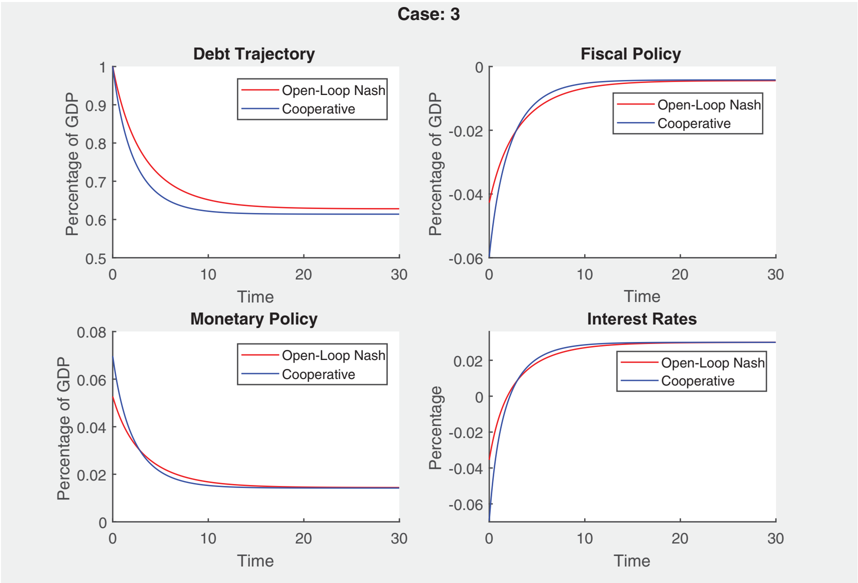

FIGURE 3. Debt, deficits, money growth, and risk premia under cooperative (blue lines) and noncooperative (red lines) policy making: nonlinear model case, $\alpha = 0.0$ and $\beta = 0.5$.

more discipline (compared to the linear model): debt is lower than in the absence of risk premia, as both monetary and fiscal players are contributing more to debt stabilization. This holds both for the cooperative and noncooperative case. The debt level in the risk premium acts as a disciplining device in the debt stabilization problem, a role that is also emphasized in the theory of market-based fiscal discipline: the threat of risk premium by financial markets—reflected by the parameter α—is internalized by the policy makers and creates incentives to keep debt under control.

In the third case, we consider the effect of the risk premium depending on the change in the debt level, $\beta = 0.5$ (instead of 0.0 as in the baseline case) in Figure 3.

Here, debt in the long run is lower in the cooperative than in the noncooperative case as in the linear model. In this case, the risk premium is reduced initially due to effect of the change in the debt level: the decline of debt contributes to a lower/negative risk premium which itself contributes to debt stabilization: this is the rewarding element in the risk premium of a declining debt trajectory. But after the initial drop, the effect dissipates over time as debt approaches equilibrium. This gives a more general insight: the effect from the change in debt in the risk premium function contributes most in the short-run when debt is declining most (or increasing most in case we would start with a debt path with very low initial—but rapidly increasing—debt) because it is far from the steady-state. In the long run, this effect fades out. Comparing Figures 1–3, it is observed that the level of debt in the risk premium (the "stick" whose effects continue to work on the long-run

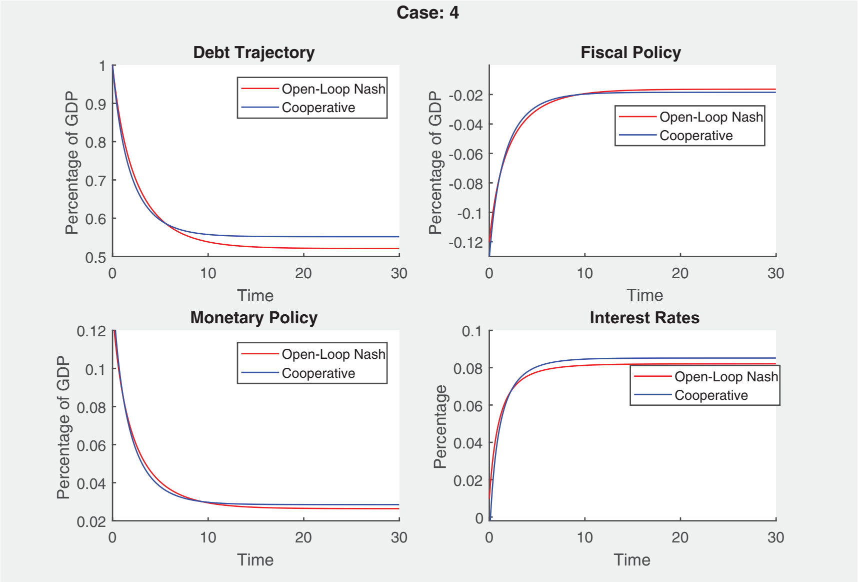

FIGURE 4. Debt, deficits, money growth, and risk premia under cooperative (blue lines) and noncooperative (red lines) policy making: nonlinear model case, $\alpha = 0.1$ and $\beta = 0.5$.

risk premium) contributes more to debt stabilization than the change of debt in the risk premium (the "carrot" whose effects dissipate in the long run). This holds not only for this specific choice of α and β, but for a broad range of combinations.

In the fourth case, we consider the effect of the risk premium depending on both the level and the change in the debt level, $\alpha = 0.1$ and $\beta = 0.5$ in Figure 4.

We observe that government debt is stabilized in the long run at an even lower level due to the interaction of both parameters in the risk premium function. That risk premia are not only depending on the level but also on the change of government debt is therefore an important feature since it contributes to reducing long-run government debt. This reflects the stronger disciplining force exerted by financial markets that is associated with a higher value of α, i.e., the stronger nonlinearities in debt dynamics in the debt stabilization game. β contributes in the short-run the stabilization of debt by reducing the risk premium. Still, in the long run, the effect of the level of debt will dominate the effect of the change in debt as we observe that the interest rate rises from a low value initially to its high long-run value of around eight percent. Note also that the long-run debt is again higher under cooperation than under noncooperation as in Case 2.

5.2. Effects of Reducing the Priority Given to Debt Stabilization

In Case 5, we analyze the case where monetary and fiscal policy makers are less concerned about debt stabilization: the parameters that measure the weight given

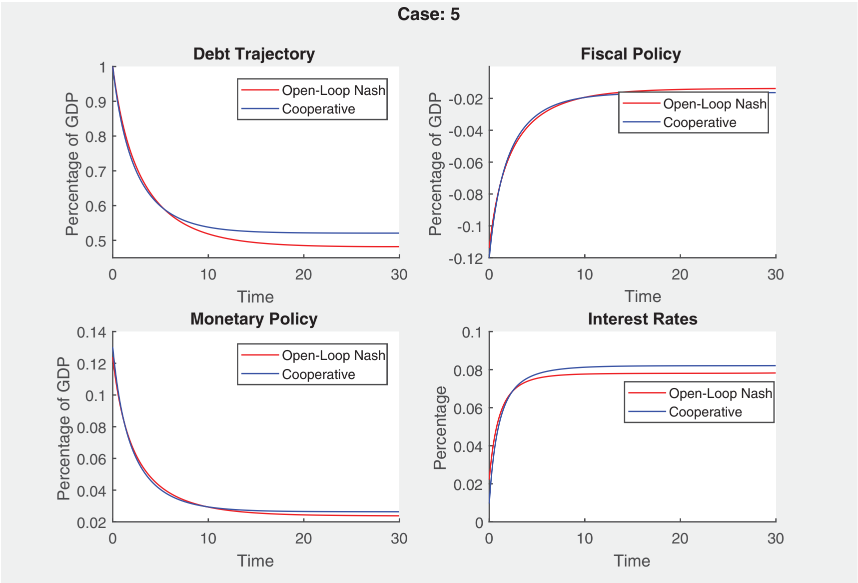

FIGURE 5. Debt, deficits, money growth, and risk premia under cooperative (blue lines) and noncooperative (red lines) policy making with reduced priority to debt stabilization, $\kappa_F = 0.01$, $\kappa_M = 0.01$.

to debt stabilization are reduced to, $\kappa_F = 0.01$, $\kappa_M = 0.01$. As in Case 4, it is assumed that $\alpha = 0.1$ and $\beta = 0.5$. It is easily verified that also for this case the discriminant $\Delta(\alpha, \beta)$ is negative for all $(\alpha, \beta) \in [0.01, 0.5] \times [0.01, 0.5]$, implying there is a unique steady state for all relevant risk premium parameter values and that debt converges to this steady state value.

This choice of parameters enables to look at the consequences of policy makers being less concerned about the accumulation of government debt and being more focused on their policy instruments. In the linear model, such a reduction in the priority attached to debt stabilization would lead to an increase in the long-run level of debt,—both in the noncooperative and cooperative case—as shown by Tabellini (1986). Figure 5 displays the resulting adjustment dynamics.

Effects are relatively small, but it can be observed that long-run debt, interest rate/the risk premium acts both as a disciplining device (a "stick"), deterring high levels of debt and a reward (a "carrot") for declining trajectories of indebtedness: if debt is decreasing, the reduction in interest rates is acting as a reward. This reduction on its turn is helping the debt level to decrease and, consequently, implies less future actions are needed risk-premium, fiscal surplus, and money growth are all smaller when the weight on debt stabilization is reduced (compare Figures 4 and 5). This therefore leads again to a difference between the linear model and the model with an endogenous risk premium: in case of an endogenous risk premium the result that a lower priority given to debt stabilization will lead to higher long-run debt does no longer hold necessarily. This could be explained by the strength of the market-based discipline from the risk premium. In case the priority to debt stabilization is reduced (but not to zero), the risk premium still deters authorities to let debt accumulation go out of control and the nonlinearity of the risk premium more than compensates the effect from lower priority given to debt stabilization.

5.3. Effects of Fiscal Shortsightedness

In this final Case 6, we consider a scenario where the fiscal authority and the Central Bank have different time preferences. We consider a setting where the fiscal authority is much less concerned about future outcomes than the Central Bank, reflecting e.g., the presence of politicians that are focussing on current outcomes rather than outcomes more further away in the future. This is expressed by choosing a discount factor for the Central Bank of $\theta_M = 0.1$ and for the fiscal authority of $\theta_F = 0.3$. As in Case 4 and 5, it is assumed that $\alpha = 0.1$ and $\beta = 0.5$. In the Appendix, we show how in this case equilibrium debt can be obtained under a noncooperative open-loop information setting. To find the relevant equilibrium for this case, roots of a fourth-order polynomial have to be calculated. For the above mentioned choice of parameters, this polynomial always has two real roots. One root is associated with a dynamically stable, and the other one with a dynamically unstable equilibrium debt. In the graphs below, we just consider the stable equilibrium point. In case policy makers have different time preferences, i.e., $\theta_F > \theta_M$, it follows (see Appendix, Theorem 11) that the equilibrium variables in the cooperative case are given by:

$$d^e = \bar{d}_M,\ m^e = \bar{m},\ \text{and}\ f^e = \bar{m} - (\bar{r} + \alpha \bar{d}_M)\bar{d}_M. \tag{19}$$

That is, independent of the coordination parameter ω, debt converges to the target value set by the monetary authority where, in the end, the monetary authority uses its policy target value as well. The monetary authority takes a larger adjustment burden in the short run from the lack of adjustment by the fiscal authority, as shows the high money growth in both the noncooperative and cooperative case. Furthermore, we observe that the steady state fiscal policy is not affected by the β parameter. Only the level parameter α plays a role here.

Figure 6 provides the adjustment dynamics that result in this case.

The increased short-sightedness increases the "size" of the short-term "carrot" and even more important reduces the "size" of the long-run "stick." In particular in the noncooperative case, the fiscal player loses sight on the need for long-run debt stabilization. As a result, the adjustment burden is passed on the monetary authority. The spirit of this example, in other words, is close to (or confirms) the logic in Leeper's (1991) analysis of "active" and "passive" monetary and fiscal policies and Sargent and Wallace (1981) "unpleasant monetarist/fiscal arithmetic." Postponing debt stabilization increases the adjustment burden later on for either

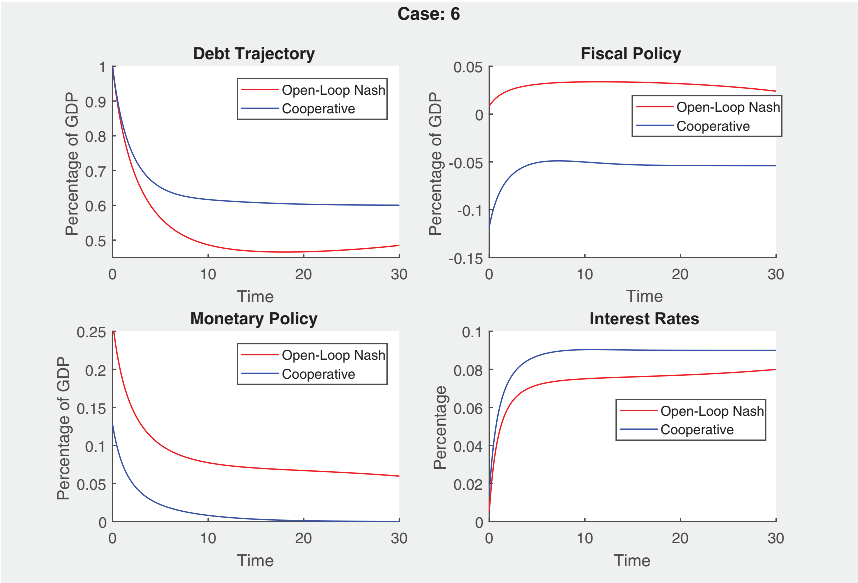

FIGURE 6. Debt, deficits, money growth, and risk premia under cooperative (blue lines) and noncooperative (red lines) policy making with fiscal short-sightedness, $\theta_F = 0.3$.

both authorities or at least one of them in case of a fundamental dominance of the other. Here in this case, this adjustment burden is placed on the "patient" monetary authority by an "impatient" fiscal player, in particular in the noncooperative case.

5.4. Comparing the Costs of Cooperative and Noncooperative Policy Making

It is insightful to analyze the losses the players incur in the cases we analyzed in this section. In Table 2, these are given. In this example, the benefits from cooperation in terms of debt stabilization (lower debt, lower monetary financing, and lower deficits) are considerable in the sense that cooperation results in more optimal adjustments of debt, deficits, and money growth than under noncooperative policies, in fact welfare losses under cooperation are typically half those of noncooperation, reflecting the logic of Remark 8.

We observe that the cost increase if α increases (Case 2) and decrease if β increases (Case 3). As discussed above the long-run effects from α will dominate over time in Case 4. In the fiscal short-sightedness Case 6, a complete different cost structure occurs under noncooperation, we see that (conform the graphs) the monetary player bears most of the cost. Furthermore, we see that in all cases except Case 6, the cost for the policy makers (approximately) coincide and that cost under cooperation are lower than under a noncooperative mode of play. Roughly speaking these cost differ by a factor of 2.

TABLE 2. Welfare losses monetary (M) and fiscal (F) authorities in different scenarios, noncooperative Nash open-loop (nc), and cooperative policies (co)

	L^F_{nc}	L^M_{nc}	L^F_{co}	L^M_{co}
Case 1	0.178	0.178	0.084	0.085
Case 2	0.431	0.431	0.215	0.215
Case 3	0.118	0.118	0.057	0.057
Case 4	0.304	0.304	0.151	0.151
Case 5	0.290	0.290	0.144	0.144
Case 6	0.059	2.653	0.087	0.067

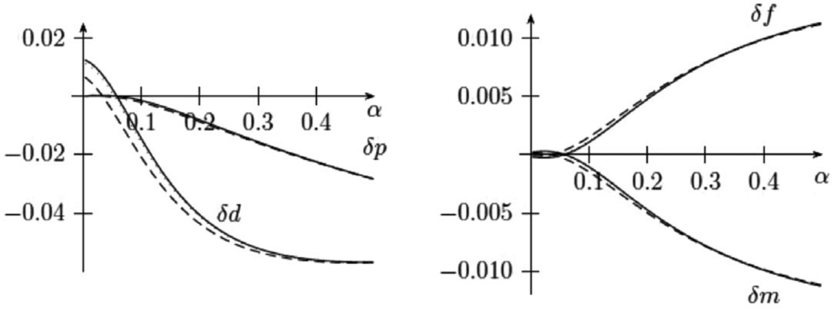

FIGURE 7. Steady-state differences in debt, deficits, money growth, and risk premia as a function of α for the benchmark parameters. Dashed: $\beta = 0.5$, Dotted: $\beta = 0.1$, Line: $\beta = 0.01$.

5.5. Effects of α and β on Steady-State Differences

It is interesting to analyze in some more detail the steady-state differences between the noncooperative and cooperative cases and the role played by α and β. We will use the shorthand notation:

$$\delta d := d^e - d^c_e;\ \delta p := \alpha(d^e - d^c_e);\ \delta f := f^e - f^c_e;\ \text{and } \delta m := m^e - m^c_e. \quad \textbf{(20)}$$

$\delta d > 0$, $\delta p > 0$, $\delta f > 0$, and $\delta m > 0$ in other words would mean that—in steady-state—respectively, debt, risk premium, deficit, and money growth in the noncooperative case exceed debt, risk premium, deficit, and money growth in the cooperative case. Figure 7 displays these steady-state differences for the baseline set of parameters.

Panel (a) illustrates that for small values of the level parameter α, steady-state debt is larger under a noncooperative regime than under a cooperative one and that, for larger values of α, the picture reverses. Then, the steady-state cooperative debt is smaller under a noncooperative setting. As explained in Engwerda et al. (2013), this may be due to the fact that under a noncooperative mode of play some overreaction occurs by the players to control debt to its target value (see panel (b)), which brings on a lower steady-state equilibrium debt. We see that this effect is somewhat intensified when parameter β increases. For larger values of α the impact of β on the attained steady-state value of debt and instruments disappears.

6. CONCLUDING REMARKS

In this paper, we analyzed the impact of risk premia in interest rates on the evolution of government debt, deficits, and money growth in a dynamic game of debt stabilization between the fiscal and monetary authorities. The endogenous risk premium changes the intrinsic dynamics of government debt and thereby the dynamic debt stabilization game. In addition to the level of debt, we introduced the rate of change of debt as a component of the risk premium and analyzed its impact on debt in a simple debt game between a monetary and fiscal player. The risk premium acts both as a disciplining device (a "stick"), deterring high levels of debt and a reward (a "carrot") for declining trajectories of indebtedness: if debt is decreasing, the reduction in interest rates is acting as a reward. This reduction on its turn is helping the debt level to decrease and, consequently, implies less future actions are needed. The inclusion of debt change as a variable is motivated by the fact that financial institutions will be more reassured that debt will be repayed when debt is declining or, in the opposite case, feel less comfortable that debt will be repaid when debt rises.

We analyzed both a noncooperative mode of play and a cooperative mode of play. Including a risk premium into the original Tabellini (1986) model brings on that the game always has at least one equilibrium and at most three equilibria. In principle, by changing either one of the involved risk premium parameters, the number of equilibria may fluctuate. The scenarios we considered in this study always had a unique steady-state equilibrium, implying that the dependency on initial debt is only temporary and that for all values of initial debt, debt converges to the same long-run value.

We performed a simulation study in which we analyzed a number of different scenarios in detail. We considered a baseline scenario, modeling a situation where both policy makers are concerned about debt stabilization. We showed that some of the logic of the model without risk premia needs to be modified in the presence of risk premia. In particular, cooperation—while leading to faster adjustment and lower losses—is no longer a guarantee for lower long-run debt and a declining weight attached to debt stabilization is not leading to lower long-run debt necessarily.

We considered also a case where both policy makers are primarily interested in targeting their own policy instrument and get less concerned about debt stabilization. Finally, we considered a case with fiscal short-sightedness where the fiscal authority is concentrated on the short-run and disregarding mostly consequences from a long-term view. These last two cases convey the same logic as well-known results of Sargent and Wallace (1981) "unpleasant monetarist/fiscal arithmetic" and Leeper's (1991) "active" and "passive" monetary and fiscal policies, concerning the consequences from postponing debt adjustment by monetary and fiscal authorities.

Clearly, the model considered is quite stylized. To obtain a model more close to real life a number of extensions are called for. First, the relationship with financial markets could be modeled more explicitly. That is, in the current setup the risk premium parameters are assumed to be imposed by the financial markets. Therefore, it seems interesting to include financial markets as a separate player in this setup too. Second, a multicountry setting is called for if the model should reflect, e.g., the EMU, where risk premia are differentiated between countries. Due to the nonlinearities already present in the current model this probably invokes the study of multiple equilibria in case countries are nonsymmetric. Some first experiments performed by the authors in this direction show that the choice of parameters plays an important role in the analysis then. Third, the current model neglects the relationship with output (real economy). Obviously, to stay more close to reality, this connection should be incorporated too. Finally, informational aspects might play a role too in the results obtained. For instance, we assumed for the noncooperative setting an open-loop information structure. To test how robust the results are w.r.t. this assumption, one might consider the problem under different information structures like, e.g., a feedback information structure, a moving horizon perspective, or different decision structures (hierarchical).

NOTES

1. A financial analyst puts these mechanisms in a concrete case: "The Nikkeis 7.3% fall in one day is an example of the herding behavior seen in financial markets. All players stampeded for the exit at the same time. But why? There are good reasons: Most fund managers are compensated for relative performance. Investors are influenced by others decisions fearing they know something they do not. And humans generally like the safety of conformity. But it is not those that are part of the herd that make the money. The real money is made by going against the crowd. It is not education and brains that make a great investor or trader, it is attitude. Those unique few who are comfortable without the safety net of conformity." [Cooper (2013)].

2. See e.g., Bikchandani and Sharma (2001) for a detailed overview on theory and evidence on herding in bond and other financial markets and the Bayesian updating by traders of ambiguous beliefs in a model of sequential trading in financial markets.

3. In the EMU case e.g., the concern for increasing debt is implicit in the debt criterion of 60% of GDP: government debt should remain below this threshold or decreasing "at a sufficient rate" toward this target.

4. Note that in the Euro Area case an additional aspect arises: the members of the Union have a common monetary policy that is implemented by the ECB. Fiscal policy that is implemented by each member country itself, subject to the budgetary restrictions of the Stability and Growth Pact that seeks to prevent excessive deficits.

5. In Engwerda (2016b), this analysis is extended to a multicountry setting, with emphasis on the symmetric country case, yielding similar results.

6. The model can be readily transformed to discrete time. The use of continuous time has a number of notational and computational advantages.

7. The rate of output growth will be assumed constant throughout the analysis. In an alternative interpretation, (5) describes the effects from the debt level and the change in the debt level on output growth. In a much discussed contribution, Reinhart and Rogoff (2010) provided empirical evidence for a negative relation between output growth and debt levels.

8. The rapid deterioration of conditions during the euro-area debt crisis of 2010–2013 and the ensuing increase in risk premia on government debt of peripheral, high debt euro-area countries are a good illustration of this point: in the model, the crisis could be interpreted as a swift increase in α and β, increasing the risk premia, and offsetting debt instability problems.

9. It might be that we still miss some Pareto solutions [see, e.g., Reddy and Engwerda(2014) or Engwerda (2010) for the finite planning horizon case]. However, a more detailed analysis of this issue is beyond the scope of this paper. We will restrict therefore the analysis to the set of Pareto solutions obtained by minimizing a weighted sum of the players' cost functions.

10. A comparable result is also obtained by Hughes Hallett and Weymark (2007) who compare the noncooperative game of monetary and fiscal policy interaction with outcomes under fiscal leadership. They argue that in the real world in several cases the noncooperative interaction with independent Central Banks and Treasuries seems realistic but that in several cases conditions resemble the case of fiscal leadership. In the setting of their model, it is shown that fiscal leadership will lead to improved outcomes compared to the noncooperative case.

11. It is easily numerically verified that for Case 1–3 the discriminant (18), $\Delta(\alpha, \beta)$, is negative for all $(\alpha, \beta) \in [0.01, 0.5] \times [0.01, 0.5]$. So, there is a unique steady state for all relevant risk premium parameter values and the dynamic behavior of the system (6, 7). This implies in particular that, independent of the choice of the risk premium parameters, the steady-state value of debt does not depend on the initial value of debt.

12. That policy coordination contributes to welfare losses and speeds up adjustment is well established throughout the entire policy coordination literature. Hughes Hallett (1986) summarizes this as follows: "..coordination restores policy effectiveness, and also cuts the cost of intervention by speeding up policy responses" and "..Coordination produces better results both because it allows governments greater freedom to specialize in those policy instruments that have comparative advantage and because it allows governments to coordinate the timing of their policy impacts. Hence, coordination should aim to exploit the differences between economies, and to organize the sequencing of policy actions rather than to promote parallel policies."

13. By introducing another variable $\nu(t) := \mu_F(t) - \mu_M(t)$ we have that $(d(t), \mu_F(t), \mu_M(t))$ solve (A.9-A.11) if and only if $(d(t), \mu(t), \nu(t))$ solve (A.13,A.14) and an another differential equation $\dot{\nu}(t) = f_3(d(t), \mu(t), \nu(t))$. So, by considering this coordination transformation we triangularized the system dynamics.

REFERENCES

Alessandrini, P., M. Fratianni, A. Hughes Hallett, and A. Presbitero (2014) External imbalances and fiscal fragility in the Euro Area. *Open Economies Review* 25, 3–34.

Baldacci, E. and M. Kumar (2012) Fiscal Deficits, Public Debt, and Sovereign Bond Yields. IMF working paper 10/184.

Başar, T. and G. J. Olsder (1999) *Dynamic Noncooperative Game Theory*. Philadelphia: SIAM.

Bayoumi, T., M. Goldstein, and G. Woglom (1995) Do credit markets discipline sovereign borrowers? Evidence from U.S. states. *Journal of Money, Credit and Banking* 27(4), 1046–1059.

Bernoth, K., J. von Hagen, and L. Schuknecht (2013) Sovereign risk premiums in the European govesrnment bond market. *Journal of International Money and Finance* 31, 975–995.

Bikchandani, S. and S. Sharma (2001) Herd behavior in financial markets. *IMF Staff Papers* 47(3), 279–310.

Blanchard, O., J. Chouraqui, R. Hangemann, and N. Sartor (1990) The sustainability of fiscal policy: New answers to an old question. *OECD Economic Studies* 15(1), 7–36.

Blueschke, D. and R. Neck (2011) "Core" and "Periphery" in a monetary union: A macroeconomic policy game. *International Advances in Economic Research* 17, 334–346.

Castren, O. (1998) Optimal fiscal and monetary policy under debt reduction programmes. Fiscal-Monetary Policy Coordination and Central Bank Independence. *Ph.D. thesis*, Bank of Finland Studies E:13, 1998 (Chapter 4).

Cooper, L. (2013) Herding behaviour in financial markets. http://coopercity.co.uk/herding-behaviour-in-financial-markets/

Day, R. (2000) *Complex Economic Dynamics*, vol. 2. London: MIT Press.

De Grauwe, P. and Y. Ji (2012) Mispricing of Sovereign Risk and Multiple Equilibria in the Eurozone. CEPS working paper 361.

Di Bartolomeo, G. and D. Di Gioacchino (2008) Fiscal-monetary policy coordination and debt management: A two-stage analysis. *Empirica* 35, 433–448.

Domar, E. (1944) The "Burden of the Debt" and the national income. *American Economic Review* 34(4), 798–827.

Engwerda, J. C. (2005) *LQ Dynamic Optimization and Differential Games*. Chichester: John Wiley & Sons.

Engwerda, J. C. (2010) Necessary and sufficient conditions for Pareto optimal solutions of cooperative differential games. *SIAM Journal on Control and Optimization* 48, 3859–3881.

Engwerda, J., B. van Aarle, J. Plasmans, and A. Weeren (2013) Debt stabilization games in the presence of risk premia. *Journal of Economic Dynamics and Control* 37, 2525–2546.

Engwerda, J. C., D. Mahmoudinia, and R. D. Isfahani (2016) Government and central bank interaction under uncertainty: A differential games approach. *Iranian Economic Review* 20, 225–259.

Engwerda, J., B. van Aarle, A. Weeren, and J. Plasmans (2015) Debt stabilization games in a monetary union with endogenous risk premia. In N. Wohlgemuth and D. Bluschke (eds.), *Festschrift für Reinhard Neck zum 65. Geburtstag*, pp. 149–180. Peter Lang: Frankfurt.

Figuieres, C. (2008) European Central Bank, international strategic relations and the stabilisation of debt. *Recherches Economiques de Louvain* 74(2), 167–190.

Foresti, P. (2017) Monetary and fiscal policies interaction in monetary unions. *Journal of Economic Surveys*. doi: 10.1111/joes.12194

Ghosh, A., J. Kim, E. Mendoza, J. Ostry, and M. Qureshi (2013) Fiscal fatigue, fiscal space and debt sustainability in advanced economies. *The Economic Journal* 123, F4–F30.

Hagen, J. von, L. Schuknecht, and G. Wolswijk (2011) Government bond risk premiums in the EU revisited: The impact of the financial crisis. *European Journal of Political Economy* 27, 36–43.

Haugh, D., P. Ollivaud, and D. Turner (2009) What Drives Sovereign Risk Premiums?: An Analysis of Recent Evidence from the Euro Area. OECD Economics Department working papers no. 718.

Heinemann, F., S. Osterloh, and A. Kalb (2014) Sovereign risk premia: The link between fiscal rules and stability culture. *Journal of International Money and Finance* 41, 110–127.

Hughes Hallett, A. (1986) Autonomy and the choice of policy in asymmetrically dependent economies. An investigation of the gains from international policy co-ordination. *Oxford Economic Papers* 38, 516–544.

Hughes Hallett, A. and D. Weymark (2007) Fiscal leadership and central bank design. *Journal of Canadian Economics* 40(2), 607–627.

Hughes Hallett, A. J. (2008) Debt targets and fiscal sustainability in an era of monetary independence. *International Economics and Economic Policy* 5(1), 165–187.

Irving, R. S. (2004) *Integers, Polynomials, and Rings*. Berlin: Springer-Verlag.

Leeper, E. M. (1991) Equilibria under "active" and "passive" monetary and fiscal policies. *Journal of Monetary Economics* 27, 129–147.

Lukkezen, J. and H. Rojas-Romagosa (2012) When is Debt Sustainable? CPB discussion paper 212, CPB Netherlands Bureau for Economic Policy Analysis.
Neck, R. and J. E. Sturm (2008) *Sustainability of Public Debt*. Cambridge, Massachusetts: MIT Press.
Reddy, P. V. and J. C. Engwerda (2014) Necessary and sufficient conditions for Pareto Optimality in infinite horizon cooperative differential games. *IEEE Transactions on Automatic Control* 59, 2536–2543.
Reinhart, C. M. and K. Rogoff (2010) Growth in times of debt. *American Economic Review* 100, 573–578.
Sargent, T. J. and N. Wallace (1981) Some unpleasant monetarist arithmetic. *Federal Reserve Bank of Minneapolis Quarterly Review*, Fall, pp. 1–17.
Staehler, N. (2013) Recent developments in quantitative models of sovereign default. *Journal of Economic Surveys* 27(4), 605–633.
Tabellini, G. (1986) Money, debt and deficits in a dynamic game. *Journal of Economic Dynamics and Control* 10, 427–442.
Unctad, 2009. *Compendium on Debt Sustainability and Development*. United Nations, New York and Geneva, pp. 1–218.
van Aarle, B., L. Bovenberg, and M. Raith (1997) Is there a tragedy of a common Central Bank? A dynamic analysis. *Journal of Economic Dynamics and Control* 21(2–3), 417–447.
van den Broek, B. (2002) Moving horizon control in dynamic games. *Journal of Economic Dynamics and Control* 26(6), 937–961.

APPENDIX A: THE OPEN LOOP CASE

Proof of Theorem 1. Let $(f^*(.), m^*(.)) \in \mathcal{U}$ be a set of open-loop Nash strategies and $d^*(.)$ the corresponding debt trajectory.
The Hamiltonians for the fiscal and monetary player for (1)–(12) with endogenous risk premium (5) are as follows:

$$H_F := \frac{1}{2}e^{-\theta t}(f-\bar{f})^2 + \frac{1}{2}e^{-\theta t}\kappa_F(d-\bar{d}_F)^2 + \lambda_F\frac{\bar{r}d+\alpha d^2+f-m}{1-\beta d} \qquad \textbf{(A.1)}$$

and

$$H_M := \frac{1}{2}e^{-\theta t}(m-\bar{m})^2 + \frac{1}{2}e^{-\theta t}\kappa_M(d-\bar{d}_M)^2 + \lambda_M\frac{\bar{r}d+\alpha d^2+f-m}{1-\beta d}, \qquad \textbf{(A.2)}$$

respectively. It is easily verified that this is a normal problem and by Pontryagin's maximum principle there exist continuous and piecewise continuous differentiable functions $\lambda_i^*(.),\ i=F,M,\ f^*(.), m^*(.)$, and $d^*(.)$ that satisfy the equations:

$$\dot{d}(t) = \frac{\bar{r}d(t)+\alpha d^2(t)+f(t)-m(t)}{1-\beta d(t)},\ d(0)=d_0, \qquad \textbf{(A.3)}$$

$$-\dot{\lambda}_F = \frac{\partial H_F}{\partial d} = e^{-\theta t}\kappa_F(d-\bar{d}_F) + \lambda_F\frac{(\bar{r}+2\alpha d)(1-\beta d)+\beta(\bar{r}d+\alpha d^2+f-m)}{(1-\beta d)^2}, \qquad \textbf{(A.4)}$$

$$-\dot{\lambda}_M = \frac{\partial H_M}{\partial d} = e^{-\theta t}\kappa_M(d-\bar{d}_M)$$
$$+\lambda_M \frac{(\bar{r}+2\alpha d)(1-\beta d)+\beta(\bar{r}d+\alpha d^2+f-m)}{(1-\beta d)^2}, \quad \textbf{(A.5)}$$

where $f^*(t)=\bar{f}-e^{\theta t}\frac{\lambda_F^*(t)}{1-\beta d(t)}$ and $m^*(t)=\bar{m}+e^{\theta t}\frac{\lambda_M^*(t)}{1-\beta d(t)}$ (since $\frac{\partial H_F}{\partial f}=\frac{\partial H_M}{\partial m}=0$ and both $\frac{\partial^2 H_F}{(\partial f)^2}>0$ and $\frac{\partial^2 H_M}{(\partial m)^2}>0$).

Since by assumption $f^*(t)$ and $m^*(t)$ converge, it follows that $\lambda_i^*(t),\ i=F,M$, converge exponentially to zero. In particular, this implies that the limiting transversality conditions $\lim_{t\to\infty}\lambda_i^*(t)(d(t)-d^*(t))\geq 0$ hold. Consequently, the above necessary conditions are sufficient too in case, for instance, both minimized Hamiltonians are convex in d along the optimal shadow price paths. Some elementary calculations show that this is the case iff $\kappa_F(1-\beta d(t))^2+2(\bar{f}-f^*(t))[\alpha+\beta(1-\beta d(t))(\bar{r}+2\alpha d(t)+\beta\dot{d}(t))]\geq 0$ and $\kappa_M(1-\beta d(t))^2+2(\bar{m}-m^*(t))[\alpha+\beta(1-\beta d(t))(\bar{r}+2\alpha d(t)+\beta\dot{d}(t))]\geq 0$, respectively.

Substitution of f and m into the above equations shows that $f^*(.)$, $m^*(.)$, and $d^*(.)$ solve the set of nonlinear differential equations:

$$\dot{d}(t)=\frac{\bar{r}d(t)+\alpha d^2(t)+\bar{f}-\bar{m}-e^{\theta t}\frac{\lambda_F+\lambda_M}{1-\beta d(t)}}{1-\beta d(t)},\ d(0)=d_0, \quad \textbf{(A.6)}$$

$$\dot{\lambda}_F=-e^{-\theta t}\kappa_F(d-\bar{d}_F)$$
$$-\lambda_F\frac{(\bar{r}+2\alpha d)(1-\beta d)+\beta(\bar{r}d+\alpha d^2+\bar{f}-\bar{m}-e^{\theta t}\frac{\lambda_F+\lambda_M}{1-\beta d(t)})}{(1-\beta d)^2}, \quad \textbf{(A.7)}$$

$$\dot{\lambda}_M=-e^{-\theta t}\kappa_M(d-\bar{d}_M)$$
$$-\lambda_M\frac{(\bar{r}+2\alpha d)(1-\beta d)+\beta(\bar{r}d+\alpha d^2+\bar{f}-\bar{m}-e^{\theta t}\frac{\lambda_F+\lambda_M}{1-\beta d(t)})}{(1-\beta d)^2}. \quad \textbf{(A.8)}$$

Or, introducing $\mu_i(t):=e^{\theta t}\frac{\lambda_i(t)}{1-\beta d(t)},\ i=F,M$,

$$\dot{d}(t)=\frac{\bar{r}d(t)+\alpha d^2(t)+\bar{f}-\bar{m}-(\mu_F(t)+\mu_M(t))}{1-\beta d(t)},\ d(0)=d_0, \quad \textbf{(A.9)}$$

$$\dot{\mu}_F(t)=\left(\theta-\frac{\bar{r}+2\alpha d(t)}{1-\beta d(t)}\right)\mu_F(t)+\frac{\kappa_F(\bar{d}_F-d(t))}{1-\beta d(t)}, \quad \textbf{(A.10)}$$

$$\dot{\mu}_M(t)=\left(\theta-\frac{\bar{r}+2\alpha d(t)}{1-\beta d(t)}\right)\mu_M(t)+\frac{\kappa_M(\bar{d}_M-d(t))}{1-\beta d(t)}. \quad \textbf{(A.11)}$$

Notice that, since by assumption $d^*(.)$, $f^*(.)$, $m^*(.)$ converge and β is such that $1-\beta d^e>0$, $\mu_i^*(t),\ i=F,M$, converges too and

$$f^e=\bar{f}-\mu_F^e \text{ and } m^e=\bar{m}+\mu_M^e,\ \text{respectively.} \quad \textbf{(A.12)}$$

Consequently, with $\mu(t):=\mu_F(t)+\mu_M(t)$, $\lim_{t\to\infty}\mu^*(t)$ also exists. So, both $\frac{d\{d^*(t)\}}{dt}$ and $\frac{d\{\mu^*(t)\}}{dt}$ converge to zero. Using this in (A.9–A.11) it follows, by adding (A.10) and (A.11), that debt and $\mu(t)$ solve the differential equation:[13]

$$\dot{d}(t) = \frac{\bar{r}d(t) + \alpha d^2(t) + \bar{f} - \bar{m} - \mu(t)}{1 - \beta d(t)}, \; d(0) = d_0, \qquad \textbf{(A.13)}$$

$$\dot{\mu}(t) = \left(\theta - \frac{\bar{r} + 2\alpha d(t)}{1 - \kappa_F d(t)}\right)\mu(t) + \frac{\beta_F(\bar{d}_F - d(t)) + \kappa_M(\bar{d}_M - d(t))}{1 - \beta d(t)}. \qquad \textbf{(A.14)}$$

Notice that $\frac{\partial f_1}{\partial d} + \frac{\partial f_2}{\partial \mu} = \theta$. So, if $\theta \neq 0$, by Bendixson's theorem [see e.g., Engwerda (2005)(p.88)], this system of differential equations has no periodic solutions. Furthermore, the steady-state values are obtained as the solutions d^e of

$$[\theta(1 - \beta d) - (\bar{r} + 2\alpha d)](\bar{r}d + \alpha d^2 + \bar{f} - \bar{m}) + \kappa_F(\bar{d}_F - d) + \kappa_M(\bar{d}_M - d) = 0 \qquad \textbf{(A.15)}$$

with $\mu^e := \bar{r}d^e + \alpha d^{e^2} + \bar{f} - \bar{m}$. Some rewriting of (A.15) shows that d^e is the solution of the third-order polynomial equation (9). ■

COROLLARY 9. *From (A.9)–(A.11), it follows that the steady-state values are* $\mu_i^e = \frac{\kappa_i(d^e - \bar{d}_i)}{\theta(1-\beta d^e)-(\bar{r}+2\alpha d^e)}$, $i = F, M$. *So, by (A.12), the corresponding equilibrium actions converge to the steady-state values*

$$f^e := \bar{f} - \frac{\kappa_F(d^e - \bar{d}_F)}{\theta(1 - \beta d^e) - (\bar{r} + 2\alpha d^e)}, \text{ and } m^e := \bar{m} + \frac{\kappa_M(d^e - \bar{d}_M)}{\theta(1 - \beta d^e) - (\bar{r} + 2\alpha d^e)},$$

respectively.
If we assume additionally $\kappa_M = \kappa_F$ *and* $\bar{d}_F = \bar{d}_M$, *we get* $\mu_F^e = \mu_M^e$. *Furthermore, we have then that* $f^e + m^e = \bar{f} + \bar{m}$ *and, consequently,* $\frac{\partial f^e}{\partial \alpha} = -\frac{\partial m^e}{\partial \alpha}$; $\frac{\partial f^e}{\partial \beta} = -\frac{\partial m^e}{\partial \beta}$; *and* $\frac{\partial f^e}{\partial \theta} = -\frac{\partial m^e}{\partial \theta}$.

In case there is a steady state, d^e, let $\tilde{d}^e := 1 - \beta d^e$ and $f(d) := \bar{r}d + \alpha d^2 + \bar{f} - \bar{m}$. Then, some rewriting of (A.15) shows that d^e solves the polynomial equation

$$g(d) := \kappa_F(\bar{d}_F - d) + \kappa_M(\bar{d}_M - d) - f' f + \theta f \tilde{d}^e = 0. \qquad \textbf{(A.16)}$$

From (A.13), (A.14), it follows that the linearized system around this steady state is described by

$$\dot{y} = Ly := \frac{1}{\tilde{d}^{e^2}} \begin{bmatrix} f'\tilde{d}^e - \beta f & -1 \\ (-\kappa_F - \kappa_M - 2\alpha f_1)\tilde{d}^{e^2} - 2\beta f' f\tilde{d}^e + \beta^2 f^2 & \theta\tilde{d}^{e^2} - (f'\tilde{d}^e - \beta f) \end{bmatrix} y. \qquad \textbf{(A.17)}$$

Straightforward calculations show that the eigenvalues of matrix L are $\frac{1}{2}(\theta \pm \sqrt{\theta^2 - \frac{4}{\tilde{d}^{e^2}}\frac{\partial g}{\partial d}})$. As $g(d)$ is a third-order polynomial with a negative leading coefficient [see (9)], it follows that if there is a unique equilibrium, this will be a saddle-point. Similarly, it follows that in case there are three equilibria, the middle one will be unstable (ignoring the special case that one eigenvalue becomes zero) and the other two will be saddle-points again. Notice that the unstable equilibrium point will be a focus if $\theta^2 - 4\frac{\partial g}{\partial d} < 0$ and a node if $\theta^2 - 4\frac{\partial g}{\partial d} > 0$. In case there are two equilibria, it follows that $\frac{\partial g}{\partial d} = 0$ at the equilibrium that has multiplicity two. So, in that case matrix L has one positive eigenvalue and one zero eigenvalue. the

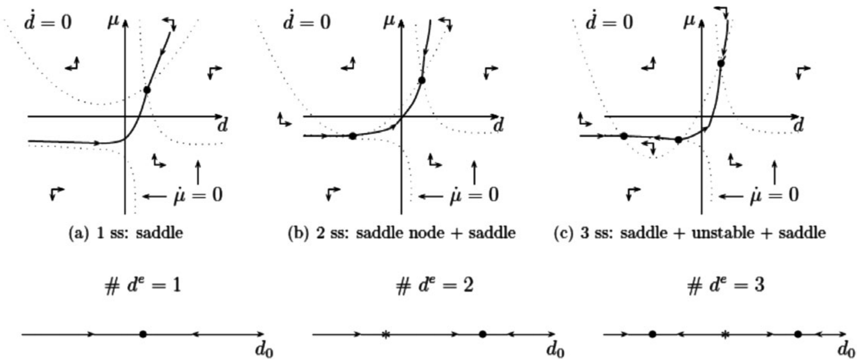

FIGURE A.1. Phase plane diagram of system (5, 6) if $\bar{f} \geq \bar{m}$ and $\theta > \bar{r}$ (upper part). Behavior of initial debt (lower part).

phase plane diagraM in thIS case displays one equilibrium that will be a saddle-node, the other equilibrium will be a saddle-point.

Figure A.1 shows this phase diagram in case $\bar{f} \geq \bar{m}$ and $\theta > \bar{r}$. In that case, the isocline where $\dot{d} = 0$ intersects the μ-axis at $\bar{f} - \bar{m} > 0$, whereas the horizontal asymptote of the isocline where $\dot{\mu} = 0$ is $\mu = -\frac{\kappa_F + \kappa_M}{\beta\theta + 2\alpha} < 0$ and its vertical asymptote $d = \frac{\theta - \bar{r}}{\beta\theta + 2\alpha} > 0$.

In Figure A.1, the dynamic behavior of the system (5, 6) is visualized, in the modal case that $\bar{f} \geq \bar{m}$ and $\theta > \bar{r}$. This phase plane diagram shows that there is always a unique equilibrium at which debt is positive. Moreover, we see that whenever initial debt is positive, debt will converge to this equilibrium.

APPENDIX B: THE SOLUTIONS TO THE THIRD ORDER POLYNOMIAL EQUATION (9)

Here, we recall [see, e.g., Irving (2004)] in some more detail the solutions d^e of the third-order polynomial equation (9)

$$g(d) := \gamma_3 d^3 + \gamma_2 d^2 + \gamma_1 d + \gamma_0 = 0, \quad \textbf{(B.1)}$$

where $\gamma_3 := -2\alpha^2 - \alpha\beta\theta$, $\gamma_2 := \alpha(\theta - 3\bar{r}) - \bar{r}\beta\theta$, $\gamma_1 := \bar{r}(\theta - \bar{r}) - \beta_F - \beta_M - 2\alpha(\bar{f} - \bar{m}) - (\bar{f} - \bar{m})\beta\theta$, and $\gamma_0 := \kappa_F \bar{d}_F + \kappa_M \bar{d}_M + (\bar{f} - \bar{m})(\theta - \bar{r})$.
Introducing the complex number $z := \frac{-1+i\sqrt{3}}{2}$ and $\bar{z}$ its conjugate, the three solutions of (B.1) are

$$d_1 := -\frac{1}{3\gamma_3}(\gamma_2 + C + \frac{\Delta_0}{C}),\ d_2 := -\frac{1}{3\gamma_3}(\gamma_2 + zC + \frac{\Delta_0}{zC}),\ \text{and } d_3$$
$$:= -\frac{1}{3\gamma_3}(\gamma_2 + \bar{z}C + \frac{\Delta_0}{\bar{z}C}).$$

Here, $C := \left(\frac{\Delta_1+\sqrt{\Delta_1^2-4\Delta_0^3}}{2}\right)^{1/3}$, with $\Delta_0 := \gamma_2^2 - 3\gamma_3\gamma_1$ and $\Delta_1 := 2\gamma_2^3 - 9\gamma_1\gamma_2\gamma_3 + 27\gamma_3^2\gamma_0$. It can be shown that $\Delta_1^2 - 4\Delta_0^3 = -27\gamma_3^2\Delta$, where Δ equals the so-called discriminant (18) whose sign determines the number of real solutions to (B.1),

$$\Delta := 18\gamma_3\gamma_2\gamma_1\gamma_0 - 4\gamma_2^3\gamma_0 + \gamma_2^2\gamma_1^2 - 4\gamma_3\gamma_1^3 - 27\gamma_3^2\gamma_0^2.$$

THEOREM 10. *Let Δ be as introduced above. Then,*
If $\Delta > 0$, then the equation (B.1) has three distinct real roots.
If $\Delta = 0$, then the equation (B.1) has a multiple root and all its roots are real.
If $\Delta < 0$, then the equation (B.1) has one real root and two nonreal complex conjugate roots.

APPENDIX C: THE SOLUTION FOR PLAYERS HAVING DIFFERENT TIME PREFERENCES

In case fiscal and monetary players have different time preferences, θ_F and θ_M, respectively, equations (A.4) and (A.5) have to be replaced by

$$-\dot{\lambda}_F = e^{-\theta_F t}\kappa_F(d - \bar{d}_F) + \frac{\lambda_F}{s}\left(\bar{r} + 2\alpha d + \beta\dot{d}\right), \tag{C.1}$$

$$-\dot{\lambda}_M = e^{-\theta_M t}\kappa_M(d - \bar{d}_M) + \frac{\lambda_M}{s}\left(\bar{r} + 2\alpha d + \beta\dot{d}\right), \tag{C.2}$$

where $s(t) := 1 - \beta d(t)$, $f^*(t) = \bar{f} - e^{\theta_F t}\frac{\lambda_F^*(t)}{s(t)}$, and $m^*(t) = \bar{m} + e^{\theta_M t}\frac{\lambda_M^*(t)}{s(t)}$. Introducing $\mu_i(t) := e^{\theta_i t}\lambda_i(t),\ i = F, M$, above equations can be rewritten as

$$\dot{d} = \frac{1}{s}(\bar{r}d + \alpha d^2 + \bar{f} - \bar{m} - \frac{\mu_F + \mu_M}{s}), \tag{C.3}$$

$$\dot{\mu}_F = \theta_F\mu_F - \kappa_F(d - \bar{d}_F) - \frac{\mu_F}{s}\left(\bar{r} + 2\alpha d + \beta\dot{d}\right), \tag{C.4}$$

$$\dot{\mu}_M = \theta_M\mu_M - \kappa_M(d - \bar{d}_M) - \frac{\mu_M}{s}\left(\bar{r} + 2\alpha d + \beta\dot{d}\right), \tag{C.5}$$

from which we obtain the equilibrium conditions:

$$0 = \bar{r}d + \alpha d^2 + \bar{f} - \bar{m} - \frac{\mu_F + \mu_M}{s}, \tag{C.6}$$

$$0 = \theta_F\mu_F - \kappa_F(d - \bar{d}_F) - \frac{\mu_F}{s}\left(\bar{r} + 2\alpha d\right), \tag{C.7}$$

$$0 = \theta_M\mu_M - \kappa_M(d - \bar{d}_M) - \frac{\mu_M}{s}\left(\bar{r} + 2\alpha d\right). \tag{C.8}$$

Assuming some regularity conditions are met, substitution of μ_F from (C.7) and μ_M from (C.8) into (C.6) shows that equilibrium debt, d^e, satisfies:

$$0 = \bar{r}d + \alpha d^2 + \bar{f} - \bar{m} - \frac{\kappa_F(d - \bar{d}_F)}{\theta_F s - (\bar{r} + 2\alpha d)} - \frac{\kappa_M(d - \bar{d}_M)}{\theta_M s - (\bar{r} + 2\alpha d)},$$

from which we conclude that d^e solves the fourth-order polynomial equation:

$$g(d) := \tilde{\gamma}_4 d^4 + \tilde{\gamma}_3 d^3 + \tilde{\gamma}_2 d^2 + \tilde{\gamma}_1 d + \tilde{\gamma}_0 = 0, \tag{C.9}$$

where, with $\alpha_i := 2\alpha + \beta\theta_i$ and $\theta_i^r := \theta_i - \bar{r},\ i = F, M$,
$\tilde{\gamma}_4 := \alpha\alpha_F\alpha_M$; $\tilde{\gamma}_3 := \bar{r}\alpha_F\alpha_M - \alpha(\theta_M^r\alpha_F + \theta_F^r\alpha_M)$;
$\tilde{\gamma}_2 := \alpha\theta_F^r\theta_M^r + \alpha_F\alpha_M(\bar{f} - \bar{m}) - \bar{r}\theta_F^r\alpha_M - \bar{r}\theta_M^r\alpha_F + \kappa_F\alpha_M + \kappa_M\alpha_F$;
$\tilde{\gamma}_1 := \bar{r}\theta_F^r\theta_M^r - (\theta_F^r\alpha_M + \theta_M^r\alpha_F)(\bar{f} - \bar{m}) - \kappa_F\theta_M^r - \kappa_M\theta_F^r - \kappa_F\bar{d}_F\alpha_M - \kappa_M\bar{d}_M\alpha_F$;
$\tilde{\gamma}_0 := \theta_F^r\theta_M^r(\bar{f} - \bar{m}) + \kappa_F\theta_M^r\bar{d}_F + \kappa_M\theta_F^r\bar{d}_M$.
In case players cooperate and we assume that, e.g., $\theta_F > \theta_M$, and $\dot{m}(t) \to 0$ if $t \to \infty$ we obtain next result.

THEOREM 11. *Consider the cooperative game with $\theta_F > \theta_M$. If $(f^*(.), m^*(.)) \in \mathcal{U}$ is a set of Pareto efficient strategies for (1–5) then steady-state values of the debt and policy variables are as follows:*

$$d^e = \bar{d}_M,\ m^e = \bar{m},\ \text{and}\ f^e = \bar{m} - (\bar{r} + \alpha\bar{d}_M)\bar{d}_M.$$

Proof. Consider the Hamiltonian:

$$H := \frac{1}{2}\omega e^{-\theta_F t}\{(f - \bar{f})^2 + \kappa_F(d - \bar{d}_F)^2\} + \frac{1}{2}(1 - \omega)e^{-\theta_M t}\{(m - \bar{m})^2 + \kappa_M(d - \bar{d}_M)^2\}$$
$$+ \frac{\lambda}{1 - \beta d}\{(\bar{r} + \alpha d)d + f - m\}.$$

Then, according to Pontryagin's maximum principle, $(d^*, f^*(.), m^*(.))$ satisfy the necessary conditions:

$$\dot{d} = \frac{1}{1 - \beta d}\{(\bar{r} + \alpha d)d + f - m\}. \tag{C.10}$$

$$-\dot{\lambda} = \omega e^{-\theta_F t}\kappa_F(d - \bar{d}_F) + (1 - \omega)e^{-\theta_M t}\kappa_M(d - \bar{d}_M)$$
$$+ \frac{\beta\lambda}{(1 - \beta d)^2}\{(\bar{r} + \alpha d)d + f - m\} + \frac{\lambda}{1 - \beta d}\{\bar{r} + 2\alpha d\}. \tag{C.11}$$

$$f = \bar{f} - \frac{1}{\omega}\frac{\lambda e^{\theta_F t}}{1 - \beta d}. \tag{C.12}$$

$$m = \bar{m} + \frac{1}{1 - \omega}\frac{\lambda e^{\theta_M t}}{1 - \beta d}. \tag{C.13}$$

Since, by assumption, $\lim_{t\to\infty} f(t) = f^e$ exists, it follows from (C.12) that $\lim_{t\to\infty} \frac{\lambda e^{\theta_F t}}{1-\beta d}$ exists. From (C.13), we next conclude, as $\theta_F > \theta_M$, that $\lim_{t\to\infty} m(t) - \bar{m} = 0$. That is, $\lim_{t\to\infty} m(t) = \bar{m}$.

Multiplication of (C.11) by $e^{\theta_M t}$ and taking limits on both sides of the equation shows next that

$$\lim_{t\to\infty} -\dot{\lambda}(t)e^{\theta_M t} = (1-\omega)\kappa_M(d^e - \bar{d}_M).$$

By differentiation of (C.13), we have

$$\frac{d\{(1-\omega)(m(t)-\bar{m})\}}{dt} = \frac{\dot{\lambda}e^{\theta_M t} + \theta_M \lambda e^{\theta_M t}}{1-\beta d} + \frac{\beta \dot{d}}{1-\beta d}\frac{\lambda e^{\theta_M t}}{1-\beta d}.$$

As $\lim_{t\to\infty} \dot{m}(t) = 0$, it follows by taking limits on both side of above equation that $0 = (1-\omega)\kappa_M(d^e - \bar{d}_M)$. That is, $d^e = \bar{d}_M$. From which we have from (C.10), by taking limits again, that $f^e = \bar{m} - (\bar{r} + \alpha\bar{d}_M)\bar{d}_M$. ■