Self-Consistent and Manipulative Behavior in Social Choice as a Repeated Nash Game on a Graph

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Overview

Introduction

- A social choice procedure as a repeated Nash game between the social agents
- Stability analysis of the derived strategies
- Topology design, which aims to the stabilization of these strategies

Motivation and novelties

- Develop a model for social choice procedures, which captures the conformity and manipulative behaviors
- A similar model has been proposed by R.Etesami et al.('17).
- We have introduced dynamically changing internal beliefs.
- Develop a graph topology design methodology in order to influence the effects of manipulative behaviors.
- Applicability of the design methodology to many problems in other areas of interest.

Model formulation

- Each agent has an internal belief or opinion ($\theta_i(k)$) and an expressed opinion or action ($u_i(k)$) at each time step k
- The opinion shaping criteria model conformity, parameterised by c_i.
- The action shaping criteria model manipulative behaviors, parameterised by g_i.

Notation

- Let A be the adjacency matrix of the communication graph.
- d_i is the degree of node i and $D = diag\{d_i\}$ is the degree matrix.
- $C = diag\{c_i\}$ is a diagonal matrix containing the parameters c_i.
- $G = diag\{g_i\}$ is also a diagonal matrix containing the parameters g_i.
- N_i denotes the neighborhood of agent i, ie $N_i = \{j : (i, j) \in E\}$, where E is the set of edges.
- Let $\mathbf{1}$ stands for a vector with all its coordinates equal to 1 and $\mathbf{I}$ for the identity matrix, of proper dimension

Opinion Dynamics

At each step of the procedure, the opinion of each agent is formed so as to minimize the following criterion:

$$Jo_i(k+1) = c_i(\theta_i(k+1) - \theta_i(k))^2 + d_i(\theta_i(k+1) - \frac{\sum_{j \in N_i} \theta_j(k)}{d_i})^2 \quad (1)$$

Thus, these dynamics arise:

$$\theta_i(k+1) = \frac{c_i}{d_i + c_i}\theta_i(k) + \frac{1}{d_i + c_i}\sum_{j \in N_i} \theta_j(k) \quad (2)$$

Equivalently, with matrix notation:

$$\theta(k+1) = (D + C)^{-1}(A + C)\theta(k) \quad (3)$$

which are known to converge to a limit vector θ^c under two non restrictive assumptions, as shown for example by M.DeGroot ('74) or by A.Olshevsky and J.Tsitsiklis ('09).

Action Dynamics

At each step of the procedure each agent knows only the previous actions of her neighbors, so her estimation of the social outcome is:

$$\tilde{u}_i(k) = \frac{\sum_{j \in N_i} u_j(k-1) + u_i(k)}{d_i + 1} \tag{4}$$

Thus, her action shaping criterion has the following form:

$$Ja_i(k) = (u_i(k) - \phi(\theta_i(k)))^2 + g_i(\tilde{u}_i(k) - \phi(\theta_i(k)))^2 \tag{5}$$

and the action dynamics which derive form its minimization are:

$$u_i(k+1) = \left(1 + \frac{d_i g_i}{g_i + (d_i+1)^2}\right)\phi(\theta_i(k+1)) - \frac{g_i}{g_i + (d_i+1)^2} \sum_{j \in N_i} u_j(k) \tag{6}$$

or with matrix notation:

$$u(k+1) = G_\theta \Phi(\theta(k+1)) - G_u A u(k) \tag{7}$$

Coupled dynamics

In the last equation (7):

$$G_\theta = diag\{1 + \frac{d_i g_i}{g_i + (d_i + 1)^2}\} \tag{8}$$

$$G_u = diag\{\frac{g_i}{g_i + (d_i + 1)^2}\} \tag{9}$$

Let $z(k) = [\theta_1(k)...\theta_N(k), u_1(k)...u_N(k)]^T$
and the resulting augmented system describing its dynamics:

$$z(k+1) = \begin{bmatrix} (D+C)^{-1}(A+C) & 0 \\ G_\theta \Phi \circ (D+C)^{-1}(A+C) & -G_u A \end{bmatrix} z(k) \tag{10}$$

Decoupling lemma

Lemma

If the matrix $A_u = G_u A$ *is asymptotically stable, ie* $|\lambda_i(A_u)| < 1, \forall i$ *and the function* Φ *is continuous in* $\mathcal{R}^n$ *and locally Lipschitz in a neighborhood of* θ^c *with a Lipschitz constant* L_Φ*, then the coupled dynamics* (10) *will be stable.*

Sufficient Condition

Since the matrix $(D+\mathcal{I})^{-1}A$ is a substochastic matrix and thus a stable one, a simple but restrictive sufficient condition for the stability of the whole system is the spectral radius $\rho(G_u(D+\mathcal{I})) = max\{\|\lambda_i(G_u(D+\mathcal{I}))\|, i=1...N\}$ to be less than one as well or equivalently

$$\frac{(d_i+1)g_i}{g_i+(d_i+1)^2} < 1 \Rightarrow g_i < d_i+2, \forall i \tag{11}$$

Thus, a useful remark for the topology design initialisation can be stated here. If we consider a graph topology with:

$$d_i > g_i - 2, \forall i \tag{12}$$

then the dynamics are stable on this topology.

Simulation parameters

- We consider a group of 20 social agents, which communicate with each other over several different graph structures.
- Their manipulation parameters g_i and their conformity parameters c_i remain the same in all simulations.
- Their initial opinions are randomly chosen from the $[0, 10]$ interval.
- Their initial actions $u_i(0) = \phi(\theta_i(0))$, where $\phi(\theta) = 10tanh(\frac{\theta}{10})$.

A random graph

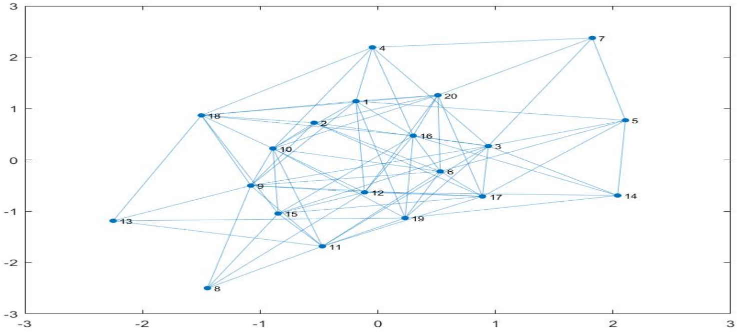

Figure: A random graph, as introduced by P.Erdős and A.Rényi (1959), with edge probability $p = 0.4$.

1

[1] This random graph has $|E| = 114$ edges.

Stable example

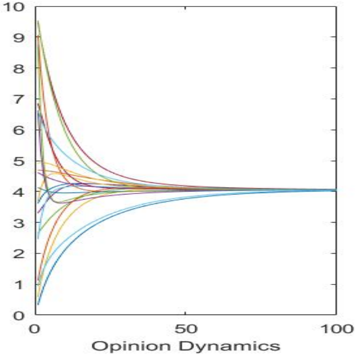

Figure: Opinion and action dynamics on a random graph with edge probability $p = 0.4$.

Unstable example

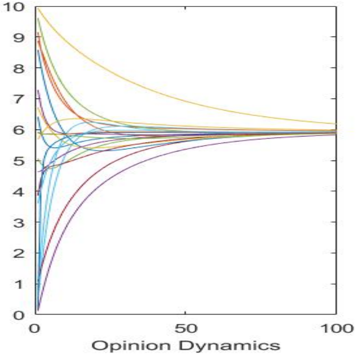

Figure: Opinion and unstable action dynamics on a random graph with edge probability $p = 0.3$.

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[2] The random graph considered in this simulation has $|E| = 52$ edges.

Stabilization ideas

Considering the problem of choosing a proper graph structure, which would result in stable dynamics, and be as close as possible (here with respect to the edge number $|E|$) to the aforementioned unstable one, we make several experiments beginning from an L^*-lattice which satisfies our sufficient condition ($L^* > g_max - 2$), $L^* = 14$ in this example, and relaxing it as shown in the following table:

Graph structure	$\lambda_{max}(A_u)$	$\lvert E\rvert$
L^*-lattice	0.4042	140
8-lattice	0.7758	80
6-lattice	1.0114	60
Small-world	0.9491	60

Table: Stability of several graph structures

The 8-lattice

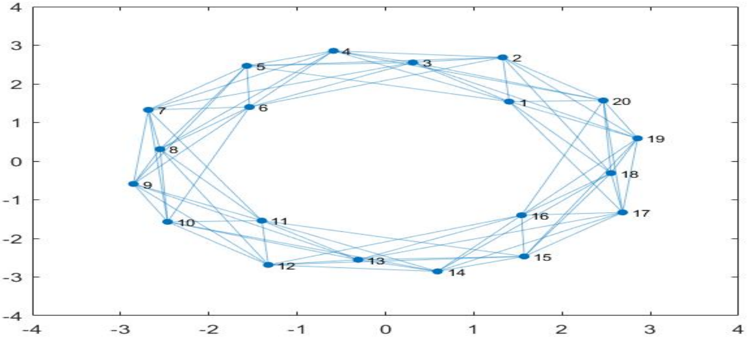

Figure: A lattice graph of node degree 8 .

The small-world graph

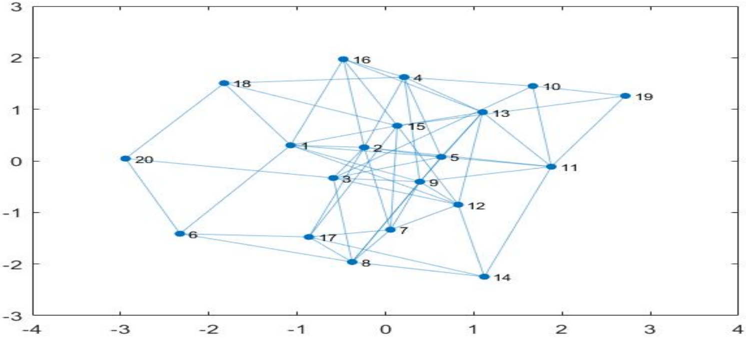

Figure: A small-world graph, as introduced by J. Watts and S.Strogatz (1998), derived from a 6-lattice with rewiring probability $p_r = 0.5$.

Stable dynamics on a small-world graph

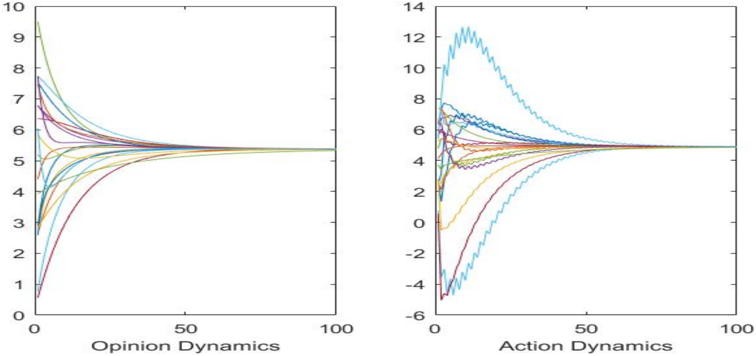

Figure: Opinion and action dynamics on a small world graph resulting form a 6-lattice.

Topology design problem statement

- Let A_0 be the adjacency matrix of the initial topology.
- Let $\{P^k, k = 1 ... \frac{n(n-1)}{2}\}$ be a basis for the symmetric matrices of size n, with zero diagonal elements. Each P^k stands for the representation of the k-th edge at the adjacency matrix of the graph, for a proper enumeration of all the possible edges.
- Let the vector x_0 be the coordinates of A_0 with respect to this basis.
- Let also $x \in \{0,1\}^{\frac{n(n-1)}{2}}$ be the vector of changes. If x(k)=1 a change occurs at position k, else no change occurs at this position.
- Let finally the sign function $\mathcal{S}_{x_0}(k) = 1$ if $x_0(k) = 0$ and $\mathcal{S}_{x_0}(k) = -1$ if $x_0(k) = 1$. This function indicates if the possible change corresponds to the addition of a new edge or to the removal of an existing one.

So the adjacency matrix of the designed topology has the following form:

$$A(x) = A_0 + \sum_{k=1}^{n(n-1)/2} x(k) P^k \mathcal{S}_{x_0}(k) \tag{13}$$

Topology design problem statement (cont'd)

The degree matrix changes accordingly:

$$D(x) = \sum_{i=1}^{n} e_i (A(x)\mathbf{1})^T P_d^i \tag{14}$$

Subsequently, we define the matrix function, in accordance with the equation (9):

$$G_u(x) = G(G + (D(x) + I)^2)^{-1} \tag{15}$$

and consequently

$$A_u(x) = G_u(x)A(x) \tag{16}$$

which are nonlinear with respect to the decision variables x.

The criterion for choosing a design is the minimum change form the initial graph structure. Thus, we have to minimize $\|x\|_1$, which is equivalent to the minimization of the linear objective $\mathbf{1}^T x$.

Topology design problem statement (cont'd)

The resulting problem is:

$$\underset{x,P}{\text{minimize}}\{\mathbf{1}^T x\} \tag{17}$$

$$x \in \{0,1\}^{n(n-1)/2} \tag{18}$$

$$P > 0 \tag{19}$$

$$A(x)G_u(x)PG_u(x)A(x) - P < 0 \tag{20}$$

The last two constraints can be relaxed as follows in order for the feasible region to become closed:

$$P \geq \epsilon \mathbf{I} \tag{21}$$

$$A(x)G_u(x)PG_u(x)A(x) - P \leq -\delta \mathbf{I} \tag{22}$$

The parameters ϵ and δ must be carefully chosen to be very small so as to not reject many feasible solutions.

In this problem formulation several linear constraints may be added without changing its difficulty.

Characteristics of the design problem

The last constraint $A(x)G_u(x)PG_u(x)A(x) - P \leq -\delta\mathbf{I}$ is nonlinear in the decision variables x. So, we consider the change of variables $Z = G_u(x)PG_u(x)$ (it holds that if $P > 0$ then $Z > 0$ and vice versa):

$$A(x)ZA(x) - G_u^{-1}(x)ZG_u^{-1}(x) \leq -\delta\mathbf{I} \tag{23}$$

This last constraint (23) is polynomial in the decision variables x and with a proper change of variables it can be converted to a Bilinear Matrix Inequality (BMI).

The feasibility of a BMI is known to be a nonconvex problem in the general case [M.Mesbahi, M.G.Safonov,G.P.Papavassilopoulos(2000)] So the same holds for our initial problem (17)-(20).

Research Directions

Our research direction now is to develop a proper algorithm to deal with the BMI constrained integer programming problem for the network topology design. Specifically, some kind of organised random search algorithm will be considered, such as:

- Genetic algorithms
- Particle swarm optimization
- Simulated annealing

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The End