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On the Probability of Existence of Pure Equilibria in Matrix Games[1]

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Abstract. We examine the probability that a randomly chosen matrix game admits pure equilibria and its behavior as the number of actions of the players or the number of players increases. We show that, for zero-sum games, the probability of having pure equilibria goes to zero as the number of actions goes to infinity, but it goes to a nonzero constant for a two-player game. For many-player games, if the number of players goes to infinity, the probability of existence of pure equilibria goes to zero even if the number of actions does not go to infinity.

Key Words. Matrix games, pure equilibria, mixed equilibria.

1. Introduction

The possible lack of existence of pure equilibria in game problems is a long known fact which led to the admittance and examination of mixed equilibria. The existence of mixed equilibria for any matrix zero-sum game was first proven by Von Neuman in 1936 (Ref. 1) and later on was extended to the nonzero-sum game by Nash in 1950 (Ref. 2). Using infinite-dimensional versions of the Brower fixed-point theorem enabled the extension of this basic existence result to other classes of games with infinite action spaces (Ref. 3). The fact that pure equilibria may fail to exist for games with a deterministic description in contrast to what holds for classical optimization problems (single player games) under similar convexity or compactness assumptions is a discomforting result, albeit a deep and conceptually fascinating one. The objective of the present paper is to examine the probability that, for a randomly chosen game, a pure equilibrium exists.

We consider the following questions: (i) If the elements of an $m \times n$ matrix are chosen independently and randomly according to a uniform

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distribution over an interval, what is the probability that the resulting zero-sum game admits a pure solution. (ii) How does this probability change as m, n tend to infinity. (iii) Similarly, if two $m \times n$ matrices A, B are randomly chosen, what is the probability of existence of a pure equilibrium and how does it change as m, n tend to infinity. The extension of these questions to the case of an arbitrary number of players and the study of the behavior of the probability of existence of pure equilibria as the number of players or the number of available actions to the players tends to infinity are also considered.

These questions are answered in the rest of this paper. We provide explicit formulae for the probability of existence of pure equilibria for each case and study their limiting behavior. The following interesting facts are shown. For randomly chosen zero-sum games, the probability of existence of pure equilibria tends to zero as the number of actions of the players goes to infinity. In contrast, for nonzero-sum two player games, the limit of the probability of existence of pure equilibria as the number of actions of the two players goes to infinity, tends to the number $1 - e^{-1} \cong 0.63$. For many-player games, if the number of players is fixed and the number of their actions goes to infinity, the probability of existence of pure equilibria tends to a fixed nonzero number. Finally, if the number of players goes to infinity, then the probability of existence of a pure equilibrium tends to zero.

The structure of this paper is as follows. In Section 2, we examine the zero-sum case; in Section 3, the two-player case; and in Section 4, the many-player cases. In Section 5, we present some interesting directions for extending the results.

Note to the Reader. Although the issues taken up in this paper could have been considered many years ago, our own search in the game literature, far beyond the references cited at the end, did not reveal any related paper addressing these issues. If the reader is aware of some related work, we would be grateful if he/she could communicate it to us.

2. Zero-Sum Case

Consider the zero-sum game

$$\min_y \max_x (x'Ay), \tag{1}$$

where $A = (a_{ij})$ is an $m \times n$ real matrix and

$$x' = (x_1, \dots, x_m), \qquad x_i \geq 0, \qquad \sum_{i=1}^m x_i = 1,$$

$$y' = (y_1, \dots, y_n), \qquad y_i \geq 0, \qquad \sum_{i=1}^n y_i = 1,$$

are the mixed equilibria of the two players. Consider that the a_{ij} are chosen independently and randomly, according to a uniform distribution from an interval $[\delta_1, \delta_2]$. If $\bar{x}=(1, 0, \ldots, 0)$, $\bar{y}=(1, 0, \ldots, 0)$ constitute a pure equilibrium, it will hold that

$$a_{m1}, a_{m-1,1}, \ldots, a_{21} \leq a_{11} \leq a_{12}, a_{13}, \ldots, a_{1n}. \tag{2}$$

Let us now calculate the probability that, for the $n+m-1$ numbers present in (2), chosen independently and randomly, (2) holds. The possible orderings of $n+m-1$ numbers are $(n+m-1)!$ in multitude. The orderings for which (2) holds equals the product of the multitude of possible orderings of $a_{12}, a_{13}, \ldots, a_{1n}$ [which is $(n-1)!$] by the product of the multitude of possible orderings of $a_{m1}, a_{m-1,1}, \ldots, a_{21}$ [which is $(m-1)!$]. Thus, the multitude of possible orderings of the $n+m-1$ numbers appearing in (2) for which (2) holds is $(n-1)!(m-1)!$, consequently, the probability that (2) holds if $a_{11}, \ldots, a_{1n}, a_{21}, \ldots, a_{m1}$ are chosen independently and randomly is

$$(m-1)!(n-1)!/(m+n-1)!. \tag{3}$$

Actually, the quantity in (3) is the ratio of the volume of the hypercube in R^{n+m-1} specified by (2) over the whole volume of the hypercube. Clearly, it is independent of the length of the side of the hypercube and thus holds for $\delta_1 \to -\infty$, $\delta_2 \to +\infty$. Also notice that this ratio of volumes does not change if strict inequalities are considered in (2), since an equality such as $a_{m1}=a_{11}$ corresponds to an $n+m-2$ dimensional manifold which has zero volume in R^{n+m-1}.

Considering now the other cases where a_{ij} $(i,j)\neq(1, 1)$, is a pure equilibrium solution and that there is a total nm of such cases, we conclude that the probability that the game (1) has a pure equilibrium solution is

$$[(m-1)!(n-1)!/(m+n-1)!]mn = m!n!/(m+n-1)! = (m+n)\Big/\binom{m+n}{n}. \tag{4}$$

Finally, let us point out that, if the game has two pure equilibria (for example a_{11}, a_{23}), it will hold that $a_{11}=a_{23}=a_{13}=a_{21}$; then, the corresponding subset of the hypercube in R^{n+m-1} is of dimension less than $n+m-1$, and consequently has zero volume. In conclusion, we have proven the following theorem.

Theorem 2.1. If the elements of an $m \times n$ real matrix A are chosen independently and randomly with a uniform probability distribution over any interval, the probability of having a pure equilibrium is

$$P_{nm}(S) = P_{mn}(S) = (m+n) \Big/ \binom{m+n}{n} = m!n!/(m+n-1)!. \tag{5}$$

Remark 2.1. Notice that the interval $[\delta_1, \delta_2]$, $\delta_i < \delta_2$, does not influence the result.

Remark 2.2. If $m=1$, then

$$P_{1n}(s) = 1, \qquad n = 1, 2, 3, \ldots. \tag{6}$$

Remark 2.3. Let m be fixed, and let $n \geq m > 1$. It holds that

$$\begin{aligned} P_{mn}(S) &= m!n!/(m+n-1)! \\ &= m![1 \cdot 2 \cdot 3 \ldots n]/[1 \cdot 2 \ldots n \cdot (n+1) \ldots (n+m-1)] \\ &= m!/[(n+1)(n+2) \ldots (n+m-1)] \downarrow 0, \qquad \text{as } n \to +\infty. \end{aligned} \tag{7}$$

Also,

$$P_{mn}(S) = m!/[(n+1) \ldots (n+m-1)] \cong m!/n^{m-1}, \qquad \text{as } n \to \infty. \tag{8}$$

Remark 2.4. Let $m = n$. Then,

$$P_{nn}(S) = n!n!/(2n-1)! = n!n!/(2n)!^{2n}.$$

Using the Stirling formula for $n \to \infty$,

$$n! \cong n^n e^{-n} \sqrt{2\pi n},$$

we have, for $n \to +\infty$,

$$P_{nn}(S) \cong [(n^n e^{-n} \sqrt{2\pi n})^2 (2n)/[(2n)^{2n} e^{-2n} \sqrt{2\pi(2n)}] = 2\sqrt{\pi}(n\sqrt{n}/4^n) \to 0. \tag{9}$$

The above remarks yield the following theorem.

Theorem 2.2. If m and $n \to \infty$, then

$$P_{mn}(S) \to 0. \tag{10}$$

Remark 2.5. Let us consider that we choose randomly a zero-sum game in the following manner: we first choose an integer n between 1 and

N with a uniform probability distribution and an integer m between 1 and M with a uniform probability distribution; N and M are fixed positive integers; n and m are chosen independently. Next, we choose a_{ij}, $i=1, \ldots,$ n and $j=1, \ldots, m$, independently with a uniform probability distribution in $[\delta_1, \delta_2]$, where $\delta_1 < \delta_2$ are arbitrarily fixed numbers. The resulting zero-sum game admits a pure equilibrium with probability

$$(1/(MN)] \sum_{n=1}^{N} \sum_{m=1}^{M} P_{mn}(S) = \bar{P}_{MN}(s).$$

Since $P_{mn}(s) \to 0$ as m and $n \to +\infty$, it holds that

$$\bar{P}_{MN}(s) \to 0.$$

This can be interpreted as follows. Consider all the zero-sum matrix games, i.e., any n, m, a'_{ij}. Then, the probability that a randomly chosen game admits a pure equilibrium is zero.

3. Nash Game with Two Players

Let us consider a two-player, nonzero-sum game where

$$J_1(x, y) = x'Ay, \qquad J_2(x, y) = x'By$$

are the costs of the two players, A and B are $n \times m$ real matrices, and

$$x' = (x_1, \ldots, x_n), \qquad x_i \geq 0, \qquad \sum_{i=1}^{n} x_i = 1,$$

$$y' = (y_1, \ldots, y_m), \qquad y_i \geq 0, \qquad \sum_{i=1}^{m} y_i = 1,$$

are the mixed equilibria. In order to calculate the probability that the game admits a pure equilibrium solution when the elements of the matrices A and B are chosen randomly and independently with uniform distributions over the same interval, we work as follows. Consider that Player 2 (namely, y) chooses for each row of A the position of the minimal element of this row. Similarly, Player 1 (namely, x) chooses for each column of B the position of the maximal element of the column. If two choices of the two players occupy the same (ij) position in A and B, respectively, then we have a pure equilibrium solution. Let us consider first the choices of Player 2. We set one at the position of the row where the maximal element of the row is located and set zero at the other positions. Considering that choosing two or more elements of the row to have the same value has probability zero, we assume that each row will have only one 1. For example, the following pattern may appear:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \tag{11a}$$

with

$$m=4, \qquad n=7, \tag{11b}$$

$$n_1=2, \qquad n_2=1, \qquad n_3=3, \qquad n_4=1. \tag{11c}$$

The multitude of ones is n_1 in the first column, n_2, in the second, and so on. It holds that

$$n_1 \geq 0, \ldots, n_m \geq 0, \tag{12a}$$

$$n_1+n_2+\cdots+n_m=n. \tag{12b}$$

The multitude of possible first columns which have n_1 ones is $\binom{n}{n_1}$. Given the position of the ones in the first column, the n_2 ones in the second column will have to be in n_2 of the remaining $n-n_1$ positions. Having set the ones of the first, second, and so on up to the kth column, we can set the ones of the $(k+1)$th column in n_{k+1} of the remaining $n-n_1-n_2 \ldots -n_k$ positions, and thus we have

$$\binom{n-n_1-n_2 \ldots -n_k}{n_{k+1}}$$

possible choices for the $(k+1)$th column. Thus, the total number of possible choices of Player 2 with n_k ones in the kth column for $k=1, \ldots, m$ is

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1 \ldots -n_{m-1}}{n_m}$$
$$=n!/(n_1!n_2! \ldots n_m!). \tag{13}$$

Using the formula

$$\sum_{n_1+\cdots+n_m=n} [n!/(n_1!\ldots n_m!)]p_1^{n_1}\ldots p_m^{n_m}=(p_1+\cdots+p_m)^n, \tag{14}$$

which holds for any numbers $p_1, \ldots, p_m$, yields that the multitude of matrices of dimension $n\times m$ which have at most a one in each row and zeros elsewhere is

$$\sum_{n_1+\cdots+n_m=n} n!/(n_1!\ldots n_m!)=m^n. \tag{15}$$

For a matrix chosen by Player 2 with n_k ones in the kth column, the probability that Player 1 places a one in each column, and his one does not coincide in position with the ones put by Player 2, is

$$(1-n_1/n)(1-n_2/n)\ldots(1-n_m/n). \tag{16}$$

Thus, the probability that a pure equilibrium does not arise is

$$P_{mn}(F)=(1/m^n)\sum_{\substack{n_1+\cdots+n_m=n\\ n_1\geq 0,\ldots,n_m\geq 0}} [n!(1-n_1/n)\ldots(1-n_m/n)]/(n_1!\ldots n_m!). \tag{17}$$

The following lemma will be used in this section and later on in Section 4.

Lemma 3.1. Consider the sum

$$S_\alpha=(1/m^n)\sum_{\substack{n_1+\cdots+n_m=n\\ n1\geq 0,\ldots,n_m\geq 0}} [n!(1-\alpha n_1/n)\ldots(1-\alpha n_m/n)]/(n_1!\ldots n_m!), \tag{18}$$

where α is a fixed real number. The following relations hold:

(i) $S_\alpha=\sum_{k=0}^{\min(m,n)}[-\alpha/(mn)]^k\binom{m}{k}\binom{n}{k}k!$;

(ii) if $0\leq\alpha\leq 1$ and $m\leq n$, then $S_\alpha\leq(1-\alpha/m)^m$;

(iii) if $\alpha\geq 0$, then $\lim_{m,n\to\infty} S_\alpha=e^{-\alpha}$.

Proof.

(i) Carrying out the multiplication in (18), we have

$$S_\alpha=(1/m^n)\Big\{\sum (n!/(n_1!\dots n_m!)]+\cdots+(-\alpha)^k$$

$$\times\sum [n!/(n_1!\dots n_m!)](n_1/n)\dots(n_k/n)\binom{m}{k}+\cdots\Big\}$$

$$=(1/m^n)\Big\{\cdots+(-\alpha)^k\binom{m}{k}(n!/n^k)$$

$$\times\sum 1/[(n_1-1)!\dots(n_k-1)!n_{k+1}!\dots n_m!]+\cdots\Big\}$$

$$=(1/m^n)\Big\{\cdots+(-\alpha)^k\binom{m}{k}[n!/(n-k)!](1/n^k)$$

$$\times\sum (n-k)!/[(n_1-1)!\dots(n_k-1)!n_{k+1}!\dots n_m!]+\cdots\Big\}$$

$$=(1/m^n)\Big\{\cdots+(-\alpha)^k\binom{m}{k}\binom{n}{k}k!(1/n^k)m^{n-k}+\cdots\Big\}, \tag{19}$$

and thus,

$$S_\alpha=\sum_{k=0}^{m}(-\alpha/mn)^k\binom{m}{k}\binom{n}{k}k!. \tag{20}$$

In the last step of the derivation in (19), we made use of the formula (14). Using the formula

$$\binom{n}{m}=0, \qquad \text{if } m>n, \tag{21}$$

we can rewrite (20) in the equivalent form

$$S_\alpha=\sum_{k=0}^{\min(n,m)}(-\alpha/mn)^k\binom{m}{k}\binom{n}{k}k!, \tag{22}$$

which is symmetric in m and n, as it should be expected.

(ii) It is easy to verify that the maximum of the product $(1-\alpha z_1)\dots(1-\alpha z_m)$, subject to $z_1+\cdots+z_m=1$, $z_1\geq 0,\dots,z_m\geq 0$, with

$0 \leq \alpha \leq 1$, is achieved at

$$z_1^* = \cdots = z_m^* = 1/m.$$

Thus, since

$$n_1 + \cdots + n_m = n \quad \text{and} \quad n_i \geq 0,$$

if holds that

$$(1 - \alpha n_1/n)(1 - \alpha n_2/n) \ldots (1 - \alpha n_m/n) \leq [1 - (n/m)(\alpha/n)]^m = (1 - \alpha/m)^m. \tag{23}$$

Using Inequality (23) in (18), we obtain

$$S_\alpha \leq (1/m^n)(1 - \alpha/m)^m \sum n!/(n_1! \ldots n_m!)$$
$$= (1/m^n)(1 - \alpha/m)^m m^n = (1 - \alpha/m)^m.$$

(iii) Let $m \leq n$ and α be a fixed nonnegative real number. Consider the sum

$$S_\alpha = \sum_{k=0}^{m} (-\alpha/mn)^k \binom{m}{k} \binom{n}{k} k!$$
$$= \sum_{k=0}^{m} (-\alpha/m)^k \binom{m}{k} [n(n-1) \ldots (n-k+1)]/n^k$$
$$= \sum_{k=\text{even}} (\alpha/m)^k \binom{m}{k} [n(n-1) \ldots (n-k+1)]/n^k$$
$$- \sum_{k=\text{odd}} (\alpha/m)^k \binom{m}{k} [n(n-1) \ldots (n-k+1)]/n^k. \tag{24}$$

It holds that

$$(1/n^k) \binom{n}{k} k! = [n(n-1) \ldots (n-k+1)]/n^k$$
$$\geq (n-k+1)^k/n^k = [1 - (k-1)/n]^k;$$

and since for any $b > 0$ it holds that

$$(1-b)^k = 1 - \binom{k}{1} b + \binom{k}{2} b^2 \cdots \geq 1 - \binom{k}{1} b, \tag{25}$$

we have

$$(1/n^k)\binom{n}{k}k! \geq 1-\binom{k}{1}[(k-1)/n]=1-k(k-1)/n. \tag{26}$$

It also holds that

$$(1/n^k)\binom{n}{k}k! = [n(n-1)\ldots(n-k+1)]/n^k \leq n^k/n^k = 1.$$

Using (25) and (26) in (24), we have

$$\begin{aligned} S_\alpha &\geq \sum_{k=\text{even}} (\alpha/m)^k \binom{m}{k}[1-k(k-1)/n] - \sum_{k=\text{odd}} (\alpha/m)^k \binom{m}{k} \\ &= \sum_{k=0}^{m} (\alpha/m)^k \binom{m}{k} - (1/n) \sum_{k=\text{even}} (\alpha/m)^k \binom{m}{k} k(k-1) \\ &= (1-\alpha/m)^m - (1/n) \sum_{k=\text{even}} (\alpha/m)^k \binom{m}{k} k(k-1). \end{aligned} \tag{27}$$

Let us consider the second term of (27). It holds that

$$\begin{aligned} &\sum_{k=\text{even}} (\alpha/m)^k \binom{m}{k} k(k-1) \\ &= \sum_{k=2,4,6,\ldots}^{k \leq m} (\alpha/m)^k \binom{m}{k} k(k-1) \\ &= \sum_{k=2,4,6,\ldots}^{k \leq m} (\alpha/m)^{k-2}(\alpha/m)^2 m!/[(k-2)!(m-k)!] \\ &= (\alpha/m)^2 \sum_{k=2,4,6,\ldots}^{k \leq m} (\alpha/m)^{k-2} \\ &\quad \times [m!/[(k-2)!(m-(k-2))!](m-k+2)(m-k+1)] \\ &\leq (\alpha/m)^2 \sum_{k=2,4,6,\ldots}^{k<m} (\alpha/m)^{k-2} \binom{m}{k-2} m(m-1) \\ &= [\alpha^2(m-1)/m] \sum_{k=2,4,6,\ldots}^{k<m} (\alpha/m)^{k-2} \binom{m}{k-2}. \end{aligned}$$

Using the binomial identity with $\delta = \alpha/m$, we have

$$[(1+\delta)^m + (1-\delta)^m]/2 = \begin{cases} 1+\delta^2\binom{m}{2}+\delta^4\binom{m}{4}+\cdots+\delta^m\binom{m}{m}, & m=\text{even}, \\ 1+\delta^2\binom{m}{2}+\cdots+\delta^{m-1}\binom{m}{m-1}, & m=\text{odd}, \end{cases} \tag{28}$$

which yields

$$\sum_{k=2,4,6,\ldots}^{k\le m} (\alpha/m)^{k-2}\binom{m}{k-2}$$
$$= \begin{cases} 1\binom{m}{0}+(\alpha/m)^2\binom{m}{4}+\cdots+\binom{m}{m-2}(\alpha/m)^{m-2}, & m=\text{even}, \\ 1\binom{m}{0}+(\alpha/m)^2\binom{m}{4}+\cdots+\binom{m}{m-3}(\alpha/m)^{m-3}, & m=\text{odd}, \end{cases}$$

$$= \begin{cases} [(1+\alpha/m)^m+(1-\alpha/m)^m]/2-(\alpha/m)^m\binom{m}{m}, & m=\text{even}, \\ [(1+\alpha/m)^m+(1-\alpha/m)^m]/2-(\alpha/m)^{m-1}\binom{m}{m-1}, & m=\text{odd}. \end{cases} \tag{29}$$

Using (28) and (29) in (27) yields

$$S_\alpha \ge (1-\alpha/m)^m-(1/n)[\alpha^2(m-1)/m](1+\alpha/m)^m+(1-\alpha/m)^m](1/2)$$
$$+\begin{cases} (1/n)[\alpha^2(m-1)/m](\alpha/m)^m\binom{m}{m}, & m=\text{even}, \\ (1/n)[\alpha^2(m-1)/m](\alpha/m)^{m-1}\binom{m}{m-1}, & m=\text{odd}, \end{cases}$$

or

$$S_\alpha \ge (1-\alpha/m)^m-(1/n)[\alpha^2(m-1)/m](1/2)[(1+\alpha/m)^m+(1-\alpha/m)^m]$$
$$+\begin{cases} (1/n)\alpha^{2+m}[(m-1)/m^{m+1}], & m=\text{even}, \\ (1/n)\alpha^{1+m}[(m-1)/m^{m-1}], & m=\text{odd}. \end{cases} \tag{30}$$

In either case, it holds that

$$\liminf_{m,n\to\infty} S_\alpha \geq e^{-\alpha}.$$

Using now part (ii), we have the desired result. □

Let us now take $\alpha=1$. Applying Lemma 3.1, we have the following theorem.

Theorem 3.1. If the elements of the $m\times n$ real matrices A and B are chosen independently and randomly with the same uniform distribution over any interval, then the probability that the resulting two-player Nash game has a pure equilibrium solution is $1-P_{mn}(F)$, where

$$P_{mn}(F)=(1/m^n)\sum_{\substack{n_1+\cdots+n_m=n\\ n_1\geq 0,\ldots,n_m\geq 0}} [n!(1-n_1/n)\ldots(1-n_m/n)]/(n_1!\ldots n_m!)$$

$$=\sum_{k=0}^{\min(m,n)}[-1/(mn)]^k\binom{m}{k}\binom{n}{k}k!.$$

Applying part (iii) of Lemma 3.1 for $\alpha=1$, we also have the following theorem.

Theorem 3.2. We have

$$\lim_{m,n\to+\infty} P_{mn}(F)=e^{-1}.$$

Remark 3.1. The importance of Theorem 3.2 is that it implies that, if we consider all the two-player nonzero-sum games in the sense of Remark 2.5, then since $(1-1/e)\cong 0.63$, this means that 63% of them have pure equilibrium solutions in contrast with the zero-sum game, where 0% have pure equilibrium solutions. It should be recalled that another basic difference between zero-sum and nonzero-sum is that the former have a unique value, whereas the latter have different pairs of values for different solutions. Thus in some sense, the nonzero-sum games, having more solutions than the zero-sum ones, allow a higher possibility of admitting pure equilibria.

Remark 3.2. Let $n\geq m$. Using $\alpha=1$, and part (i) of Lemma 3.1, we have

$$P_{mn}(F)=\sum_{k=0}^{m}[-1/(mn)]^k\binom{m}{k}\binom{n}{k}k!. \tag{31}$$

For the kth term of (31), it holds that

$$[-1/(mn)]^k \binom{m}{k}\binom{n}{k} k!$$

$$=[(-1)^k/(m^k n^k)]\binom{m}{k}[n!/k!(n-k)!]k!$$

$$=[(-1)^k/m^k]\binom{m}{k}[n!/n^k(n-k)!]$$

$$=(-1/m)^k\binom{m}{k}[n(n-1)\ldots(n-(k-1))]/n^k.$$

Since $n(n-1)\ldots(n-(k-1))$ has k terms, it behaves like n^k as $n\to\infty$; thus,

$$[n(n-1)\ldots(n-(k-1))]/n^k \cong n^k/n^k \to 1, \qquad \text{as } n\to+\infty.$$

Therefore,

$$[-1/(nm)]^k\binom{m}{k}\binom{n}{k}k! \to (-1/m)^k\binom{m}{k}, \qquad \text{as } n\to\infty;$$

thus,

$$P_{mn}(F)\cong\sum_{k=0}^{m}(-1/m)^k\binom{m}{k}=(1-1/m)^m, \qquad \text{for } n\to+\infty.$$

Letting now $m\to\infty$, we conclude that

$$\lim_{m\to\infty}\lim_{n\to\infty}P_{mn}(F)\to e^{-1}, \qquad \text{as } n \text{ and } m\to\infty. \tag{32}$$

Thus, since

$$P_{mn}(F)\le(1-1/m)^m\to e^{-1}$$

and

$$\lim_{m\to\infty}\lim_{n\to\infty}P_{mn}(F)=e^{-1},$$

we have a good indication that

$$\lim_{m,n\to\infty}P_{mn}(F)=e^{-1}.$$

To prove it, we need the more complex arguments used in proving part (iii) of Lemma 3.1.

Remark 3.3. Since

$$e^{-\alpha}=\sum_{k=0}^{\infty}(-\alpha)^k/k!,$$

it is reasonable to expect that, for the sum

$$\hat{S}_\alpha(m,n)=\sum_{k=0}^{\infty}[(-\alpha)^k/k!]\delta(k,m,n)$$

to converge to $e^{-\alpha}$, it is necessary that

$$\lim_{m,n\to\infty}\delta(k,m,n)=1. \tag{33}$$

For the formula of S_α in part (i) of Lemma 3.1, $\delta(k,m,n)$ is given by

$$\delta(k,m,n)=\begin{cases}(1/mn)^k\binom{m}{k}\binom{n}{k}k!, & \text{if } k\le\min(m,n),\\ 0, & \text{if } k>\min(m,n),\end{cases} \tag{34}$$

for which (33) holds, since

$$\delta(k,m,n)=[m(m-1)(m-k+1)/m^k]\ldots[n(n-1)\ldots(n-k+1)/n^k]\to 1,$$

as m and $n\to+\infty$ for fixed k. Thus, it is reasonable to expect that the sum S_α, as given in part (i) of Lemma 3.1, converges to $e^{-\alpha}$. Clearly, additional conditions on $\delta(k,m,n)$ beyond (33) are needed in order to guarantee convergence of $\hat{S}_\alpha(m,n)$ to $e^{-\alpha}$. Finally, it should be pointed out that $\delta(k,m,n)$ may depend on more arguments and not just m and n. For example, instead of (34), we could have

$$\delta(k,m,n,l)$$
$$=\begin{cases}\binom{m}{k}k!(1/m^k)\binom{n}{k}k!(1/n^k)\binom{l}{k}k!(1/l^k), & \text{if } k\le\min(l,m,n),\\ 0, & \text{if } k>\min(l,m,n).\end{cases}$$

Clearly,

$$\begin{aligned}\delta(k,m,n,l)&=[m(m-1)\ldots(m-k+1)/m^k]\\&\quad\times[n(n-1)\ldots(n-k+1)/n^k]\cdot[l(l-1)\ldots(l-k+1)/l^k]\\&\quad\to 1\cdot 1\cdot 1=1, \qquad \text{as } m,n,l\to\infty, \text{ for each fixed } k.\end{aligned}$$

Remark 3.4. Part (ii) of Lemma 3.1 and (29) yield, for $m \leq n$,

$$\begin{aligned}(1-1/m)^m &\geq P_{mn}(F)\\ &\geq (1-1/m)^m-(1/n)[(m-1)/m](1/2)\\ &\times[(1+1/m)^m+(1-1/m)^m]\\ &+(1/n)\begin{cases}(m-1)/m^{m+1}, & m=\text{even},\\ (m-1)/m^{m-1}, & m=\text{odd},\end{cases}\end{aligned} \tag{35}$$

which can be used to estimate the rapidity of approximation of $P_{mn}(F)$ to its limit.

4. Nash Game with More Than Two Players

In this section, we will first consider the three-player case. It will be clear how the derived formulas generalize to the more-than-three-player case. The cost of player i is given by

$$J_i(x, y, z)=\sum \alpha^i_{kw\sigma}x_k y_w z_\sigma, \qquad i=1, 2, 3,$$

where

$$\begin{aligned}&k=1,\ldots,n, \qquad w=1,\ldots,m, \qquad \sigma=1,\ldots,e,\\ &x_k\geq 0, \qquad x_1+\cdots+x_n=1,\\ &y_w\geq 0, \qquad y_1+\cdots+y_m=1,\\ &z_\sigma\geq 0, \qquad z_1+\cdots+z_e=1.\end{aligned}$$

Let $P_{lm}(F)$ denote the probability that a two-player nonzero-sum game with $l\times m$ matrices does not have a pure equilibrium solutions. Note that $P_{lm}(F)$ was calculated in the previous section; see Theorem 3.1.

In order to calculate the probability that a randomly chosen game with three players does not have a pure solution, we consider the following experiment. Consider the three-dimensional grid of Fig. 1.

The possible pure equilibrium may arise as follows. For each fixed value of n, say $n=k$, we are faced with a two-player nonzero-sum game with $l\times m$ dimensional matrices. If this $l\times m$ game has a pure equilibrium between the two corresponding players, this will happen if the choices of the two players meet at a particular point of the grid with k=fixed. The third player will choose σ_1 points of the grid with $k=1$, σ_2 points of the grid with $k=2$, and

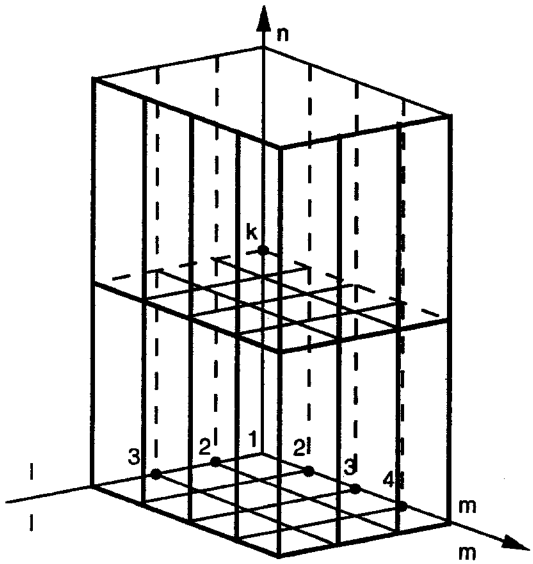

Fig. 1. Three-dimensional grid for a Nash game with three players.

so on, until he also chooses σ_n points of the grid with $k=n$. It will also be

$$\sigma_1+\sigma_2+\cdots+\sigma_n=lm.$$

For a particular configuration $(\sigma_1, \ldots, \sigma_n)$, the probability of failing to have the choice of the third player coincide with the choices of the other two players on the subgrid with k fixed is

$$P_{lm}(F)+P_{lm}(S)[(lm-\sigma_1)/lm]=1-(\sigma_1/lm)P_{lm}(S).$$

The probability of failing for each k is

$$[1-(\sigma_1/lm)P_{lm}(S)]\ldots[1-(\sigma_n/lm)P_{lm}(S)].$$

The possible positions of the choices of the third player if $\sigma_1, \ldots, \sigma_n$ choices correspond to each horizontal subgrid (i.e., $k=1, \ldots, n$) are

$$\binom{lm}{\sigma_1}\binom{lm-\sigma_1}{\sigma_2}\cdots\binom{lm-\sigma_1\ldots\sigma_{n-1}}{\sigma_n}=(lm)!/(\sigma_1!\sigma_2!\ldots\sigma_n!).$$

It holds that [recall (14)]

$$\sum_{\sigma_1+\cdots+\sigma_n=lm}(lm)!/(\sigma_1!\ldots\sigma_n!)=n^{lm}.$$

Thus, the probability of failing to have a pure equilibrium for the three-player game is

$$P_{lmn}(F) = \sum_{\sigma_1 + \cdots + \sigma_n = lm} \{[1 - P_{lm}(S)(\sigma_1/lm)] \ldots \times [1 - P_{lm}(S)(\sigma_n/lm)]/(\sigma_1! \ldots \sigma_n!)\}[(lm)!/n^{lm}]. \qquad (36)$$

Using Lemma 3.1 (i), we can rewrite $P_{lmn}(F)$ as

$$P_{lmn}(F) = \sum_{k=0}^{n} \{[1 - P_{l,m}(F)]/lmn\}^k \binom{n}{k}\binom{lm}{k} k!. \qquad (37)$$

Let us summarize the above results in the following theorem.

Theorem 4.1. If the $a^i_{kw\sigma}$ are chosen randomly and independently with the same uniform distribution over any interval, then the probability that the resulting three-player Nash game has a pure equilibrium solution is $1 - P_{lmn}(F)$, where $P_{lmn}(F)$ is given by (36) or (37).

It should be clear to the reader how this result can be generalized to the N-player case. In what follows, we concentrate on studying the limit behavior of this probability as the number of actions or number of players goes to infinity. Let us first consider an N-player game where the number of actions goes to infinity. Let us denote by $P^N(F)$ the probability of failing to have pure equilibria for an N-player game with infinite action sets for the players. Since

$$\lim_{lm \to +\infty} P_{lm}(F) = e^{-1} = P^2(F),$$

using Lemma 3.1 yields the formula

$$P^3(F) = \exp[-1 + P^2(F)], \qquad P^2(F) = e^{-1},$$

and more generally

$$P^{N+1}(F) = \exp[-1 + P^N(F)], \qquad P^2(F) = e^{-1}, \qquad N \geq 2. \qquad (38)$$

It is easy to see that the recursion (38) yields a monotonically increasing sequence $P^N(F)$ which converges to 1 as $N \to +\infty$. This proves the following theorem.

Theorem 4.2. If the number of actions available to the players tends to infinity and the number of players tends to infinity, the probability of having a pure equilibrium solution goes to zero.

Next, we examine the case where the number of actions available to all the players is fixed and equal to n, but the number of players N increases to infinity. Let us denote by $P_n^N(F)$ the probability that the randomly chosen game does not admit a pure equilibrium solution. It holds that (see Theorem 3.1)

$$P_n^2(F) = \sum_{k=0}^{n} (-1/n^2)^k \binom{n}{k}\binom{n}{k} k!. \tag{39}$$

Using (37), we have

$$P_n^3(F) = \sum_{k=0}^{n} \{-[1-P_n^2(F)]/n^3\}^k \binom{n}{k}\binom{n^2}{k} k!, \tag{40}$$

and using (38),

$$P_n^{N+1}(F) = \sum_{k=0}^{n} \{-[1-P_n^N(F)]/n^{N+1}\}^k \binom{n}{k}\binom{n^N}{k} k!. \tag{41}$$

It holds that

$$\begin{aligned}
&P_n^{N+1}(F)\\
&= \sum_{k=0}^{n} (-1)^k [1-P_n^N(F)]^k [n!/k!(n-k)!][(n^N)!/k!(n^N-k)!](k!/n^{N_{k+k}})\\
&= \sum_{k=0}^{n} (-1)^k [1-P_n^N(F)]^k (1/k!)[n!/(n-k)!n^k][(n^N)!/(n^N-k)!n^{Nk}]\\
&= P_n^N(F) + \sum_{k=2}^{n} (-1)^k [1-P_n^N(F)]^k (1/k!)[n!/(n-k)!n^k][(n^N)!/(n^N-k)!n^{N_k}]\\
&= P_n^N(F) + \sum_{k=2}^{n} (-1)^k [1-P_n^N(F)]^k\\
&\times (1/k!)(1-1/n)\dots[1-(k-1)/n](1-1/n^N)\dots[1-(k-1)/n^N].
\end{aligned} \tag{42}$$

The inequality

$$\begin{aligned}
&[1-P_n^N(F)]^k (1/k!)(1-1/n)\dots[1-(k-1)/n]\\
&\times(1-1/n^N)\dots[1-(k-1)/n^N]\\
&\geq [1-P_n^N(F)]^{k+1}[1/(k+1)!](1-1/n)\dots[1-(k-1)/n]\\
&\times(1-k/n)(1-1/n^N)\dots[1-(k-1)/n^N](1-k/n^N)
\end{aligned} \tag{43}$$

is equivalent to

$$1 \geq [1-P_n^N(F)][1/(k+1)](1-k/n)(1-k/n^N),$$

which obviously holds, since $P_n^N(F) \in [0, 1]$ and $k \leq n$, $k \leq n^N$. Thus, for $k =$ even, the sum of the consecutive kth and $(k+1)$th terms of (42) [the kth term is positive and the $(k+1)$th term is negative] is nonnegative. Therefore, the summation in (42) is nonnegative; consequently,

$$P_n^{N+1}(F) \geq P_n^N(F). \tag{44}$$

Note that the monotonicity of $P_n^N(F)$ in N is important in its own sake. (44) implies that the sequence $P_n^N(F)$ converges as $N \to \infty$. Letting now $N \to \infty$, we conclude that, as $N \to \infty$, a limit point of $P_n^N(F)$, say x^*, will satisfy [see (42)]

$$0 = \sum_{k=2}^{n} (-1)^k (1-x^*)^k (1/k!)(1-1/n) \ldots [1-(k-1)/n]$$

$$= (1-x^*)^2 \sum_{k=2}^{n} (-1)^k (1-x^*)^{k-2} (1/k!)(1-1/n) \ldots [1-(k-1)/n]. \tag{45}$$

Clearly, $x^* = 1$ satisfies (45). Since for $x^* \neq 1$, $x^* \in [0, 1)$, it holds that

$$(1-x^*)^{k-2}(1/k!)(1-1/n) \ldots [1-(k-1)/n] > (1-x^*)^{k-1}$$
$$\times [1/(k+1)!](1-1/n) \ldots [1-(k-1)/n](1-k/n), \tag{46}$$

the sum

$$\sum_{k=2}^{n} (-1)^k (1-x^*)^{k-2} (1/k!)(1-1/n) \ldots [1-(k-1)/n] \tag{47}$$

is strictly positive [recall a similar argument in (43)]. Thus, the only root of (45) is $x^* = 1$. To sum up, we have proven the following theorem.

Theorem 4.3. The sequence $P_n^N(F)$ converges monotonically to 1 as $N \to +\infty$; i.e., the probability of having pure solutions, for a randomly chosen N-player Nash game with a fixed finite multitude n of player actions, converges to zero as the multitude of players goes to infinity.

It should be noticed that, in some sense, Theorem 4.3 includes Theorem 4.2, since Theorem 4.2 corresponds to the extreme case of Theorem 4.3 where the multitude of actions is not fixed finite n, but infinite.

5. Conclusions and Extensions

In this paper, we examined the probability that a randomly chosen matrix game admits pure equilibrium solutions. We provided explicit formulae for the probability of this occurrence for zero-sum and nonzero-sum games and studied their behavior as the number of actions available to the players or as the number of players goes to infinity. We showed that, for zero-sum two-player games, this probability goes to zero as the number of actions goes to infinity. In contrast, for nonzero-sum Nash games, this does not hold. Nonetheless, for Nash games, if the number of players goes to infinity, this probability goes to zero independently of whether the number of actions is finite or infinite.

Several extensions of these results can be examined. For example, matrix games of particular structure correspond to the extensive form of dynamic games; in particular, examination of the limiting behavior of the probability of existence of pure equilibria as the time length goes to infinity is of particular interest. Another extension recommended by J. B. Cruz, Jr. is the following: Cruz Problem. Considering the result presented here that a randomly chosen zero-sum two-player game with infinite action spaces almost never has pure equilibrium solutions, whereas a corresponding two-player Nash game randomly chosen does have pure solutions with nonzero probability, the following well-posedness problem arises. If we chose randomly a two-player Nash game with infinite action spaces, subject to the restriction $\|B+A\| \leq \epsilon$, is it true that the probability that it has pure solutions goes to zero as $\epsilon \to 0$, so as to recover the zero-sum result? This problem is similar in spirit to the examination of lack of well-posedness for dynamic differential zero-sum games with respect to their limiting behavior as the time interval goes to infinity, demonstrated in Ref. 4.

Another direction for investigation is the following. It is known that, between zero-sum and nonzero-sum cases, there are important differences. Perhaps, the most basic one is that the solutions of zero-sum games produce a unique cost value for the players, whereas the solutions of nonzero-sum games may result in different cost for each player. Another difference that surfaced from the analysis presented in this paper is that the probability of having pure equilibria in randomly chosen games is zero for zero-sum games and nonzero for nonzero-sum games. Nonetheless, it was also shown that, as the number of player tends to infinity, the probability of appearance of pure equilibria tends to zero, i.e., recovers the zero-sum case characteristic. An immediate question arises due to this behavior and it is the following: Although an N-player nonzero-sum game may have many solutions which result in differnt costs for each player, is it true that, as N tends to infinity, although multiplicity of solutions may still be present, the resulting costs to each player tend to a unique value?

A more general issue is the study of multiplicity of the solution cost values for each player, its dependence on the number of actions and players, and its limiting behavior as the number of actions or players goes to infinity.

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