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University-Students Game

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Abstract The purpose of this paper is to formulate and study a game where there is a player who is involved for a long time interval and several small players who stay in the game for short time intervals. Examples of such games abound in practice. For example, a Bank is a long term player who stays in business for a very long time whereas most of its customers are affiliated with the Bank for relatively short time periods. A University and its Students provide another example and it is this model that we use here for motivating and posing the questions. The University is considered to have an infinite time horizon and the Students are considered as players who stay in the game for a fixed period of 5 years (indicative number). A class of Students who start their studies at a certain year is considered as one player/Student who is involved for 5 years. This player overlaps in action with the other students who entered at different years and with the University. We study this game in a linear quadratic, deterministic, discrete and continuous time setups, where the players use linear feedback strategies and are in Nash or Stackelberg equilibrium, and where the Students have the same cost structure independently of the year they started their studies. An important feature of the solutions derived is that they lead to Riccati type equations for calculating the gains, which are interlaced in time i.e. their evolution depends on present and past values of the gains. In the continuous time setup this corresponds to integrodifferential equations.

Keywords Nash · Stackelberg · Linear quadratic · Different and overlapping time horizons

1 Introduction

The purpose of this paper is to formulate and study a game where there is a player who is involved for a long time interval and several small players who stay in the game for short time intervals. Examples of such games abound in practice. For example, a Bank is a long

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term player who stays in business for a very long time whereas most of its customers are affiliated with the Bank for relatively short time periods. A University and its Students provide another example and it is this model that we use here for motivating and posing the questions. The University is considered to have an infinite time horizon and the Students are considered as players who stay in the game for a fixed period of 5 years (indicative number). All the formulae can be straightforwardly adapted for the case of other time horizons. (Having these formulae is important because one can use them for studying the impact of extending or shortening the time horizon of the small players.) A class of Students who start their studies at a certain year is considered as one player/Student who is involved for 5 years. This player overlaps in action with the other students who entered at different years and with the University. For example, when a Student is in the third year of his studies, he overlaps with the Student who is in the first year of his studies, with the Student who is in the second year, the Student who is in the fourth year, the Student who is in the fifth year, and of course with the University.

We will study this game in a linear quadratic, deterministic, discrete and continuous time setups, where the players use linear feedback strategies and are in Nash or Stackelberg equilibrium, and where the Students have the same cost structure independently of the year they started their studies. We seek feedback solutions that are linear functions of the current state, since it is known that they are the only ones that survive in the presence of small disturbances. An important feature of the solutions derived is that they lead to Riccati type equations for calculating the gains, which are interlaced in time, i.e., their evolution depends on present and past values of the gains. In the continuous time setup this corresponds to integrodifferential equations. Variants of this game may consider additional features of importance in practice, such as the random entry time and exit of the small players, the existence of a pool of different types of Students from which the Students who start their studies at a certain year are drawn, etc.

Earlier versions of portions of this work have been presented by the author and his coauthors in [14–17].They deal with aspects of the deterministic formulation with the exemption of [17] where the random entry/exit and pools of different types of small players is considered. Relations to the work presented here can be found with the reputation games, [11], the overlapping generations problems; see [3, 19], and the intergenerational game models introduced in [2] and further developed in [4, 12, 13].

In Sect. 1, we present the basic model. In Sect. 2, we present the Nash feedback solution, and in Sect. 3 the feedback Stackelberg solution. In Sect. 4, we present some sufficient conditions for existence, and in Sect. 5 the continuous time analogues are given. In Sect. 6, some conceptual numerical algorithms are delineated as to highlight some important features that deserve further consideration. Section 7 presents two scalar examples for the feedback Nash strategy, one for the discrete time and the other for the continuous time. These examples exhibit some very interesting features of the games at hand. Conclusions are in Sect. 8.

2 State Equations and Costs

The state evolves according to

$$x_{k+1} = Ax_k + Bu_k + B_1u_k^1 + B_2u_k^2 + B_3u_k^3 + B_4u_k^4 + B_5u_k^5, \quad k = 0, 1, 2, \dots \tag{1}$$

The cost of the University is

$$J = \frac{1}{2}\sum_{k=0}^{\infty}\left(x_k^T Q x_k + u_k^T R u_k\right) \tag{2}$$

The cost of the Student who enters the University at time k is

$$\begin{aligned} J_s[k,k+5] &= \frac{1}{2}\{x_k^T Q_1 x_k + x_{k+1}^T Q_2 x_{k+1} + x_{k+2}^T Q_3 x_{k+2} + x_{k+3}^T Q_4 x_{k+3} \\ &\quad + x_{k+4}^T Q_5 x_{k+4} + x_{k+5}^T Q_6 x_{k+5} + (u_k^1)^T R_1 u_k^1 + (u_{k+1}^2)^T R_2 u_{k+1}^2 \\ &\quad + (u_{k+2}^3)^T R_3 u_{k+2}^3 + (u_{k+2}^4)^T R_4 u_{k+2}^4 + (u_{k+2}^5)^T R_5 u_{k+2}^5\} \\ &= \frac{1}{2} x_{k+5}^T Q_6 x_{k+5} + \frac{1}{2}\sum_{l=0}^{4}\left(x_{k+l}^T Q_{l+1} x_{k+l} + (u_{k+l}^{l+1})^T R_{l+1} u_{k+l}^{l+1}\right) \\ & x_k \in R^n, u_k \in R^m, u_k^i \in R^{m_i} \end{aligned} \tag{3}$$

The matrices involved have dimensions:

$$\begin{aligned} &A(n\times n),\quad B(n\times m),\quad B_i(n\times m_i) \\ &Q(n\times n),\quad Q_i(n\times n),\quad R(m\times m),\quad R_i(m_i\times m_i) \\ &Q = Q^T \geq 0,\ Q_i = Q_i^T \geq 0,\ R = R^T > 0,\ R_i = R_i^T > 0,\ i = 1,2,3,4,5 \end{aligned}$$

The Student who enters the University at time k sees a state evolution as follows, where his control actions are u_k^1, u_{k+1}^2, u_{k+2}^3, u_{k+3}^4, u_{k+4}^5, (marked in bold):

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + B_1\boldsymbol{u_k^1} + B_2u_k^2 + B_3u_k^3 + B_4u_k^4 + B_5u_k^5 \\ x_{k+2} &= Ax_{k+1} + Bu_{k+1} + B_1u_{k+1}^1 + B_2\boldsymbol{u_{k+1}^2} + B_3u_{k+1}^3 + B_4u_{k+1}^4 + B_5u_{k+1}^5 \\ x_{k+3} &= Ax_{k+2} + Bu_{k+2} + B_1u_{k+2}^1 + B_2u_{k+2}^2 + B_3\boldsymbol{u_{k+2}^3} + B_4u_{k+2}^4 + B_5u_{k+2}^5 \\ x_{k+4} &= Ax_{k+3} + Bu_{k+3} + B_1u_{k+3}^1 + B_2u_{k+3}^2 + B_3u_{k+3}^3 + B_4\boldsymbol{u_{k+3}^4} + B_5u_{k+3}^5 \\ x_{k+5} &= Ax_{k+4} + Bu_{k+4} + B_1u_{k+4}^1 + B_2u_{k+4}^2 + B_3u_{k+4}^3 + B_4u_{k+4}^4 + B_5\boldsymbol{u_{k+4}^5} \end{aligned} \tag{4}$$

At time instant k, say for example, $k = 46$, the costs involved are: J, $J_s[46,51]$, $J_s[45,50]$, $J_s[44,49]$, $J_s[43,48]$, $J_s[42,47]$, which means that the players involved are the University with current action u_k and cost J, the Student who is a first year student at time $k = 46$ with current action u_k^1 and cost $J_s[46,51]$, the Student who is a second year student at time $k = 46$ with current action u_k^2 and cost $J_s[45,50]$, the Student who is a third year student at time $k = 46$ with current action u_k^3 and cost $J_s[44,49]$, the Student who is a fourth year student at time $k = 46$ with current action u_k^4 and cost $J_s[43,48]$, and the Student who is a fifth year student at time $k = 46$ with current action u_k^5 and cost $J_s[42,47]$. Thus, the controls to be characterized by backward induction for the feedback Nash or the feedback Stackelberg equilibria are u_{46}, u_{46}^1, u_{46}^2, u_{46}^3, u_{46}^4, u_{46}^5.

3 The Feedback Nash Solution

In this section, we will derive the feedback Nash solution. The open loop Nash solution is also of interest but it will be considered elsewhere. The feedback Nash solution is obtained by using dynamic programming; see [1, 6, 8, 11, 22, 23]. The feedback solutions we are after are restricted to be linear functions of the current state, since it is known that they are the only ones that survive in the presence of small disturbances. At time k, the University solves the problem:

$$\min_{u_k} \frac{1}{2}\left(x_k^T Q x_k + u_k^T R u_k + x_{k+1}^T K x_{k+1}\right)$$

$$\min_{u_k} \frac{1}{2}\left\{x_k^T Q x_k + u_k^T R u_k + \left(A x_k + B u_k + B_1 u_k^1 + B_2 u_k^2 + B_3 u_k^3 + B_4 u_k^4 + B_5 u_k^5\right)^T \right. \\ \left. \times K\left(A x_k + B u_k + B_1 u_k^1 + B_2 u_k^2 + B_3 u_k^3 + B_4 u_k^4 + B_5 u_k^5\right)\right\}$$

and the solution satisfies

$$R u_k + B^T K\left(A x_k + B u_k + B_1 u_k^1 + B_2 u_k^2 + B_3 u_k^3 + B_4 u_k^4 + B_5 u_k^5\right) = 0$$

At time k, the first year student solves:

$$\min_{u_k^1}\left(x_k^T Q_1 x_k + \left(u_k^1\right)^T R_1 u_k^1 + x_{k+1}^T K_1 x_{k+1}\right)$$

$$\min_{u_k^1}\left[x_k^T Q_1 x_k + \left(u_k^1\right)^T R_1 u_k^1 + \left(A x_k + B u_k + B_1 u_k^1 + B_2 u_k^2 + B_3 u_k^3 + B_4 u_k^4 + B_5 u_k^5\right)^T \right. \\ \left. \times K_1\left(A x_k + B u_k + B_1 u_k^1 + B_2 u_k^2 + B_3 u_k^3 + B_4 u_k^4 + B_5 u_k^5\right)\right]$$

where

$$\frac{1}{2} x_{k+1}^T K_1 x_{k+1}$$

is the cost to go and the solution is

$$R_1 u_k^1 + B_1^T K_1\left(A x_k + B u_k + B_1 u_k^1 + B_2 u_k^2 + B_3 u_k^3 + B_4 u_k^4 + B_5 u_k^5\right) = 0$$

At time k, the second year student solves:

$$\min_{u_k^2}\left(x_k^T Q_2 x_k + \left(u_k^2\right)^T R_2 u_k^2 + x_{k+1}^T K_2 x_{k+1}\right)$$

$$\min_{u_k^2}\left[x_k^T Q_2 x_k + \left(u_k^2\right)^T R_2 u_k^2 + \left(A x_k + B u_k + B_1 u_k^1 + B_2 u_k^2 + B_3 u_k^3 + B_4 u_k^4 + B_5 u_k^5\right)^T \right. \\ \left. \times K_2\left(A x_k + B u_k + B_1 u_k^1 + B_2 u_k^2 + B_3 u_k^3 + B_4 u_k^4 + B_5 u_k^5\right)\right]$$

where

$$\frac{1}{2} x_{k+1}^T K_2 x_{k+1}$$

is the cost to go and the solution is

$$R_2 u_k^2 + B_2^T K_2\left(A x_k + B u_k + B_1 u_k^1 + B_2 u_k^2 + B_3 u_k^3 + B_4 u_k^4 + B_5 u_k^5\right) = 0$$

At time k, the third year student solves:

$$\min_{u_k^3}\left(x_k^T Q_3 x_k + \left(u_k^3\right)^T R_3 u_k^3 + x_{k+1}^T K_3 x_{k+1}\right)$$

$$\min_{u_k^3}\left[x_k^T Q_3 x_k + \left(u_k^3\right)^T R_3 u_k^3 + \left(A x_k + B u_k + B_1 u_k^1 + B_2 u_k^2 + B_3 u_k^3 + B_4 u_k^4 + B_5 u_k^5\right)^T \right. \\ \left. \times K_3\left(A x_k + B u_k + B_1 u_k^1 + B_2 u_k^2 + B_3 u_k^3 + B_4 u_k^4 + B_5 u_k^5\right)\right]$$

where

$$\frac{1}{2}x_{k+1}^T K_3 x_{k+1}$$

is the cost to go and the solution is

$$R_3 u_k^3 + B_3^T K_3\big(Ax_k + Bu_k + B_1 u_k^1 + B_2 u_k^2 + B_3 u_k^3 + B_4 u_k^4 + B_5 u_k^5\big) = 0$$

At time k, the fourth year student solves:

$$\min_{u_k^4}\big(x_k^T Q_4 x_k + \big(u_k^4\big)^T R_4 u_k^4 + x_{k+1}^T K_4 x_{k+1}\big)$$

$$\min_{u_k^4}\big[x_k^T Q_4 x_k + \big(u_k^4\big)^T R_4 u_k^4 + \big(Ax_k + Bu_k + B_1 u_k^1 + B_2 u_k^2 + B_3 u_k^3 + B_4 u_k^4 + B_5 u_k^5\big)^T \times K_4\big(Ax_k + Bu_k + B_1 u_k^1 + B_2 u_k^2 + B_3 u_k^3 + B_4 u_k^4 + B_5 u_k^5\big)\big]$$

where

$$\frac{1}{2}x_{k+1}^T K_4 x_{k+1}$$

is the cost to go and the solution is

$$R_4 u_k^4 + B_4^T K_4\big(Ax_k + Bu_k + B_1 u_k^1 + B_2 u_k^2 + B_3 u_k^3 + B_4 u_k^4 + B_5 u_k^5\big) = 0$$

At time k, the fifth year student solves:

$$\min_{u_k^5}\big(x_k^T Q_5 x_k + \big(u_k^5\big)^T R_5 u_k^5 + x_{k+1}^T K_5 x_{k+1}\big)$$

$$\min_{u_k^5}\big[x_k^T Q_5 x_k + \big(u_k^5\big)^T R_5 u_k^5 + \big(Ax_k + Bu_k + B_1 u_k^1 + B_2 u_k^2 + B_3 u_k^3 + B_4 u_k^4 + B_5 u_k^5\big)^T \times K_5\big(Ax_k + Bu_k + B_1 u_k^1 + B_2 u_k^2 + B_3 u_k^3 + B_4 u_k^4 + B_5 u_k^5\big)\big]$$

where

$$\frac{1}{2}x_{k+1}^T K_5 x_{k+1}$$

is the cost to go and the solution is

$$R_5 u_k^5 + B_5^T K_5\big(Ax_k + Bu_k + B_1 u_k^1 + B_2 u_k^2 + B_3 u_k^3 + B_4 u_k^4 + B_5 u_k^5\big) = 0$$

Under the appropriate invertibility assumptions, the system of equations has a solution of the form:

$$u_k = Lx_k, \qquad u_k^1 = L_1 x_k, \qquad u_2^k = L_2 x_k,$$
$$u_3^k = L_3 x_k, \qquad u_k^4 = L_4 x_k, \qquad u_k^5 = L_5 x_k$$

where the gains L, L_1, L_2, L_3, L_4, L_5 satisfy:

$$RL + B^T K(A + BL + B_1 L_1 + B_2 L_2 + B_3 L_3 + B_4 L_4 + B_5 L_5) = 0$$

$$R_i L_i + B_i^T K_i(A + BL + B_1 L_1 + B_2 L_2 + B_3 L_3 + B_4 L_4 + B_5 L_5) = 0, \quad i = 1, 2, \ldots, 5 \tag{5}$$

Or

$$\begin{aligned} &A_c = A + BL + B_1L_1 + B_2L_2 + B_3L_3 + B_4L_4 + B_5L_5 \\ &RL + B^T K A_c = 0 \\ &R_i L_i + B_i^T K_i A_c = 0, \quad i = 1, 2, \ldots, 5 \end{aligned} \tag{6}$$

For the costs to go $\frac{1}{2} x_{k+1}^T K_i x_{k+1}$ of the students hold:

$$\begin{aligned} &K_5 = Q_6 \\ &K_i = Q_{i+1} + L_{i+1}^T R_{i+1} L_{i+1} + A_c^T K_{i+1} A_c, \quad i = 0, 1, \ldots, 4 \end{aligned} \tag{7}$$

where

$$J_s^*[k, k+5] = \frac{1}{2} x_k K_0 x_k$$

is the optimal cost of the student who entered the University at year k. For the cost to go of the University, using the first three equations of the present section in conjunction with the fact that the University's problem is infinite time, time-invariant yields:

$$K = Q + L^T R L + A_c^T K A_c$$

and

$$J^* = \frac{1}{2} x_0 K x_0$$

is the optimal cost of the University. From (5), (6), we have

$$\begin{aligned} &L = -R^{-1} B^T K A_c \\ &L_i = -R_i^{-1} B_i^T K_i A_c, \quad i = 1, 2, \ldots, 5 \end{aligned} \tag{8a}$$

and substituting in (7) we get:

$$\begin{aligned} &K = Q + A_c^T \big(K + K B R^{-1} B^T K\big) A_c \\ &K_5 = Q_6, \\ &K_i = Q_{i+1} + A_c^T \big(K_{i+1} + K_{i+1} B_{i+1} R_{i+1}^{-1} B_{i+1}^T K_{i+1}\big) A_c, \quad i = 0, 1, \ldots, 4 \\ &A = \big(I + B R^{-1} B^T K + B_1 R_1^{-1} B_1^T K_1 + B_2 R_2^{-1} B_2^T K_2 + B_3 R_3^{-1} B_3^T K_3 \\ &\qquad + B_4 R_4^{-1} B_4^T K_4 + B_5 R_5^{-1} B_5^T K_5\big) A_c \end{aligned} \tag{8b}$$

Let us formalize the above results in the form of a proposition.

Proposition 1 *Let us assume that the system of* (8b) *has a solution*: $K_5 \geq 0$, $K_4 \geq 0$, $K_3 \geq 0$, $K_2 \geq 0$, $K_1 \geq 0$, $K_0 \geq 0$, $K \geq 0$, A_c *that satisfy also* (9). *Then the* (*linear*) *feedback Nash solution of the problem* (1)–(3) *is* $u = L x_k$ *for the University and* $u_i = L_i x_k$, $i = 1, 2, \ldots, 5$ *for the Students*, *where* (8a) *and* (8b) *give the* L, K, L_i, K_i, $i = 0, 1, 2, \ldots, 5$. *The optimal cost of the University is* $J^* = \frac{1}{2} x_0 K x_0$ *and for the Student who starts at year k is* $J_s^*[k, k+5] = \frac{1}{2} x_k K_0 x_k$.

Essentially, (8b) is the system of equations that has to be solved for the K_0, K_1, K_2, K_3, K_4, K_5, K and the closed loop matrix A_c. Notice that it has to hold: $K_5 \geq 0$, $K_4 \geq 0$, $K_3 \geq 0$,

$K_2 \geq 0$, $K_1 \geq 0$, $K_0 \geq 0$ which will obviously hold since all the $Q_i \geq 0$. In addition it has to hold that: $K \geq 0$. For having asymptotic stability and thus finite University cost it suffices:

$$\left|\text{eigenvalues}(A_c)\right| < 1 \tag{9}$$

(If there is no University player, the sufficiency condition (9) that guarantees finiteness of the University cost is not needed. The closed loop matrix does not need to be asymptotically stable, since then the costs of the Students, being calculated during finite time periods, are finite even though the system may become unstable as time goes to infinity.) Notice that we have an interlaced (in time) system of quadratic type difference equations, since the evolution of the difference equation for K_k depends on past and future values of the unknown K_k. This type of equations is beyond the usual coupled Riccati equations that appear in several Linear Quadratic Games.

Remarks It would be interesting to isolate some interesting cases and study them on their own, such as:

1. Only Students are present; no University is present. This is model where coordination can exist through time without the permanent presence of a coordinator. The role of the coordinator is assumed by a succession of overlapping generations that although they individually have a finite life time, they are interlaced by succession, and thus a permanent sustenance is manifested. We examine this issue in the context of an example in Sect. 7.
2. Increase the number of years a Student stays in the University, and study the limiting behavior on the costs and the closed loop matrix.. Study the impact of increasing the number of years a Student stays in the University in conjunction with the absence of the University, i.e., absence of a coordinator.
3. Take: $Q_1 \leq Q_2 \leq Q_3 \leq Q_4 \leq Q_4 \leq Q_6 = Q$, i.e., as far as the state is concerned; the objective of the student as he matures in years of study coincides with that of the University (Q). Take $R = I$ for the University and the R_i's of the Students to increase toward the $R = I$ of the University: $R_1 \geq R_2 \geq R_3 \geq R_4 \geq R_5 = I$. (As the Students mature tend to agree with University's overall goals.) In this case, we can take $B_1 = B_2 = B_3 = B_4 = B_5$. We do not need to take $B = B_1$ since University and Students do not affect the state in an identical manner.

4 The Feedback Stackelberg Solution

Let us now derive the feedback Stackelberg solution. This type of solution was introduced in [7, 20, 21], and it uses dynamic programming for deriving the solutions. For further insights into this solution concept, see [5, 6]. We consider that the University is the Leader and the Students are Followers who play Nash among themselves. Here, as for the previous case, we seek feedback solutions that are linear functions of the current state, since it is known that they are the only ones that survive in the presence of small disturbances. At time k, the students solve the problems (1), (3) where they consider the u_k as given, and the following equations result:

$$R_i u_k^i + B_i^T K_i \left(Ax_k + Bu_k + B_1 u_k^1 + B_2 u_k^2 + B_3 u_k^3 + B_4 u_k^4 + B_5 u_k^5\right) = 0, \quad i = 1, 2, \ldots, 5 \tag{10}$$

The University solves the problem:

$$\min_{u_k,u_k^1,u_k^2,u_k^3,u_k^4,u_k^5} \frac{1}{2}\left(x_k^T Q x_k + u_k^T R u_k + x_{k+1}^T K x_{k+1}\right)$$

$$\min_{u_k,u_k^1,u_k^2,u_k^3,u_k^4,u_k^5} \frac{1}{2}\left\{x_k^T Q x_k + u_k^T R u_k + \left(Ax_k + Bu_k + B_1u_k^1 + B_2u_k^2 + B_3u_k^3 + B_4u_k^4 + B_5u_k^5\right)^T \right.$$
$$\left. \times K\left(Ax_k + Bu_k + B_1u_k^1 + B_2u_k^2 + B_3u_k^3 + B_4u_k^4 + B_5u_k^5\right)\right\}$$

subject to the constraints (10).

The constraints are linear and, therefore, Lagrange multipliers $\lambda_i \in R^{m_i}$, $i = 1, 2, 3, 4, 5$ exist and we can append the constraints to the objective function and form the Lagrangian L. Since the objective function is convex, setting the gradient of the Lagrangian with respect to the unknowns u_k, u_k^1, u_k^2, u_k^3, u_k^4, u_k^5 equal to zero provides together with the constraints (10) necessary and sufficient conditions for the minimization of the University's problem.

The Lagrangian is:

$$\begin{aligned}
&L\left(u_k, u_k^1, u_k^2, u_k^3, u_k^4, u_k^5, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\right)\\
&\quad= \frac{1}{2}x_k^T Q x_k + \frac{1}{2}u_k^T R u_k + \frac{1}{2}\left(Ax_k + Bu_k + B_1u_k^1 + B_2u_k^2 + B_3u_k^3 + B_4u_k^4 + B_5u_k^5\right)^T\\
&\qquad \times K\left(Ax_k + Bu_k + B_1u_k^1 + B_2u_k^2 + B_3u_k^3 + B_4u_k^4 + B_5u_k^5\right)\\
&\qquad + \lambda_1^T\left(R_1u_k^1 + B_1^T K_1\left(Ax_k + Bu_k + B_1u_k^1 + B_2u_k^2 + B_3u_k^3 + B_4u_k^4 + B_5u_k^5\right)\right)\\
&\qquad + \lambda_2^T\left(R_2u_k^2 + B_2^T K_2\left(Ax_k + Bu_k + B_1u_k^1 + B_2u_k^2 + B_3u_k^3 + B_4u_k^4 + B_5u_k^5\right)\right)\\
&\qquad + \lambda_3^T\left(R_3u_k^3 + B_3^T K_3\left(Ax_k + Bu_k + B_1u_k^1 + B_2u_k^2 + B_3u_k^3 + B_4u_k^4 + B_5u_k^5\right)\right)\\
&\qquad + \lambda_4^T\left(R_4u_k^4 + B_4^T K_4\left(Ax_k + Bu_k + B_1u_k^1 + B_2u_k^2 + B_3u_k^3 + B_4u_k^4 + B_5u_k^5\right)\right)\\
&\qquad + \lambda_5^T\left(R_5u_k^5 + B_5^T K_5\left(Ax_k + Bu_k + B_1u_k^1 + B_2u_k^2 + B_3u_k^3 + B_4u_k^4 + B_5u_k^5\right)\right)\\
&\quad \lambda_i \in R^{m_i}, \quad i = 1, 2, 3, 4, 5
\end{aligned}$$

Setting the gradients of L with respect to u_k, u_k^1, u_k^2, u_k^3, u_k^4, u_k^5 equal to zero, we get respectively:

$$\begin{aligned}
&\nabla_{u_k} L\left(u_k, u_k^1, u_k^2, u_k^3, u_k^4, u_k^5, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\right)\\
&\quad = Ru_k + B^T K\left(Ax_k + Bu_k + B_1u_k^1 + B_2u_k^2 + B_3u_k^3 + B_4u_k^4 + B_5u_k^5\right)\\
&\qquad + B^T (K_1B_1\lambda_1 + K_2B_2\lambda_2 + K_3B_3\lambda_3 + K_4B_4\lambda_4 + K_5B_5\lambda_5) = 0\\
&\nabla_{u_k^1} L\left(u_k, u_k^1, u_k^2, u_k^3, u_k^4, u_k^5, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\right)\\
&\quad = B_1^T K\left(Ax_k + Bu_k + B_1u_k^1 + B_2u_k^2 + B_3u_k^3 + B_4u_k^4 + B_5u_k^5\right)\\
&\qquad + R_1\lambda_1 + B_1^T (K_1B_1\lambda_1 + K_2B_2\lambda_2 + K_3B_3\lambda_3 + K_4B_4\lambda_4 + K_5B_5\lambda_5) = 0\\
&\nabla_{u_k^2} L\left(u_k, u_k^1, u_k^2, u_k^3, u_k^4, u_k^5, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\right)\\
&\quad = B_2^T K\left(Ax_k + Bu_k + B_1u_k^1 + B_2u_k^2 + B_3u_k^3 + B_4u_k^4 + B_5u_k^5\right)\\
&\qquad + R_2\lambda_2 + B_2^T (K_1B_1\lambda_1 + K_2B_2\lambda_2 + K_3B_3\lambda_3 + K_4B_4\lambda_4 + K_5B_5\lambda_5) = 0
\end{aligned}$$

$$\nabla_{u_k^3} L\big(u_k, u_k^1, u_k^2, u_k^3, u_k^4, u_k^5, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\big)$$
$$= B_3^T K\big(Ax_k + Bu_k + B_1u_k^1 + B_2u_k^2 + B_3u_k^3 + B_4u_k^4 + B_5u_k^5\big)$$
$$+ R_3\lambda_3 + B_3^T(K_1B_1\lambda_1 + K_2B_2\lambda_2 + K_3B_3\lambda_3 + K_4B_4\lambda_4 + K_5B_5\lambda_5) = 0$$
$$\nabla_{u_k^4} L\big(u_k, u_k^1, u_k^2, u_k^3, u_k^4, u_k^5, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\big)$$
$$= B_4^T K\big(Ax_k + Bu_k + B_1u_k^1 + B_2u_k^2 + B_3u_k^3 + B_4u_k^4 + B_5u_k^5\big)$$
$$+ R_4\lambda_4 + B_4^T(K_1B_1\lambda_1 + K_2B_2\lambda_2 + K_3B_3\lambda_3 + K_4B_4\lambda_4 + K_5B_5\lambda_5) = 0$$
$$\nabla_{u_k^5} L\big(u_k, u_k^1, u_k^2, u_k^3, u_k^4, u_k^5, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\big)$$
$$= B_5^T K\big(Ax_k + Bu_k + B_1u_k^1 + B_2u_k^2 + B_3u_k^3 + B_4u_k^4 + B_5u_k^5\big)$$
$$+ R_5\lambda_5 + B_5^T(K_1B_1\lambda_1 + K_2B_2\lambda_2 + K_3B_3\lambda_3 + K_4B_4\lambda_4 + K_5B_5\lambda_5) = 0$$

The system of equations that we have to solve for the unknowns u_k, u_k^1, u_k^2, u_k^3, u_k^4, u_k^5, λ_1, λ_2, λ_3, λ_4, λ_5 is:

$$\begin{aligned} &R_iu_k^i + B_i^T K_i\big(Ax_k + Bu_k + B_1u_k^1 + B_2u_k^2 + B_3u_k^3 + B_4u_k^4 + B_5u_k^5\big) = 0, \quad i = 1, 2, \ldots, 5 \\ &Ru_k + B^T K\big(Ax_k + Bu_k + B_1u_k^1 + B_2u_k^2 + B_3u_k^3 + B_4u_k^4 + B_5u_k^5\big) \\ &\quad + B^T(K_1B_1\lambda_1 + K_2B_2\lambda_2 + K_3B_3\lambda_3 + K_4B_4\lambda_4 + K_5B_5\lambda_5) = 0 \\ &R_i\lambda_i + B_i^T K\big(Ax_k + Bu_k + B_1u_k^1 + B_2u_k^2 + B_3u_k^3 + B_4u_k^4 + B_5u_k^5\big) \\ &\quad + B_i^T(K_1B_1\lambda_1 + K_2B_2\lambda_2 + K_3B_3\lambda_3 + K_4B_4\lambda_4 + K_5B_5\lambda_5) = 0, \quad i = 1, 2, \ldots, 5 \end{aligned} \tag{11}$$

Under the appropriate invertibility assumptions, the solutions for the unknowns will be linear in x_k, i.e.,

$$u_k = Lx_k, \qquad u_k^1 = L_1x_k, \qquad u_k^2 = L_2x_k$$
$$u_k^3 = L_3x_k, \qquad u_k^4 = L_4x_k, \qquad u_k^5 = L_5x_k$$

We set

$$A_c = A + BL + B_1L_1 + B_2L_2 + B_3L_3 + B_4L_4 + B_5L_5$$
$$\lambda = K_1B_1\lambda_1 + K_2B_2\lambda_2 + K_3B_3\lambda_3 + K_4B_4\lambda_4 + K_5B_5\lambda_5$$

Then Eqs. (11) can be written as:

$$L_i = -R_i^{-1}B_i^T K_i A_c, \quad i = 1, 2, \ldots, 5$$
$$Lx_k = -R^{-1}B^T(KA_cx_k + \lambda)$$
$$\lambda_i = -R_i^{-1}B_i^T(KA_cx_k + \lambda), \quad i = 1, 2, \ldots, 5$$

Substituting the λ_i in (11) we get:

$$\big(I + K_1B_1R_1^{-1}B_1^T + K_2B_2R_2^{-1}B_2^T + K_3B_3R_3^{-1}B_3^T + K_4B_4R_4^{-1}B_4^T + K_5B_5R_5^{-1}B_5^T\big)\lambda$$
$$+ \big(K_1B_1R_1^{-1}B_1^T + K_2B_2R_2^{-1}B_2^T + K_3B_3R_3^{-1}B_3^T + K_4B_4R_4^{-1}B_4^T$$
$$+ K_5B_5R_5^{-1}B_5^T\big)KA_cx_k = 0$$

It is clear that the basic condition for existence of a unique solution is the invertibility of the matrix: $I + W$ where

$$W = K_1B_1R_1^{-1}B_1^T + K_2B_2R_2^{-1}B_2^T + K_3B_3R_3^{-1}B_3^T + K_4B_4R_4^{-1}B_4^T + K_5B_5R_5^{-1}B_5^T$$

Therefore, if the matrix $I + W$ is invertible, have

$$\lambda = -(I + W)^{-1} W K A_c x_k$$
$$L = -R^{-1} B^T (I + W)^{-1} K A_c$$

In conclusion, we have the following formulae for the control gains:

$$\begin{aligned} &L_i = -R_i^{-1} B_i^T K_i A_c, \quad i = 1, 2, \ldots, 5 \\ &L = -R^{-1} B^T (I + W)^{-1} K A_c \\ &W = K_1 B_1 R_1^{-1} B_1^T + K_2 B_2 R_2^{-1} B_2^T + K_3 B_3 R_3^{-1} B_3^T + K_4 B_4 R_4^{-1} B_4^T + K_5 B_5 R_5^{-1} B_5^T \\ &\big(I + B R^{-1} B^T (I + W)^{-1} K + B_1 R_1^{-1} B_1^T K_1 + B_2 R_2^{-1} B_2^T K_2 \\ &\quad + B_3 R_3^{-1} B_3^T K_3 + B_4 R_4^{-1} B_4^T K_4 + B_5 R_5^{-1} B_5^T K_5\big) A_c = A \end{aligned} \tag{12}$$

As far as the updating of the costs to go is concerned, the same methodology and formulae hold as in (8a), (8b), which concern the Nash solution, but we have to use the new A_c of the formula (12) above. Thus, the final formulae for finding the feedback Stackelberg solution are:

$$\begin{aligned} &K = Q + A_c^T \big(K + K B R^{-1} B^T K\big) A_c \\ &K_5 = Q_6 \\ &K_i = Q_{i+1} + A_c^T \big(K_{i+1} + K_{i+1} B_{i+1} R_{i+1}^{-1} B_{i+1}^T K_{i+1}\big) A_c, \quad i = 0, 1, \ldots, 4 \\ &W = K_1 B_1 R_1^{-1} B_1^T + K_2 B_2 R_2^{-1} B_2^T + K_3 B_3 R_3^{-1} B_3^T + K_4 B_4 R_4^{-1} B_4^T + K_5 B_5 R_5^{-1} B_5^T \\ &\big(I + B R^{-1} B^T (I + W)^{-1} K + B_1 R_1^{-1} B_1^T K_1 + B_2 R_2^{-1} B_2^T K_2 \\ &\quad + B_3 R_3^{-1} B_3^T K_3 + B_4 R_4^{-1} B_4^T K_4 + B_5 R_5^{-1} B_5^T K_5\big) A_c = A \end{aligned} \tag{13}$$

Let us formalize the above results in the form of a proposition.

Proposition 2 *Let us assume that the system of* (13) *has a solution*: $K_5 \geq 0$, $K_4 \geq 0$, $K_3 \geq 0$, $K_2 \geq 0$, $K_1 \geq 0$, $K_0 \geq 0$, $K \geq 0$, A_c *that satisfy* (14) *and that* $I + W$ *is invertible. Then the* (*linear*) *feedback Stackelberg solution of the problem* (1)–(3) *is* $u = L x_k$ *for the University and* $u_i = L_i x_k$, $i = 1, 2, \ldots, 5$ *for the Students*, *where* (12) *and* (13) *give the* L, K, L_i, K_i, $i = 0, 1, 2, \ldots, 5$. *The optimal cost of the University is* $J^* = \frac{1}{2} x_0 K x_0$ *and for the Student who starts at year k is* $J_s^*[k, k+5] = \frac{1}{2} x_k K_0 x_k$.

For having asymptotic stability and thus finite University it suffices:

$$\big|\text{eigenvalues}(A_c)\big| < 1 \tag{14}$$

(If there is no University player, the closed loop matrix does not need to be asymptotically stable, since then the costs of the Students, being calculated during a finite time period, are finite). The system (13) is a system of coupled nonlinear equations similar to the one derived for the feedback Nash solution, with the only difference being the term $B R^{-1} B^T (I + W)^{-1} K$ in the formula (13) for calculating A_c instead of the $B R^{-1} B^T K$ term in (8b) for calculating A_c for the Nash case.

Notice that the matrix on the left-hand side of (13) is:

$$
\begin{aligned}
&I + BR^{-1}B^T(I+W)^{-1}K + B_1R_1^{-1}B_1^TK_1 + B_2R_2^{-1}B_2^TK_2 + B_3R_3^{-1}B_3^TK_3 \\
&\quad + B_4R_4^{-1}B_4^TK_4 + B_5R_5^{-1}B_5^TK_5 \\
&= I + W^T + BR^{-1}B^T(I+W)^{-1}K \\
&= \big(I+W^T\big)\big(I + \big(I+W^T\big)^{-1}BR^{-1}B^T(I+W)^{-1}K\big)
\end{aligned}
$$

and with

$$
X = \big(I+W^T\big)^{-1}BR^{-1}B^T(I+W)^{-1}, \qquad Y = K
$$

We have that $I + XY$ is invertible since both X, Y are symmetric and positive semidefinite.

5 Sufficient Conditions for the Existence of Nash Solutions

We will derive some sufficient conditions for the existence of solutions to Eqs. (8a), (8b) and (9). The reason for this derivation is to demonstrate that there are classes of problems for which Nash solutions exist, and thus the problem at hand and the derived solutions are not vacuous. These conditions are based on the contraction mapping and are restrictive. They do not deprive the least from the need to study in full the geometric structure these equations. Less stringent conditions could be derived by using Brower's fixed-point theorem in a way similar to the one employed in [18]. For insights into methodologies for studying such types of coupled matrix equations appearing in games, see [1].

The system of equations that we want to be solvable is the following:

$$
\begin{aligned}
&K = Q + A_c^T\big(K + KBR^{-1}B^TK\big)A_c \\
&K_5 = Q_6 \\
&K_i = Q_{i+1} + A_c^T\big(K_{i+1} + K_{i+1}B_{i+1}R_{i+1}^{-1}B_{i+1}^TK_{i+1}\big)A_c, \quad i = 0, 1, \ldots, 4 \\
&A = \big(I + BR^{-1}B^TK + B_1R_1^{-1}B_1^TK_1 + B_2R_2^{-1}B_2^TK_2 + B_3R_3^{-1}B_3^TK_3 + B_4R_4^{-1}B_4^TK_4 \\
&\qquad + B_5R_5^{-1}B_5^TK_5\big)A_c
\end{aligned}
$$

In what follows, we use the usual L_2 norm of matrices. Let

$$
\begin{aligned}
&s \geq \max\big(\|BR^{-1}B^T\|, \|B_1R_1^{-1}B_1^T\|, \|B_2R_2^{-1}B_2^T\|, \|B_3R_3^{-1}B_3^T\|, \|B_4R_4^{-1}B_4^T\|, \\
&\qquad\qquad \|B_5R_5^{-1}B_5^T\|\big) \\
&q \geq \max\big(\|Q\|, \|Q_1\|, \|Q_2\|, \|Q_3\|, \|Q_4\|, \|Q_5\|\big) \\
&w \geq \max\big(\big\|BR^{-1}B^TQ\big\|, \big\|B_1R_1^{-1}B_1^TQ_1\big\|, \big\|B_2R_2^{-1}B_2^TQ_2\big\|, \big\|B_3R_3^{-1}B_3^TQ_3\big\|, \\
&\qquad\qquad \big\|B_4R_4^{-1}B_4^TQ_4\big\|, \big\|B_5R_5^{-1}B_5^TQ_5\big\|\big)
\end{aligned}
$$

And let us also consider that the K, K_i are sought in some neighborhood $B(Q, Q_1, Q_2, Q_3, Q_4, Q_5; \delta)$ around Q, Q_i, respectively:

$$
\begin{aligned}
&B(Q, Q_1, Q_2, Q_3, Q_4, Q_5; \delta) \\
&\quad = \big\{(K, K_1, K_2, K_3, K_4, K_5) : \\
&\qquad \delta \geq \max\big(\|K - Q\|, \|K_1 - Q_2\|, \|K_2 - Q_3\|, \|K_3 - Q_4\|, \|K_4 - Q_5\|, \|K_5 - Q_6\|\big)\big\}
\end{aligned}
$$

It holds

$$\|K\| \leq \delta + q, \qquad \|K_i\| \leq \delta + q, \quad i = 1, 2, \ldots, 5$$

If

$$6(\delta + q)s < 1,$$

then the matrix

$$\big(I + BR^{-1}B^T K + B_1 R_1^{-1} B_1^T K_1 + B_2 R_2^{-1} B_2^T K_2 + B_3 R_3^{-1} B_3^T K_3 + B_4 R_4^{-1} B_4^T K_4 + B_5 R_5^{-1} B_5^T K_5\big)$$

is invertible and

$$\big\|\big(I + BR^{-1}B^T K + B_1 R_1^{-1} B_1^T K_1 + B_2 R_2^{-1} B_2^T K_2 + B_3 R_3^{-1} B_3^T K_3 + B_4 R_4^{-1} B_4^T K_4 + B_5 R_5^{-1} B_5^T K_5\big)^{-1}\big\| < 1$$

That this inequality holds is an immediate consequence of standard results, see p. 187 in [24]. Clearly, then

$$A_c = \big(I + BR^{-1}B^T K + B_1 R_1^{-1} B_1^T K_1 + B_2 R_2^{-1} B_2^T K_2 + B_3 R_3^{-1} B_3^T K_3 + B_4 R_4^{-1} B_4^T K_4 + B_5 R_5^{-1} B_5^T K_5\big)^{-1} A$$

and

$$\|A_c\| \leq \big\|\big(I + BR^{-1}B^T K + B_1 R_1^{-1} B_1^T K_1 + B_2 R_2^{-1} B_2^T K_2 + B_3 R_3^{-1} B_3^T K_3 + B_4 R_4^{-1} B_4^T K_4 + B_5 R_5^{-1} B_5^T K_5\big)^{-1} A\big\| \leq \|A\|$$

For the first equation of (8a), (8b) for K and the four equations for K_1, K_2, K_3, K_4, K_5 to be contractions it suffices:

$$\|A_c\|^2(\delta + 2s\delta) < \delta + 2s\delta < 1$$

Thus, if

$$6(k + q)s < 1, \qquad \delta + 2s\delta < 1, \qquad \|A\| < 1$$

or

$$\delta < \min\left(\frac{1}{1+2s}, \frac{1}{6s} - q\right), \quad \text{and} \quad \|A\| < 1$$

We will have a contraction. Clearly, we need: $6qs < 1$. Let us formalize this discussion in the form of a proposition:

Proposition 3 *Let*

$$s \geq \max\big(\|BR^{-1}B^T\|, \|B_1 R_1^{-1} B_1^T\|, \|B_2 R_2^{-1} B_2^T\|, \|B_3 R_3^{-1} B_3^T\|, \|B_4 R_4^{-1} B_4^T\|, \|B_5 R_5^{-1} B_5^T\|\big)$$

$$q \geq \max\big(\|Q\|, \|Q_1\|, \|Q_2\|, \|Q_3\|, \|Q_4\|, \|Q_5\|\big)$$

$$6qs < 1$$
$$\delta < \min\left(\frac{1}{1+2s}, \frac{1}{6s} - q\right), \quad \textit{and} \quad \|A\| < 1$$

Then in the neighborhood $B(Q, Q_1, Q_2, Q_3, Q_4, Q_5; \delta)$ *of* $K, K_1, K_2, K_3, K_4, K_5$:

$$\big\{(K, K_1, K_2, K_3, K_4, K_5) : \max\big(\|K - Q\|, \|K_1 - Q_2\|, \|K_2 - Q_3\|, \|K_3 - Q_4\|, \|K_4 - Q_5\|, \|K_5 - Q_6\|\big) < \delta\big\}$$

the mapping:

$$\bar{K} = Q + A_c^T\big(K + KBR^{-1}B^TK\big)A_c$$
$$\bar{K}_i = Q_{i+1} + A_c^T\big(K_{i+1} + K_{i+1}B_{i+1}R_{i+1}^{-1}B_{i+1}^TK_{i+1}\big)A_c, \quad i = 1, 2, \ldots, 4$$

where

$$A_c = \big(I + BR^{-1}B^TK + B_1R_1^{-1}B_1^TK_1 + B_2R_2^{-1}B_2^TK_2 + B_3R_3^{-1}B_3^TK_3 + B_4R_4^{-1}B_4^TK_4 + B_5R_5^{-1}B_5^TQ_6\big)^{-1}A$$

is a contraction and it has a solution in the above mentioned neighborhood, and the resulting A_c *is asymptotically stable.*

The meaning of this sufficient condition for existence is that if $\|A\| < 1$ and the $\|BR^{-1}B^T\|$, $\|B_iR_i^{-1}B_i^T\|$, $\|Q\|$, $\|Q_i\|$, $i = 1, 2, \ldots, 5$ are sufficiently small, then a Nash equilibrium in linear Feedback strategies exists. This does not exclude the possibility of having nonuniqueness where other solutions may and lay outside the aforementioned neighborhood.

Notice that similar stringent sufficient conditions can be derived in an identical manner for the feedback Stackelberg strategy of (13).

6 Continuous Time Analogues

6.1 Continuous Time Analogue for the Feedback Nash Solution

We start by providing a formulation for the continuous time analogue and then we present and justify carefully the linear feedback Nash equilibrium. It leads to an interesting form of Riccati type equations; see (17), (18), which is reminiscent of similar equations in delay or distributed time systems. These equations are not of the usual generalized Riccati ordinary differential equations but of the integrodifferential type, because they contain delay and noncausal terms. In order to formulate the continuous time analogue, we have to specify the times the successive Student players enter the game. Because the time is continuous, the separation in the time entry of two successive student players becomes negligible and we have to define the continuous time version as the limit of a discrete time version. To do that, we start with a continuous time model where the Students duration of studies is a fixed length, say $\bar{T}$ which we cut into (five for example) pieces representing the length of a year's study. Thus a Student may enter at time $45T$ and be a first year student during the period $[45T, 46T]$, a second year student during the period $[46T, 47T]$, a third year student during the period $[47T, 48T]$, a fourth year student during the period $[48T, 49T]$, and a fifth year student during the period $[49T, 50T]$. The whole period of studies is $5T = \bar{T}$. We will consider this model, and then we will take the limit by considering more periods of study, say N instead of 5 but with total length $NT = \bar{T}$ where the $\bar{T}$ is fixed. (Essentially, we allow the total period of study to be finite, i.e., $\bar{T}$ and we divide this $\bar{T}$ into a large number N of periods each one of length $\frac{\bar{T}}{N}$. We then let N to go to infinity and $\frac{\bar{T}}{N}$ to go to zero. We can think of $\frac{\bar{T}}{N}$ as the length of a "one year study" with a total of N such years.) The University player has a permanent presence and no special discretization procedure is needed for him. The University cost is

$$J = \frac{1}{2}\int_0^{\infty} \left(x^T(t)Q_0x(t) + u^T(t)R_0u(t)\right) dt$$

The cost of the Student who enters at the beginning of the year kT and his studies has duration of $5T$:

$$\begin{aligned} 2J^S[kT, kT+5T] = x^T(t_f)Q_f x(t_f) + \int_{kT}^{kT+T} \left(x^T Q_1 x + u_1^T R_1 u_1\right) dt \\ + \int_{kT+T}^{kT+2T} \left(x^T Q_2 x + u_2^T R_2 u_2\right) dt + \int_{kT+2T}^{kT+3T} \left(x^T Q_3 x + u_3^T R_3 u_3\right) dt \\ + \int_{kT=3T}^{kT+4T} \left(x^T Q_4 x + u_4^T R_4 u_4\right) dt + \int_{kT+4T}^{kT+5T} \left(x^T Q_5 x + u_4^T R_5 u_5\right) dt \end{aligned}$$

A, B_i, $Q_i = Q_i^T \geq 0$, $R_i = R_i^T > 0$ are constant matrices. The student who enters at the beginning of the year kT and his studies have a duration of $5T$, sees the state equation below and acts: as first year Student with control $u_1(t)$, as second year Student with control $u_2(t)$, as third year Student with control $u_3(t)$, as fourth year Student with control $u_4(t)$, and as fifth year Student with control $u_5(t)$, i.e.,

$$\begin{aligned} \frac{dx}{dt} &= Ax(t) + B_0u_0(t) + B_1u_1(t) + B_2u_2(t) + B_3u_3(t) + B_4u_4(t) + B_5u_5(t) \\ & t \in [kT, kT+T] \\ \frac{dx}{dt} &= Ax(t) + B_0u_0(t) + B_1u_1(t) + B_2u_2(t) + B_3u_3(t) + B_4u_4(t) + B_5u_5(t) \\ & t \in [kT+T, kT+2T] \\ \frac{dx}{dt} &= Ax(t) + B_0u_0(t) + B_1u_1(t) + B_2u_2(t) + B_3u_3(t) + B_4u_4(t) + B_5u_5(t) \\ & t \in [kT+2T, kT+3T] \\ \frac{dx}{dt} &= Ax(t) + B_0u_0(t) + B_1u_1(t) + B_2u_2(t) + B_3u_3(t) + B_4u_4(t) + B_5u_5(t) \\ & t \in [kT+3T, kT+4T] \\ \frac{dx}{dt} &= Ax(t) + B_0u_0(t) + B_1u_1(t) + B_2u_2(t) + B_3u_3(t) + B_4u_4(t) + B_5u_5(t) \\ & t \in [kT+4T, kT+5T] \end{aligned}$$

If we solve for the linear feedback Nash equilibria, the control of the University is

$$u_0(t) = L_0(t)x(t), \quad L_0(t) = -R_0^{-1} B_0^T K_0(t)x(t)$$

and the controls of the Students are:

$$\begin{aligned}
u_1(t) &= L_1(t)x(t), \quad L_1(t) = -R_1^{-1} B_1^T K_1(t)x(t),\ t \in [kT, kT+T],\ k = 0, 1, 2, 3, \dots \\
u_2(t) &= L_2(t)x(t), \quad L_2(t) = -R_2^{-1} B_2^T K_2(t)x(t),\ t \in [kT, kT+T],\ k = 0, 1, 2, 3, \dots \\
u_3(t) &= L_3(t)x(t), \quad L_3(t) = -R_3^T B_3^T K_3(t)x(t),\ t \in [kT, kT+T],\ k = 0, 1, 2, 3, \dots \\
u_4(t) &= L_4(t)x(t), \quad L_4(t) = -R_4^T B_4^T K_4(t)x(t),\ t \in [kT, kT+T],\ k = 0, 1, 2, 3, \dots \\
u_5(t) &= L_5(t)x(t), \quad L_5(t) = -R_5^T B_5^T K_5(t)x(t),\ t \in [kT, kT+T],\ k = 0, 1, 2, 3, \dots
\end{aligned}$$

The gains are calculated as follows:

$$\begin{aligned}
-\frac{dK_0}{dt} &= K_0\left(A + \sum_{i=1}^{5} L_i\right) + \left(A + \sum_{i=1}^{5} L_i\right)^T K_0 + Q_0 - K_0 B_0 R_0^{-1} B_0^T K_0 \\
& t \in [0, T],\ K_0(0) = K_0(T) \\
-\frac{dK_1}{dt} &= K_1\left(A + B_0 L_0 + \sum_{j=1, j\neq 1}^{5} B_j L_j\right) + \left(A + B_0 L_0 + \sum_{j=1, j\neq 1}^{5} B_j L_j\right)^T K_1 + Q_1 \\
&\quad - K_1 B_1 R_1^{-1} B_1^T K_1, \quad t \in [0, T], K_1(T) = K_2(0) \\
-\frac{dK_2}{dt} &= K_2\left(A + B_0 L_0 + \sum_{j=1, j\neq 2}^{5} B_j L_j\right) + \left(A + B_0 L_0 + \sum_{j=1, j\neq 2}^{5} B_j L_j\right)^T K_2 + Q_2 \\
&\quad - K_2 B_2 R_2^{-1} B_2^T K_2, \quad t \in [0, T], K_2(T) = K_3(0) \\
-\frac{dK_3}{dt} &= K_3\left(A + B_0 L_0 + \sum_{j=1, j\neq 3}^{5} B_j L_j\right) + \left(A + B_0 L_0 + \sum_{j=1, j\neq 3}^{5} B_j L_j\right)^T K_3 + Q_3 \\
&\quad - K_3 B_3 R_3^{-1} B_3^T K_3, \quad t \in [0, T], K_3(T) = K_4(0) \\
-\frac{dK_4}{dt} &= K_4\left(A + B_0 L_0 + \sum_{j=1, j\neq 4}^{5} B_j L_j\right) + \left(A + B_0 L_0 + \sum_{j=1, j\neq 4}^{5} B_j L_j\right)^T K_4 + Q_4 \\
&\quad - K_4 B_4 R_4^{-1} B_4^T K_4, \quad t \in [0, T], K_4(T) = K_5(0) \\
-\frac{dK_5}{dt} &= K_5\left(A + B_0 L_0 + \sum_{j=1, j\neq 5}^{5} B_j L_j\right) + \left(A + B_0 L_0 + \sum_{j=1, j\neq 5}^{5} B_j L_j\right)^T K_5 + Q_5 \\
&\quad - K_5 B_5 R_5^{-1} B_5^T K_5, \quad t \in [0, T], K_5(T) = Q_f
\end{aligned} \tag{15}$$

Notice that this system (15) is a concatenated boundary value problem. We can rewrite Eqs. (15) as:

$$
\begin{aligned}
-\frac{dK_0}{dt} &= K_0\left(A-\sum_{i=1}^{5}B_iR_i^{-1}B_i^TK_i\right)+\left(A-\sum_{i=1}^{5}B_iR_i^{-1}B_i^TK_i\right)^TK_0+Q_0\\
&\quad -K_0B_0R_0^{-1}B_0^TK_0,\quad t\in[0,T],\ K_0(0)=K_0(T)\\
-\frac{dK_i}{dt} &= K_i\left(A-B_0R_0^{-1}B_0^TK_0-\sum_{i=1}^{5}B_iR_i^{-1}B_i^TK_i\right)\\
&\quad +\left(A-B_0R_0^{-1}B_0^TK_0-\sum_{i=1}^{5}B_iR_i^{-1}B_i^TK_i\right)^TK_i+Q_i+K_iB_iR_i^{-1}B_i^TK_i\\
&t\in[0,T],\ K_i(T)=K_{i+1}(0),\ i=1,2,3,4,5,\ K_5(T)=Q_f
\end{aligned}
\tag{16}
$$

Let us now consider a limiting case where the duration of studies $5T=\bar{T}$ remains fixed but the years of study are not 5 but N, so that $NT=\bar{T}$. It is as if the academic year lasts $\frac{\bar{T}}{N}$ and we have N such years of study. Equations (15)–(16) become:

$$
\begin{aligned}
-\frac{dK_0}{dt} &= K_0\left(A-\sum_{i=1}^{N}B_iR_i^{-1}B_i^TK_i\right)+\left(A-\sum_{i=1}^{N}B_iR_i^{-1}B_i^TK_i\right)^T\\
&\quad \times K_0+Q_0-K_0B_0R_0^{-1}B_0^TK_0,\quad t\in\left[0,\frac{\bar{T}}{N}\right],\ K_0(0)=K_0\left(\frac{\bar{T}}{N}\right)\\
-\frac{dK_i}{dt} &= K_i\left(A-B_0R_0^{-1}B_0^TK_0-\sum_{i=1}^{N}B_iR_i^{-1}B_i^TK_i\right)\\
&\quad +\left(A-B_0R_0^{-1}B_0^TK_0-\sum_{i=1}^{N}B_iR_i^{-1}B_i^TK_i\right)^TK_i+Q_i+K_iB_iR_i^{-1}B_i^TK_i,\\
&t\in\left[0,\frac{\bar{T}}{N}\right],\ K_i\left(\frac{\bar{T}}{N}\right)=K_{i+1}(0),\ i=1,2,3,\ldots,N-1,\ K_N\left(\frac{\bar{T}}{N}\right)=Q_f
\end{aligned}
$$

Let us introduce the following notation:

$$
\begin{aligned}
&K_i=K(i\bar{s}),\quad \bar{s}=\frac{\bar{T}}{N},\ i=1,2,3,\ldots,N\\
&B_i=\bar{s}B(i\bar{s}),\quad i=1,2,3,\ldots,N\\
&Q_i=Q\left(i\frac{\bar{T}}{N}\right),\quad i=1,2,3,\ldots,N\\
&R_i(t)=\bar{s}R(i\bar{s}),\quad i=1,2,3,\ldots,N
\end{aligned}
$$

And we rewrite (16) as:

$$
\begin{aligned}
-\frac{dK(t)}{dt} &= K(t)\left(A-B_0R_0^{-1}B_0^TK_0-\sum_{i=1}^{N}\bar{s}B(i\bar{s})R^{-1}(i\bar{s})B^T(i\bar{s})K(i\bar{s})\right)\\
&\quad +\left(A-B_0R_0^{-1}B_0^TK_0-\sum_{i=1}^{N}\bar{s}B(i\bar{s})R^{-1}(i\bar{s})B^T(i\bar{s})K(i\bar{s})\right)^TK(t)
\end{aligned}
$$

$$
\begin{aligned}
&+ Q(t) + K(t)\bar{s}B(is)R^{-1}(i\bar{s})B^T(is)K(t)\\
&t \in \left[i\frac{\bar{T}}{N}, (i+1)\frac{\bar{T}}{N}\right]\\
&K\left((i+1)\frac{\bar{T}}{N}, i\bar{s}\right) = K\left(i\frac{\bar{T}}{N}, (i+1)\bar{s}\right), K\left(N\frac{\bar{T}}{N}\right) = Q_f,\\
&i = 1, 2, 3, \ldots, N-1
\end{aligned}
$$

Assuming continuity and differentiability of $K(t)$ in its arguments and taking the limits as $N \to \infty$(equivalently: $\frac{\bar{T}}{N} = \bar{s} \to 0$), we see that this equation becomes:

$$
\begin{aligned}
-\frac{dK(t)}{dt} &= K(t)\left(A - B_0 R_0^{-1} B_0^T K_0 - \int_0^{\bar{T}} B(s)R^{-1}(s)B^T(s)K(s)\,ds\right)\\
&\quad + \left(A - B_0 R_0^{-1} B_0^T K_0 - \int_0^{\bar{T}} B(s)R^{-1}(s)B^T(s)K(s)\,ds\right)^T K(t) + Q(t)\\
t &\in [0, \bar{T}],
\end{aligned} \tag{17}
$$

The boundary conditions give:

$$K(\bar{T}) = Q_f$$

In this case, the limiting value of the Student's cost is

$$2J^S[kT, kT+\bar{T}] = x^T(t_f)Q_f x(t_f) + \int_{kT}^{kT+\bar{\bar{T}}} \left(x^T Q(t)x + u^T R(t)u\right) dt$$

and the state equation becomes:

$$
\begin{aligned}
&\frac{dx}{dt} \approx Ax(t) + B_0 u_0(t) + \sum_{i=1}^{N} \bar{s}B(i\bar{s})u(t, i\bar{s})\\
&\to \quad \frac{dx}{dt} = Ax(t) + B_0 u_0(t) + \int_0^{\bar{T}} B(s)u(t, s)\,ds
\end{aligned}
$$

The limiting form of the equation for the University's gain becomes an algebraic Riccati equation:

$$
\begin{aligned}
0 &= K_0\left(A - \int_0^{\bar{T}} B(s)R^{-1}(s)B^T(s)K(s)\,ds\right)\\
&\quad + \left(A - \int_0^{\bar{T}} B(s)R^{-1}(s)B^T(s)K(t,s)\,ds\right)^T K_0 + Q_0 - K_0 B R^{-1} B^T K_0
\end{aligned} \tag{18}
$$

The control of the University is

$$u(t) = -R_0^{-1} B_0^T K_0 x(t) \tag{19}$$

and the control of the Student who starts his studies at time t and completes them at time $t + \bar{T}$ is:

$$u(t+s) = -R(s)^{-1} B^T(s)K(s)x(t+s), \quad s \in [0, \bar{T}] \tag{20}$$

To sum-up, Eqs. (17), (18), (19) have to be solved. The resulting closed loop matrix:

$$A_c = A - B_0 R_0^{-1} B_0^T K_0 - \int_0^{\bar{T}} B(s) R^{-1}(s) B^T(s) K(s)\, ds \tag{21}$$

has to be asymptotically stable.

Let us now state formally the continuous time analogue of the game of the linear Nash feedback strategies solution.

Proposition 4 *Consider the state equation*

$$\frac{dx}{dt} = Ax(t) + B_0 u_0(t) + \int_0^{\bar{T}} B(s) u(t,s)\, ds \tag{22}$$

and the costs

$$\begin{aligned} J &= \frac{1}{2}\int_0^{\infty} \left(x^T(t) Q_0 x(t) + u^T(t) R_0 u(t)\right) dt \\ J^S[t, t+\bar{T}] &= \frac{1}{2} x^T(t_f) Q_f x(t_f) + \frac{1}{2}\int_t^{t+\bar{}} \left(x^T Q_1 x + u^T R_1 u\right) dt \end{aligned} \tag{23}$$

for the University and the Students, respectively.

Consider also the system evolution equations:

$$\begin{aligned}
&\frac{dx}{dt} = Ax(t) + B_0u_0(t) + B_1u_1(t) + B_2u_2(t) + \cdots + B_{N-1}u_{N-1}(t) + B_N u_N(t) \\
&\quad t \in [kT, kT+T] \\
&\frac{dx}{dt} = Ax(t) + B_0u_0(t) + B_1u_1(t) + B_2u_2(t) + B_3u_3(t) + \cdots + B_{N-1}u_{N-1}(t) + B_N u_N(t) \\
&\quad t \in [kT+T, kT+2T] \\
&\frac{dx}{dt} = Ax(t) + B_0u_0(t) + B_1u_1(t) + B_2u_2(t) + B_3u_3(t) + B_4u_4(t) + \cdots + B_N u_N(t) \\
&\quad t \in [kT+2T, kT+3T] \\
&\vdots \\
&\frac{dx}{dt} = Ax(t) + B_0u_0(t) + B_1u_1(t) + B_2u_2(t) + \cdots + B_{N-2}u_{N-2}(t) \\
&\qquad + B_{N-1}u_{N-1}(t) + B_N u_N(t) \\
&\quad t \in \left[kT+NT, kT+(N-1)T\right] \\
&\frac{dx}{dt} = Ax(t) + B_0u_0(t) + B_1u_1(t) + B_2u_2(t) + \cdots + B_{N-1}u_{N-1}(t) + B_N u_N(t) \\
&\quad t \in \left[kT+(N-1)T, kT+NT\right]
\end{aligned} \tag{24}$$

and costs for the University:

$$J = \frac{1}{2}\int_0^{\infty} \left(x^T(t) Q_0 x(t) + u^T(t) R_0 u(t)\right) dt$$

and for the Student:

$$2J^S_{kT}[kT, kT + NT] = x^T(t_f)Q_f x(t_f) + \int_{kT}^{kT+T} \left(x^T Q_1 x + u_1^T R_1 u_1\right) dt$$
$$+ \int_{kT+T}^{kT+2T} \left(x^T Q_2 x + u_2^T R_2 u_2\right) dt$$
$$+ \int_{kT+2T}^{kT+3T} \left(x^T Q_3 x + u_3^T R_3 u_3\right) dt + \cdots$$
$$+ \int_{kT+(N-1)T}^{kT+NT} \left(x^T Q_5 x + u_4^T R_5 u_5\right) dt \tag{25}$$

We define the linear feedback solution of the game described by (22)–(23) *as the limit of the linear feedback solution of the game described by* (24)–(25) *as* $N \to \infty$, $T = \frac{\bar{T}}{N}$ *where we take* $[kT, NT] = [t, t + \bar{T}]$, *assuming of course that the limiting solutions exist and the resulting closed loop matrix*:

$$A_c = A - B_0 R_0^{-1} B_0^T K_0 - \int_0^{\bar{T}} B(s) R^{-1}(s) B^T(s) K(s)\, ds$$

is asymptotically stable. (*This a sufficient*, *not necessary condition for the finiteness of the leader's cost*.)

This limiting solutions are given by

$$u(t) = -R_0^{-1} B_0^T K_0 x(t)$$

for the University and

$$u(t+s) = -R(s)^{-1} B^T(s) K(s) x(t+s), \quad s \in [0, \bar{T}]$$

for the Student who starts his studies at time t *and completes them at time* $t + \bar{T}$. *The* K_0, K *are given by the equations*:

$$0 = K_0\left(A - \int_0^{\bar{T}} B(s) R^{-1}(s) B^T(s) K(s)\, ds\right)$$
$$+ \left(A - \int_0^{\bar{T}} B(s) R^{-1}(s) B^T(s) K(t,s)\, ds\right)^T K_0 + Q_0 - K_0 B R^{-1} B^T K_0$$
$$-\frac{dK(t)}{dt} = K(t)\left(A - B_0 R_0^{-1} B_0^T K_0 - \int_0^{\bar{T}} B(s) R^{-1}(s) B^T(s) K(s)\, ds\right)$$
$$+ \left(A - B_0 R_0^{-1} B_0^T K_0 - \int_0^{\bar{T}} B(s) R^{-1}(s) B^T(s) K(s)\, ds\right)^T K(t) + Q(t),$$
$$t \in [0, \bar{T}],$$

with boundary condition:

$$K(\bar{T}) = Q_f$$

The resulting closed loop matrix is

$$A_c = A - B_0 R_0^{-1} B_0^T K_0 - \int_0^{\bar{T}} B(s) R^{-1}(s) B^T(s) K(s)\, ds$$

Notice that we avoided giving a definition of the solution directly for the continuous time case, but we defined the solution as the limit of an appropriate discretization. This is a device that has been used on several occasions for control and game problems; see [5, 9, 10].

6.1.1 The Time Invariant Case

The case where the matrices $B(s)$, $R(s)$ involved are constant can be further simplified as follows. Let

$$\begin{aligned}
&A_c = A - B_0 R_0^{-1} B_0^T K_0 - B_1 R^{-1} B_1^T L \\
&L = \int_0^{\bar{T}} K(t)\, dt \\
&-\frac{dK(t)}{dt} = K(t) A_c + A_c^T K(t) + Q, \quad t \in [0, \bar{T}] \\
&-\big(K(T) - K(0)\big) = L A_c + A_c^T L + QT \\
&K(0) = L A_c + A_c^T L + QT + Q_f \\
&K(t) = e^{-A_c^T (t-T)} Q_f e^{-A_c (t-T)} + \int_T^t e^{-A_c^T (t-\tau)} (-Q) e^{-A_c (t-\tau)}\, d\tau \\
&K(0) = e^{A_c^T T} Q_f e^{A_c T} + \int_0^T e^{A_c^T \tau} Q e^{A_c \tau}\, d\tau \\
&L A_c + A_c^T L + QT + Q_f - e^{A_c^T T} Q_f e^{A_c T} = X \\
&A_c^T X + X A_c = e^{A_c^T T} Q e^{A_c T} - Q \\
&A_c^T \big(L A_c + A_c^T L + QT + Q_f - e^{A_c^T T} Q_f e^{A_c T}\big) \\
&\quad + \big(L A_c + A_c^T L + QT + Q_f - e^{A_c^T T} Q_f e^{A_c T}\big) A_c = e^{A_c^T T} Q e^{A_c T} - Q \\
&A_c^T \big(L A_c + A_c^T L\big) + \big(L A_c + A_c^T L\big) A_c + T\big(A_c^T Q + Q A_c\big) + A_c^T Q_f + Q_f A_c + Q \\
&\quad = e^{A_c^T T} \big(Q + Q_f A_c + A_c^T Q_f\big) e^{A_c T} \\
&L = \int_0^{\bar{T}} K(t)\, dt = \int_0^{\bar{T}} \left(e^{-A_c^T (t-T)} Q_f e^{-A_c (t-T)} + \int_T^t e^{-A_c^T (t-\tau)} (-Q) e^{-A_c (t-\tau)}\, d\tau \right) dt
\end{aligned}$$

After some calculations and repeated use of the formula,

$$\int_a^b e^{C^T t} Q e^{C t}\, dt = X, \quad C^T X + X C = e^{C^T b} Q e^{C b} - e^{C^T a} Q e^{C a}$$

which holds if: all eigenvalues $(C) < 0$, or all eigenvalues $(C) > 0$, we obtain the equation:

$$\begin{aligned}&A_c^T\left(A_c^T L+LA_c\right)+\left(A_c^T L+LA_c\right)A_c+\bar{T}\left(A_c^T Q+QA_c\right)+\left(A_c^T Q_f+Q_f A_c+Q\right)\\&\quad=e^{TA_c^T}\left(A_c^T Q_f+Q_f A_c+Q\right)e^{TA_c}\end{aligned}\tag{26}$$

Equation (26) is a transcendental equation for A_c, which together with:

$$\begin{aligned}0&=K_0A_c+A_cK_0+Q_0+K_0B_0R_0^{-1}B_0^TK_0\\A_c&=A-B_0R_0^{-1}B_0^TK_0-B_1R^{-1}B_1^TL\end{aligned}\tag{27}$$

constitute essentially a system of two equations that has to be solved for symmetric positive semidefinite K_0, L. The resulting A_c has to be asymptotically stable if we have a University player whose infinite time cost has to be finite. Notice that Eq. (26) contains third powers of L in the left-hand side, but also exponential terms of L in the right-hand side. Thus, we have a system of transcendental, nonalgebraic equations. This is the price we pay for substituting the Riccati differential equation for $K(t)$, which essentially has a quadratic in $K(t)$ right-hand side, with a stationary equation, (26), for L.

6.2 Continuous Time Analogue for the Feedback Stackelberg Solution

In order to find the continuous time analogue of the feedback Stackelberg solution we can follow a procedure similar to the one employed for the Nash case. The analogue of Proposition 3 can be stated for the feedback Stackelberg solution, which again can be defined as the limit of the appropriate discretization procedure. We will follow a different discretization procedure which starts from the discrete problem (1)–(3) and then takes the limit. In the procedure used for the Nash case we used a mixed type of discretization procedure, where the Students were acting in continuous time for small intervals and the state equation was not discretized. We believe that both procedures have merit as they represent different modeling approaches.

Consider the equations derived for the discrete time feedback Stackelberg:

$$\begin{aligned}&K=Q+A_c^T\left(K+KBR^{-1}B^TK\right)A_c\\&K_5=Q_6\\&K_i=Q_{i+1}+A_c^T\left(K_{i+1}+K_{i+1}B_{i+1}R_{i+1}^{-1}B_{i+1}^TK_{i+1}\right)A_c,\quad i=0,1,\ldots,4\\&W=K_1B_1R_1^{-1}B_1^T+K_2B_2R_2^{-1}B_2^T+K_3B_3R_3^{-1}B_3^T+K_4B_4R_4^{-1}B_4^T+K_5B_5R_5^{-1}B_5^T\\&\big(I+BR^{-1}B^T(I+W)^{-1}K+B_1R_1^{-1}B_1^TK_1+B_2R_2^{-1}B_2^TK_2\\&\quad+B_3R_3^{-1}B_3^TK_3+B_4R_4^{-1}B_4^TK_4+B_5R_5^{-1}B_5^TK_5\big)A_c=A\end{aligned}\tag{28}$$

Let $\bar{A}$, $\bar{A}_c$, $\bar{B}$, $\bar{B}_i$, $\bar{Q}$, $\bar{R}$, $\bar{R}_i$ be the corresponding quantities for the continuous time case. If we consider a discretization of the continuous time problem with time step δ very small positive number, it will be:

$$\begin{aligned}&A\approx I+\delta\bar{A},\qquad A_c\approx I+\delta\bar{A}_c,\qquad B\approx\delta\bar{B},\qquad B_i\approx\delta^2\bar{B}_i,\\&\bar{Q}\approx Q,\qquad Q_i\approx\bar{Q}(i\delta),\qquad R\approx\delta\bar{R},\qquad R_i\approx\delta\bar{R}_i,\qquad K_i\approx\bar{K}(i\delta)\end{aligned}$$

The summations in (13) will become integrals and instead of $i=1,2,\ldots,6$ we will take $i=1,2,\ldots,N$ and $N\delta\approx\bar{T}$ where $\bar{T}$ is the duration of the Student's studies. Also,

$$
\begin{aligned}
(I+W)^{-1} &= \left(I + K_1 B_1 R_1^{-1} B_1^T + K_2 B_2 R_2^{-1} B_2^T + K_3 B_3 R_3^{-1} B_3^T + \cdots + K_N B_N R_N^{-1} B_N^T\right)^{-1} \\
&\approx \left(I + \delta^2 \sum_{i=1}^{N} K_i B_i R_i^{-1} B_i^T\right)^{-1} \approx I - \delta^2 \sum_{i=1}^{N} K_i B_i R_i^{-1} B_i^T \\
&\approx I - \delta \int_0^{\bar{T}} K(t) B(t) R^{-1}(t) B^T(t)\, dt
\end{aligned}
$$

And using it in (13)

$$
\begin{aligned}
&\left(I + \delta B R^{-1} B^T (I+W)^{-1} K + \delta \sum_{i=1}^{N} B_1 R_1^{-1} B_1^T K_1\right) A_c \approx A \\
&\left(I + \delta B R^{-1} B^T \left(I - \int_0^{\bar{T}} K(t) B(t) R^{-1}(t) B^T(t)\, dt\right) K + \delta \sum_{i=1}^{N} B_1 R_1^{-1} B_1^T K_1\right) A_c \approx A \\
&\left(I - \delta B R^{-1} B^T \int_0^{\bar{T}} K(t) B(t) R^{-1}(t) B^T(t)\, dt + \delta B R^{-1} B^T K \right. \\
&\quad \left. + \delta \sum_{i=1}^{N} B_1 R_1^{-1} B_1^T K_1\right) A_c = A \\
&A_c \approx \left(I - \delta B R^{-1} B^T \int_0^{\bar{T}} K(t) B(t) R^{-1}(t) B^T(t)\, dt + \delta B R^{-1} B^T K \right. \\
&\quad \left. + \sum_{i=1}^{N} B_1 R_1^{-1} B_1^T K_1\right)^{-1} A \\
&A_c \approx \left(I + \delta B R^{-1} B^T \int_0^{\bar{T}} K(t) B(t) R^{-1}(t) B^T(t)\, dt - \delta B R^{-1} B^T K \right. \\
&\quad \left. - \delta \int_0^{\bar{T}} B(t) R^{-1}(t) B^T(t) K(t)\, dt\right) A \\
&I + \delta \bar{A}_c \approx \left(I + \delta B R^{-1} B^T \int_0^{\bar{T}} K(t) B(t) R^{-1}(t) B^T(t)\, dt - \delta B R^{-1} B^T K \right. \\
&\quad \left. - \delta \int_0^{\bar{T}} B(t) R^{-1}(t) B^T(t) K(t)\, dt\right) (I + \delta \bar{A}) \\
&I + \delta \bar{A}_c \approx I + \delta B R^{-1} B^T \int_0^{\bar{T}} K(t) B(t) R^{-1}(t) B^T(t)\, dt - \delta B R^{-1} B^T K \\
&\quad - \delta \int_0^{\bar{T}} B(t) R^{-1}(t) B^T(t) K(t)\, dt + \delta \bar{A} + \mathrm{Order}\left(\delta^2\right) \\
&\bar{A}_c \approx B R^{-1} B^T \int_0^{\bar{T}} K(t) B(t) R^{-1}(t) B^T(t)\, dt - B R^{-1} B^T K \\
&\quad - \int_0^{\bar{T}} B(t) R^{-1}(t) B^T(t) K(t)\, dt + \bar{A} + \mathrm{Order}\left(\delta^2\right)
\end{aligned}
$$

Plugging this A_c in (28), we get:

$$
\begin{aligned}
&K \approx \delta Q + (I + \delta A_c)^T \big(K + \delta K B R^{-1} B^T K\big)(I + \delta A_c)\\
&K \approx \delta Q + K + \delta K B R^{-1} B^T K + \delta A_c^T K + \delta K A_c + \text{Order}\big(\delta^2\big)\\
&0 \approx Q + K B R^{-1} B^T K + A_c^T K + K A_c\\
&0 \approx Q + K B R^{-1} B^T K + \Bigg(B R^{-1} B^T \int_0^{\bar{T}} K(t) B(t) R^{-1}(t) B^T(t)\, dt - B R^{-1} B^T K\\
&\quad - \int_0^{\bar{T}} B(t) R^{-1}(t) B^T(t) K(t)\, dt + \bar{A} + \text{Order}\big(\delta^2\big)\Bigg)^T K\\
&\quad + K\Bigg(B R^{-1} B^T \int_0^{\bar{T}} K(t) B(t) R^{-1}(t) B^T(t)\, dt - B R^{-1} B^T K\\
&\quad - \int_0^{\bar{T}} B(t) R^{-1}(t) B^T(t) K(t)\, dt + \bar{A} + \text{Order}\big(\delta^2\big)\Bigg)\\
&0 \approx Q - K B R^{-1} B^T K + \bar{A}^T K + K \bar{A}\\
&\quad + K\Bigg(B R^{-1} B^T \int_0^{\bar{T}} K(t) B(t) R^{-1}(t) B^T(t)\, dt - \int_0^{\bar{T}} B(t) R^{-1}(t) B^T(t) K(t)\, dt\Bigg)\\
&\quad + \Bigg(B R^{-1} B^T \int_0^{\bar{T}} K(t) B(t) R^{-1}(t) B^T(t)\, dt - \int_0^{\bar{T}} B(t) R^{-1}(t) B^T(t) K(t)\, dt\Bigg)^T K
\end{aligned}
$$

This is the steady state Riccati equation for the University's gain.

Plugging A_c in the other equations of (24) we get:

$$
\begin{aligned}
&K_3 = Q_4 + A_c^T \big(K_4 + K_4 B_4 R_4^{-1} B_4^T K_4\big) A_c\\
&K_3 \approx \delta Q_4 + (I + \delta \bar{A}_c)^T \big(K_4 + \delta K_4 B_4 R_4^{-1} B_4^T K_4\big)(I + \delta \bar{A}_c)\\
&K_3 \approx \delta Q_4 + K_4 + \delta K_4 B_4 R_4^{-1} B_4^T K_4 + \delta \bar{A}_c^T \big(K_4 + \delta K_4 B_4 R_4^{-1} B_4^T K_4\big)\\
&\quad + \big(K_4 + \delta K_4 B_4 R_4^{-1} B_4^T K_4\big)\delta \bar{A}_c\\
&K_3 \approx \delta Q_4 + K_4 + \delta K_4 B_4 R_4^{-1} B_4^T K_4 + \delta \bar{A}_c^T K_4 + K_4 \delta \bar{A}_c + \text{Order}\big(\delta^2\big)\\
&K_3 \approx \delta Q_4 + K_4 + \delta K_4 B_4 R_4^{-1} B_4^T K_4\\
&\quad + \delta\Bigg(B R^{-1} B^T \int_0^{\bar{T}} K(t) B(t) R^{-1}(t) B^T(t)\, dt - B R^{-1} B^T K\\
&\quad - \int_0^{\bar{T}} B(t) R^{-1}(t) B^T(t) K(t)\, dt + \bar{A}\Bigg)^T K_4\\
&\quad + \delta K_4\Bigg(B R^{-1} B^T \int_0^{\bar{T}} K(t) B(t) R^{-1}(t) B^T(t)\, dt - B R^{-1} B^T K\\
&\quad - \int_0^{\bar{T}} B(t) R^{-1}(t) B^T(t) K(t)\, dt + \bar{A}\Bigg) + \text{Order}\big(\delta^2\big)
\end{aligned}
$$

$$
\begin{aligned}
-\frac{dK}{dt} &\approx Q + \bar{A}^T K + K\bar{A} - KBR^{-1}B^T K \\
&\quad + K\left(BR^{-1}B^T \int_0^{\bar{T}} K(t)B(t)R^{-1}(t)B^T(t)\,dt - \int_0^{\bar{T}} B(t)R^{-1}(t)B^T(t)K(t)dt\right) \\
&\quad + \left(BR^{-1}B^T \int_0^{\bar{T}} K(t)B(t)R^{-1}(t)B^T(t)\,dt - \int_0^{\bar{T}} B(t)R^{-1}(t)B^T(t)K(t)\,dt\right)K
\end{aligned}
$$

This is the differential equation on the interval $[0, \bar{T}]$ with final condition $K(\bar{T}) = Q_f$.

Therefore, the continuous time analogue of the feedback Stackelberg strategy for the model (1)–(3) is identical with the feedback Nash strategy. This observation is in agreement with the result of [5, 6], for the linear quadratic game with no cross terms in the cost. It is interesting to note that the same observation holds for the model examined here. In case we allow cross terms where we have the state multiplying the control in the costs, we would expect that the two solutions do not coincide. This is an easy exercise to verify, which nonetheless has its own merit.

7 Remarks on Computational Issues

The computational solution of the equations that give the equilibrium strategies is of the utmost importance, not only for providing the solutions, but also because any theoretical study of their convergence is imminently related with the study of existence and uniqueness or multiplicity of solutions. Some numerical results for the scalar case are reported in [15, 16]. Reference [14] represents the first effort formulating and studying algorithms for the matrix case. In order to solve the system of Eqs. (8a), (8b) and (13) for the discrete time Nash and Stackelberg solutions, respectively, we can proceed in several ways, some of which are presented next.

A straightforward method is to consider them as a system of nonlinear equations and use any of the general methods available, taking care to operate with symmetric positive semidefinite updates. The solutions that yield asymptotically stable closed loop matrices are retained.

Another way is to iterate backwards starting with given final conditions as if we had a finite time duration and finite summation for the costs. We continue the iteration backwards until (and if) we achieve convergence or other interesting behavior such as oscillations arise. This procedure guarantees the symmetric positive semidefinite character of the iterates. Still we may have nonconvergence or convergence to different solutions for different initializing solutions. Let us recall the equations we have to solve, and let us denote them by:

$$
\begin{aligned}
&X_0 = K = Q + A_c^T\left(K + KBR^{-1}B^T K\right)A_c = \Theta_0(X_0, X_1, X_2, X_3, X_4) \\
&K_5 = Q_6 \\
&X_1 = K_4 = Q_5 + A_c^T\left(K_5 + K_5B_5R_5^{-1}B_5^T K_5\right)A_c = \Theta_1(X_0, X_1, X_2, X_3, X_4) \\
&X_2 = K_3 = Q_4 + A_c^T\left(K_4 + K_4B_4R_4^{-1}B_4^T K_4\right)A_c = \Theta_2(X_0, X_1, X_2, X_3, X_4) \\
&X_3 = K_2 = Q_3 + A_c^T\left(K_3 + K_3B_3R_3^{-1}B_3^T K_3\right)A_c = \Theta_3(X_0, X_1, X_2, X_3, X_4) \\
&X_4 = K_1 = Q_2 + A_c^T\left(K_2 + K_2B_2R_2^{-1}B_2^T K_2\right)A_c = \Theta_4(X_0, X_1, X_2, X_3, X_4)
\end{aligned}
$$

$$\begin{aligned}
K_0 &= Q_1 + A_c^T\big(K_1 + K_1 B_1 R_1^{-1} B_1^T K_1\big)A_c \\
A &= \big(I + BR^{-1}B^T X_0 + B_1 R_1^{-1} B_1^T X_4 + B_2 R_2^{-1} B_2^T X_3 \\
&\quad + B_3 R_3^{-1} B_3^T X_2 + B_4 R_4^{-1} B_4^T X_1 + B_5 R_5^{-1} B_5^T Q_6\big)A_c
\end{aligned}$$

So we have to solve the system:

$$X_i = \Theta_i(X_0, X_1, X_2, X_3, X_4), \quad i = 0, 1, \dots, 4$$

We can iterate with any initial positive semidefinite set of initial conditions for the unknowns. The updates of the matrix

$$\begin{aligned}
&\big(I + BR^{-1}B^T X_0 + B_1 R_1^{-1} B_1^T X_4 + B_2 R_2^{-1} B_2^T X_3 + B_3 R_3^{-1} B_3^T X_2 \\
&\quad + B_4 R_4^{-1} B_4^T X_1 + B_5 R_5^{-1} B_5^T Q_6\big)
\end{aligned}$$

are considered to be invertible during the operation. We can also iterate asynchronously as

$$X_i^{k+1} = \Theta_i\big(X_0^k, X_1^k, X_2^k, X_3^k, X_4^k\big), \quad i = 0, 1, \dots, 4$$

which corresponds to updating the cost functions iteratively and backward for all the players. For a limit point to be acceptable, we also need that the asymptotic stability condition (9) be satisfied in the limit.

For the continuous time Nash solution for the case where the matrices involved are time invariant, we have to solve the system:

$$\begin{aligned}
&A_c^T\big(A_c^T L + LA_c\big) + \big(A_c^T L + LA_c\big)A_c + \bar{T}\big(A_c^T Q + QA_c\big) + \big(A_c^T Q_f + Q_f A_c + Q\big) \\
&\quad = e^{TA_c^T}\big(A_c^T Q_f + Q_f A_c + Q\big)e^{TA_c} \\
&0 = K_0 A_c + A_c K_0 + Q_0 + K_0 B_0 R_0^{-1} B_0^T K_0 \\
&A_c = A - B_0 R_0^{-1} B_0^T K_0 - B_1 R^{-1} B_1^T L
\end{aligned}$$

We can iterate as follows: For $K_0^n, L^n, A_c^n = A - B_0 R_0^{-1} B_0^T K_0^n - B_1 R^{-1} B_1^T L^n$ given, the updates are:

$$\begin{aligned}
&\big(A_c^n\big)^T M^{n+1} + M^{n+1} A_c^n = -\big(A_c^n\big)^T (TQ + Q_f) - (TQ + Q_f)A_c^n - Q \\
&\qquad\qquad\qquad\qquad + e^{(TA_c^n)^T}\big(\big(TA_c^n\big)^T Q_f + Q_f A_c^n + Q\big)e^{TA_c^n} \\
&\big(A_c^n\big)^T L^{n+1} + L^{n+1} A_c^n = M^{n+1} \\
&0 = K_0^{n+1} A_c^n + \big(A_c^n\big)^T K_0^{n+1} + Q_0 + K_0^{n+1} B_0 R_0^{-1} B_0^T K_0^{n+1} \\
&A_c^{n+1} = A - B_0 R_0^{-1} B_0^T K_0^{n+1} - B_1 R^{-1} B_1^T L^{n+1}
\end{aligned}$$

The first two equations for M^{n+1}, L^{n+1} constitute a two step procedure for finding a new L^{n+1} given the K^n, L^n. They can be actually viewed as a fixed-point iteration for the solution of Eq. (26). The theoretical study of convergence of this iterative process is an interesting problem.

8 Examples

We present two scalar examples. The first explores the existence and nonuniqueness issue for the feedback Nash solution of the discrete time formulation. The second studies the feedback Nash solution of the continuous time formulation. In the case of absence of the University player, both examine the interplay between the length of the Students horizon and the stability of the closed loop matrix. (These examples pertain to the scalar case and do not constitute examples of use of algorithms for solving the general matrix-Riccati-type equations since the method by which we solve them relies heavily on the scalar character.)

8.1 Discrete Time

It is interesting that nonuniqness of solutions may appear in our problem, as it so often happens in game theory. In order to demonstrate this phenomenon and at the same time gain some insights in the possible reasons for that, and the structure of the problem, we consider the scalar case for the Feedback Nash solution. Let

$$\begin{aligned}
&K_i = k_i, \qquad K = k \\
&B_1 = B_2 = B_3 = B_4 = B_5 = B = R_1 = R_2 = R_3 = R_4 = R_5 = R = 1 \\
&Q_1 = Q_2 = Q_3 = Q_4 = Q_5 = q_s, \qquad Q = q \\
&A_c = a_c, \qquad A = a
\end{aligned}$$

Notice that since all the k_i, k are nonnegative both a_c, a have the same sign. Let $a_c > 0$, $a > 0$ w.l.o.g. Eqs. (8a), (8b) become:

$$\begin{aligned}
k &= q + a_c^2\left(k + k^2\right) \\
k_5 &= q_s \\
k_4 &= q_s + a_c^2\left(k_5 + k_5^2\right) \\
k_3 &= q_s + a_c^2\left(k_4 + k_4^2\right) \\
k_2 &= q_s + a_c^2\left(k_3 + k_3^2\right) \\
k_1 &= q_s + a_c^2\left(k_2 + k_2^2\right) \\
k_0 &= q_s + a_c^2\left(k_1 + k_1^2\right) \\
a &= (1 + k + k_1 + k_2 + k_3 + k_4 + k_5)a_c
\end{aligned}$$

Let

$$F(a_c) = (1 + k + k_1 + k_2 + k_3 + k_4 + k_5)a_c$$

We will try to solve these equations by considering the a_c as an unknown that when found, automatically determines the k_i, k (two choices for k) and this a_c is acceptable if the equation $a = (1 + k + k_1 + k_2 + k_3 + k_4 + k_5)a_c$ is satisfied.

Consider a plot of $F(a_c)$ for $a_c \in [0, 1]$. For each value of a_c, we calculate the k_i recursively using the above formulae. We also calculate the k from the first equation above to get

$$k^+ = \frac{1 - a_c^2 + \sqrt{(1 - a_c^2)^2 - 4qa_c^2}}{2a_c^2}, \qquad k^- = \frac{1 - a_c^2 - \sqrt{(1 - a_c^2)^2 - 4qa_c^2}}{2a_c^2}$$

For the solution to be acceptable, we need that it is positive and real. For the reality, we need:

$$\left(1-a_c^2\right)^2-4qa_c^2\geq 0$$

i.e.,

$$0\leq a_c\leq \frac{1}{\sqrt{q}+\sqrt{q+1}}$$

Notice that for $a_c\to 0$, $k_+\to\infty$, $k_-\to 0$. Thus, doing the plot of $F(a_c)$ for $0\leq a_c\leq \frac{1}{\sqrt{q}+\sqrt{q+1}}$ we have two branches. One starts from zero using the value k^- and rises up to the value of $F(\bar{a}_c)$ that corresponds to $\bar{a}_c=\frac{1}{\sqrt{q}+\sqrt{q+1}}$. For this branch, all of k and k_1, k_2, k_3, k_4, k_5, are increasing as $a_c\to\bar{a}_c$, and thus this branch is increasing. For the other branch that uses k^+, starts from the value $F(\bar{a}_c)$ and for $a_c\to 0$ it goes toward ∞, we have that k is decreasing whereas k_1, k_2, k_3, k_4, k_5, are increasing as $a_c\to\bar{a}_c$. We will show that this branch is not monotonic but may have rising and falling parts giving rise to nonunique solutions. We have definitely some value of $a_c=\bar{a}_c$ for which the value of $F(\bar{a}_c)=a$. We conclude that for any a, q, q_s we have at least one solution for a_c, k, k_1, k_2, k_3, k_4, k_5. It is easy to see that the branch that corresponds to $\varepsilon=-1$ is monotonically rising as a_c moves to the right. The branch that corresponds to $\varepsilon=+1$ that starts from ∞ and goes toward the $F(\bar{a}_c)$ value as a_c moves to the right can have a dive before rising again toward $F(\bar{a}_c)$, and this can happen if $q_s\gg 0$ as the following argument shows:

(In what follows, by $P_1(a_c^2)$, $P_2(a_c^2)$, we denote polynomial functions of their arguments whose coefficients are easily calculated although not needed explicitly in our discussion.)

$$k_\varepsilon=\frac{1-a_c^2+\sqrt{(1-a_c^2)^2-4qa_c^2}}{2a_c^2}$$

And

$$\begin{aligned}
&k_1+k_2+k_3+k_4+k_5=5q_s+a_c^2P_1\left(a_c^2\right)\\
&F(a_c)=\left[1+5q_s+a_c^2P_1\left(a_c^2\right)+\frac{1-a_c^2+\sqrt{(1-a_c^2)^2-4qa_c^2}}{2a_c^2}\right]a_c\\
&\qquad=F_1(a_c)+F_2(a_c)\\
&F_1(a_c)=\left(\frac{1}{2}+5q_s+a_c^2P_1\left(a_c^2\right)\right)a_c\\
&F_2(a_c)=\frac{1}{2a_c}+\sqrt{\left(\frac{1-a_c^2}{2a_c}\right)^2-q}
\end{aligned}$$

Also:

$$\begin{aligned}
&F_2(\bar{a}_c)=\frac{1-\bar{a}_c^2}{2\bar{a}_c}=\frac{1}{2}\left(\sqrt{q}+\sqrt{q+1}-\frac{1}{\sqrt{q}+\sqrt{q+1}}\right)\\
&\frac{d}{da_c}F_2(a_c)=\frac{1}{2a_c}+\sqrt{\left(\frac{1-a_c^2}{2a_c}\right)^2-q}
\end{aligned}$$

$$= -\frac{1}{2a_c^2} + \frac{1}{2}\frac{2(\frac{1-a_c^2}{2a_c})(-\frac{1}{2a_c^2}-\frac{1}{2})}{\sqrt{(\frac{1-a_c^2}{2a_c})^2-q}} = -\left(\frac{1}{2a_c^2} + \frac{(1-a_c^4)}{4a_c^3\sqrt{(\frac{1-a_c^2}{2a_c})^2-q}}\right) < 0$$

$$\frac{d}{da_c}F_1(a_c) = \frac{1}{2} + 5q_s + a_c^2 P_2(a_c^2) > 0$$

It is easy to see that if we take an $\hat{a}_c < \bar{a}_c$, we will have $\frac{d}{da_c}F_2(\hat{a}_c) < 0$ and for q_s sufficiently big $\frac{d}{da_c}F_1(\hat{a}_c) > -\frac{d}{da_c}F_2(\hat{a}_c)$ and thus $\frac{d}{da_c}F(\hat{a}_c) = \frac{d}{da_c}F_1(\hat{a}_c) + \frac{d}{da_c}F_2(\hat{a}_c) > 0$, which means that the branch that starts from infinity for a_c close to zero, ends up at the value $F(\bar{a}_c)$ for $a_c = \bar{a}_c$ not decreasing all the time but increasing around $\hat{a}_c$. This means that there are values of a for which the parallel at height a cut the curve $F(a_c)$ at least three points, and thus we have at least three solutions. (Notice that the polynomials $P_1(a_c^2)$, $P_2(a_c^2)$ have positive coefficients which are positive sums of powers of q_s.) Thus, we can conclude that nonuniqueness appears when there is a large discrepancy between the state penalizations between the University's and the Student's costs.

A case of interest, which we will also consider for the continuous time example that follows, is when there is no University player present. Since the University player has an infinite time duration cost, we needed that for the resulting a_c it holds: $|a_c| < 1$. In the absence of the University player, the closed loop system may be unstable. In addition, we do not need the condition: $0 \leq a_c \leq \frac{1}{\sqrt{q}+\sqrt{q+1}}$, which has to do with the existence of a real control gain k^- (or k^+) for the University. Thus, if no University player is present and we do not care to have a stable closed loop system we always have a solution as the plot of $F(a_c) = (1 + k_1 + k_2 + k_3 + k_4 + k_5)a_c$ for $a_c \in [0, \infty)$ is monotonically increasing and always meets the value a. On the other hand, if we want the resulting a_c to satisfy $|a_c| < 1$, we need $a < F(1) = 1 + k_1(1) + k_2(1) + k_3(1) + k_4(1) + k_5(1)$, whereby $k_i(1)$ we mean the values: $k_5(1) = q_s$, $k_4(1) = q_s + k_5(1) + k_5(1)^2, \ldots, k_1(1) = q_s + k_2(1) + k_2(1)^2$. This means a bound on a, which depends on q_s. If $|a| < 1$ the bound is automatically satisfied.

8.2 Continuous Time Feedback Nash Strategy

As an example, let us consider the scalar case where there is no University ($b_0 = r_0 = q_0 = 0$), with $b = r = 1$, $q =$ constant, the scalar solution for $K(t)$ is

$$k(t) = \left(q_f + \frac{q}{2a}\right)e^{2(a-l)(t_f-t)} - \frac{q}{2a}$$

And (22)

$$A_c^T\big(LA_c + A_c^T L + QT + Q_f\big) + \big(LA_c + A_c^T L + QT + Q_f\big)A_c + Q$$
$$= e^{A_c^T T}\big(Q + Q_f A_c + A_c^T Q_f\big)e^{A_c T}$$

results in

$$l = -\frac{q}{2(a-l)}T - \frac{1}{2(a-l)}\left(q_f + \frac{q}{2(a-l)}\right) + \frac{1}{2(a-l)}\left(q_f + \frac{q}{2(a-l)}\right)e^{2(a-l)T}$$

This equation has to be solved for l, and the solution has to satisfy $a - l < 0$. Setting $x = -2T(a-l)$, we have to solve the equivalent:

$$e^x = f(x) = \frac{qT^2 - q_f T x}{qT^2 - (q_f T + qT^2)x + ax^2 + \frac{1}{2T}x^3}$$

for $x > 0$. We calculate: $f(0) = 0$, $\frac{d}{dx} f(0) = 1$, $\frac{d^2}{dx^2} f(0) = -2\frac{a}{qT^2} + 2\frac{q_f}{qT} + 2$. In order to have that the plot of $f(x)$ cuts the plot of e^x for some $x > 0$, it suffices to have $\frac{d^2}{dx^2} f(0) = -2\frac{a}{qT^2} + 2\frac{q_f}{qT} + 2 > 1$ (or else the graph of the right-hand side is below the graph of e^x for $x > 0$). This condition means that a solution of the Nash games, which makes the closed loop system stable, exists if a is appropriately small, and this smallness depends on the length of the Students studies and the penalties on the state. (Recall a similar bound derived for the discrete time example.) If $a < 0$, the condition is always fulfilled. But if $a > 0$—(i.e. the system is unstable on its own and the University player/regulator is not existent)—then the stabilizing solutions will not necessarily exist. For that we need:

$$T > -\frac{q_f}{q} + \sqrt{\left(\frac{q_f}{q}\right)^2 + 2\frac{a}{q}} = \frac{2\frac{a}{q}}{\frac{q_f}{q} + \sqrt{(\frac{q_f}{q})^2 + 2\frac{a}{q}}}$$

Thus, for $a > 0$, the intertemporal coordinating role of the long term player can be substituted by a continuous overlapping succession of small players, if their duration T (interpreted as memory, experience, and life span presence) is sufficiently long.

For $a \gg \frac{(q_f)^2}{2q}$, we have the sufficient condition: $T > \sqrt{\frac{2a}{q}}$. (The possibility of having a Nash solution which yields an unstable closed loop system, i.e., $x = -2T(a - l) < 0$ and $e^x = f(x)$ is still possible under the appropriate existence conditions.)

That the essence of this result can be extended to the matrix case can be supported by the following argument: Since the University is not present, we look at the equation:

$$-\frac{dK(t)}{dt} = K(t)\left(A - \int_0^{\bar{T}} B(s)R^{-1}(s)B^T(s)K(s)\,ds\right) + \left(A - \int_0^{\bar{T}} B(s)R^{-1}(s)B^T(s)K(s)\,ds\right)^T K(t) + Q(t), \quad t \in [0, \bar{T}],$$

with boundary condition:

$$K(\bar{T}) = Q_f$$

For T very small, we have $K(t) \approx Q_f$ and the closed loop matrix is

$$A - \int_0^{\bar{T}} B(s)R^{-1}(s)B^T(s)K(s)\,ds \approx A - BR^{-1}B^T Q_f T \approx A$$

which is not necessarily stable except if A is stable. For T sufficiently large, if $K(t)$ has converged to some constant value L, this L will satisfy

$$0 = L\left(A - BR^{-1}B^T LT\right) + \left(A - BR^{-1}B^T LT\right)^T L + Q$$

Or equivalently:

$$0 = LA + A^T L + Q - 2LBR^{-1}B^T LT$$

and the closed loop matrix will be:

$$A - BR^{-1}B^T LT$$

Clearly, this closed loop matrix is stable if the pair (A, B) is controllable.

Let us now consider the case where the University is also present. Let $b_0 = b = r = r_0 = 1, q, q_0$ constants. The equations we have to solve are

$$0 = 2k_0(a - l) + q_0 - k_0^2$$
$$-\frac{dk(t)}{dt} = 2k(t)(a - k_0 - l) + q, \quad t \in [0, \bar{T}],\ k(\bar{T}) = q_f$$
$$l = \int_0^{\bar{T}} k(s)\, ds$$

The resulting closed loop matrix has to be stable (or else the University's cost is infinite):

$$a_c = a - k_0 - l < 0$$

We have:

$$k_0 = a - l + \sqrt{(a - l)^2 + q_0}$$
$$a_c = -\sqrt{(a - l)^2 + q_0} < 0$$

which always holds.

$$-\frac{dk(t)}{dt} = 2k(t)\big(\sqrt{(a - l)^2 + q_0}\big) + q, \quad t \in [0, \bar{T}],\ k(\bar{T}) = q_f$$
$$\frac{dk(t)}{dt} = 2a_c k(t) - q, \quad t \in [0, \bar{T}],\ k(\bar{T}) = q_f$$
$$k(t) = \left(q_f + \frac{q}{2a_c}\right) e^{2a_c(t_f - t)} - \frac{q}{2a_c}$$

Calculating the integral $l = \int_0^{\bar{T}} k(s)\, ds$, we get

$$l = -\frac{q}{2a_c}T - \frac{1}{2a_c}\left(q_f + \frac{q}{2a_c}\right) + \frac{1}{2a_c}\left(q_f + \frac{q}{2a_c}\right) e^{2a_c T}$$

Let

$$\sqrt{(a - l)^2 + q_0} = \sqrt{q_0} + x$$
$$l = \big(a + \varepsilon\sqrt{x^2 + 2x\sqrt{q_0}}\big), \quad \varepsilon = \pm 1$$

We can rewrite the equation for l:

$$\big(a + \varepsilon\sqrt{x^2 + 2x\sqrt{q_0}}\big)$$
$$= \frac{q}{2\sqrt{q_0} + x}T + \frac{1}{2\sqrt{q_0} + x}\left(q_f - \frac{q}{2\sqrt{q_0} + x}\right)$$
$$- \frac{1}{2\sqrt{q_0} + x}\left(q_f - \frac{q}{2\sqrt{q_0} + x}\right) e^{-2\sqrt{q_0}T} e^{-2xT}$$

As $x \to +\infty$, the right-hand side goes to zero and the left-hand side goes to $+\infty$ if $\varepsilon = +1$ and to $-\infty$ if $\varepsilon = -1$. Let $f(0)$ be the value of the right-hand side at $x = 0$. If

$$a < f(0) = \frac{q}{2\sqrt{q_0}}T + \frac{1}{2\sqrt{q_0}}\left(q_f - \frac{q}{2\sqrt{q_0}}\right) - \frac{1}{2\sqrt{q_0}}\left(q_f - \frac{q}{2\sqrt{q_0}}\right)e^{-2\sqrt{q_0}T}$$

we take $\varepsilon = +1$ and the plot of the left-hand side will necessarily cut the plot of the right-hand side for some $x > 0$. If $a > f(0)$, we take $\varepsilon = -1$ and the plot of the left-hand side will necessarily cut the plot of the right-hand side for some $x > 0$.

We conclude that a solution for some x and, therefore, for some l, k, k_0 always exists. So in the presence of a University player the Students can complete their studies within any finite time period T, of course multiplicity of solutions is not excluded.

9 Conclusions

We have provided the formulation of an interesting class of game problems, which besides having important applications on their own, they lead even for the Linear Quadratic case to some novel conditions beyond the quadratic Riccati type equations encountered so far in the literature for LQ Games. Further study could address among other issues: existence of solutions, uniqueness or multiplicity of solutions, computational procedures, and stochastic variants.

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