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Games on Large Networks: Information and Complexity.

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***Abstract*—In this work, we study Static and Dynamic Games on Large Networks of interacting agents, assuming that the players have some statistical description of the interaction graph, as well as some local information. Inspired by Statistical Physics, we consider statistical ensembles of games and define a Probabilistic Approximate equilibrium notion for such ensembles. A Necessary Information Complexity notion is introduced to quantify the minimum amount of information needed for the existence of a Probabilistic Approximate equilibrium. We then focus on some special classes of games for which it is possible to derive upper and/or lower bounds for the complexity. At first, static and dynamic games on random graphs are studied and their complexity is determined as a function of the graph connectivity. In the low complexity case, we compute Probabilistic Approximate equilibrium strategies. We then consider static games on lattices and derive upper and lower bounds for the complexity, using contraction mapping ideas. A LQ game on a large ring is also studied numerically. Using a reduction technique, approximate equilibrium strategies are computed and it turns out that the complexity is relatively low.**

***Index Terms*— Game Theory, Stochastic Systems, Network Analysis and Control, Statistical Physics, Information and Complexity**

I. Introduction

In the last decade, there is a large and growing interest in the study of static and dynamic games with a large number of players. In this context the theory of Mean Field Games (MFGs) [1],[2],[3] was introduced in order to study game situations, where each individual interacts with the mass of the other players (mean field interaction). Asymptotic Nash equilibrium results are usually obtained under the assumption that each player knows her own state variable, as well as the statistical distribution of the types and state variables of the rest of the players. This work aims to study games with a large number of players, under more general interaction structures, and identify cases of games in which it is possible to have an approximate equilibrium, assuming only local and statistical information.

In several game situations involving many agents, the strategic interactions depend on a Large Network. An example is an electricity transmission grid where several entities, such as producers, consumers or smart micro-grids are connected in different places of the network. In this case local, as well as global cooperation and/or competition arises (ex. [4]). Several examples of game situations, such as the selection of a telecommunication company, the opinion about an idea or a product, the selection of fashion group or the engagement in criminal behavior, involve interactions over social networks [5], [6]. In these examples, the payoff or the cost of the choices of each agent depends on her own preferences, as well as on the choices of her friends. There are also several examples of non-social interaction structures, such as the interaction among the owners of stores for renting and the gas station prices, where there is local, as well as global competition. Several features characterizing Systems of Systems [7], such as operational and managerial independence, geographical distribution, heterogeneity of systems and networks of systems can be also captured by Dynamic Games on Networks models.

Two kinds of approach have been mainly used to predict the behavior of the participants in large games. The first approach is based on equilibrium concepts and the dominant notion in this approach is the Nash equilibrium. The knowledge of a large amount of information is often needed in order to be possible to determine equilibrium or approximate equilibrium. The second kind of approach assumes limited (bounded) rationality for the participants [8], [9] and it is based on dynamic formulations. In particular some deterministic or stochastic rules, describing the future actions of the agents as a function of their current actions, are postulated and then evaluated theoretically or experimentally. Examples of dynamic rules include evolutionary dynamics, learning, adaptive control laws and best response.

This second kind of approach does not require a complete knowledge of the game. However, the dynamic rules used are not universal, in the sense that there is no reason to believe that all the players will follow some specified rule to determine their future actions. Furthermore, the cheating problem [10] may arise. That is, if a player knows dynamic/adaptation rule of another player, she may exploit this knowledge in order to manipulate her. Thus, the outcomes may be different from the predicted.

Which kind of approach should be used in order to describe/predict the behavior of the players in a large game? The full rationality assumption for the players should depend on the difficulty of the problem they have to solve, as well as the informational requirements. In this work, we define an informational complexity concept, for a certain class of games, as the minimum amount of information needed for

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a certain form of approximate equilibrium to be achievable. This complexity notion is studied asymptotically for large games. If the complexity is small, then the participants of a large game can use strategies which are in an approximate equilibrium without assuming a lot of information. In a large game with very high complexity, it is reasonable to assume that the players would use a dynamic rule. In this sense, our approach aims at the one hand to identify classes of large games which admit approximate equilibrium solutions assuming only a small portion of the information and on the other hand to identify some informational limitations of the applicability of equilibrium concepts in static or dynamic games with a large number of players.

A. Contribution

We study Dynamic Games on Large Networks, assuming some stochastic description of the Graph, such as Random Graph, Grid, Small World Network, etc. The stochasticity is divided into two parts. The first part describes the structural stochasticity and corresponds to the lack of knowledge of the players, while the second corresponds to the lack of predictability of the future.

Instead of studying a single game, we consider an ensemble (collection) of games, inspired by ideas arisen in the Statistical Physics domain, and assume that a statistical model on that ensemble is available to the players. Furthermore, we assume that each player has access to the information contained in a neighborhood of a certain order around her. As the order of neighborhoods increases the information varies from (only) statistical to perfect. We then use an analogy with a situation common in Statistical Physics to define a Probabilistic Approximate Nash (PAN) equilibrium concept, for ensembles of games. For the ensemble of games we define a complexity function, as the minimum amount of information needed for a PAN equilibrium to exist. This complexity function is studied asymptotically, as the number of the participants of the game becomes large.

We then focus on special cases of games, where upper or lower bounds for the complexity can be obtained. A class of static games on Erdos-Renyi Random Graphs is shown to be simple under some connectivity assumptions and complex under other connectivity assumptions. We also analyze a LQ opinion game on Erdos-Renyi Random Graphs and derive approximate equilibrium strategies, generalizing older work about consensus [11].

Quadratic Games on ordered interaction Structures (Grids or Rings) are then analyzed. They are found to have polynomial complexity functions, where the order of the polynomial depends on the dimension of the interaction structure. We then analyze a class of non-quadratic games on rings having chaotic best response maps and show that they have a relatively low complexity. The approximate equilibrium is possible due to some cooperation among the players. Finally, a LQ game on a ring is studied numerically. Approximate equilibrium strategies are computed, using a reduced game and the complexity function is found to be approximately linear in the order of the information neighborhood.

In the special cases analyzed, we establish low complexity results using one of the following properties: a law of large numbers, contractivity of the best response maps, cooperation among the players or low gains assigned to distant players.

B. Related Topics

The interest for the games with large number of players is not new. Probably the first works dealing with games involving a continuum of players are [12] and [13]. [12] analyzes a market with a continuum of players and [13] studies games with a continuum of players, called Oceanic Games and introduces a value for such games. The Mean Field Games [1] were recently introduced to study static and dynamic games with large number of players. The closely related methodology of Nash Certainty Equivalence was also developed, in order to obtain asymptotic approximate Nash equilibrium results, as the number of players tends to infinity [2],[3]. These works study games, where each player interacts with the mass of the other players, which is approximated by a continuum. [14] presents several extensions of the Mean Field game theory on models describing more general interactions.

Another related topic is Games with Local Interactions, in which each player interacts with some players important to her, on some organized structure. In [15], equilibria for complete and incomplete information Local Interaction Games were found, based on contraction mapping ideas. [16] studies the dynamic game counterpart and [17] studies models with discrete choice.

Quadratic games on networks are studied in [18], using centrality notions. Games on networks with incomplete information are studied in [19]. Dynamic games on evolving state dependent graphs are studied in [20] and stochastic games in [21]. [22] is a recent review of network games.

Dynamic rules for updating the actions of the agents on lattices were studied in the context of Interacting Particle Systems in [23] and [24]. In [25] several dynamic rules for games on graphs were introduced and studied analytically and computationally. Several sociological applications of evolutionary games on graphs are studied in [26].

The impact of the quality of information that the agents receive on their costs is studied in [27], in the LQG Game framework. It is shown that, as the number of players or the time horizon becomes large, better information becomes beneficial for all the participants of the game. The notion of the price of uncertainty is introduced in [28] and [29], in order to describe the difference in the costs of the players, under different information structures. The price of information is introduced in [30] to describe the difference of the cost that the players have in deterministic dynamic games under different information patterns, i.e. feedback and open loop. Finally, [31] studies incomplete information games where the players may have access to additional private information and characterizes the possible outcomes using the notion of Bayes Correlated Equilibrium.

C. Notation

Let $(\Omega, \mathcal{F}, P)$ be the underlying probability space. For a random variable X, denote by $\sigma(X)$ the σ algebra

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generated by X, i.e. the coarsest σ algebra such that X is $\sigma(X)$-measureable.

We denote by $G = (V, E)$ a directed or undirected graph, where V is the set of vertices and E the set of edges (ex. [32], [33]). For a vertex $v \in V$, the neighborhood of v is defined as $\mathcal{N}_v(G) = \{v' \in V : (v', v) \in E\}$ and the closed neighborhood of v as $\bar{\mathcal{N}}_v(G) = \{v\} \cup \{v' \in V : (v', v) \text{ or } (v, v') \in E\}$. The closed neighborhood of order n of v is defined as $\bar{\mathcal{N}}_v^n(G) = \cup_{j \in \bar{\mathcal{N}}_v(G)} \bar{\mathcal{N}}_j^{n-1}(G)$, where $\bar{\mathcal{N}}_v^1(G) = \bar{\mathcal{N}}_v(G)$ and $\bar{\mathcal{N}}_v^0(G) = \{v\}$. For a subset A of V, denote by $G_A = (A, \{(v', v) \in E : v, v' \in A\})$, the largest subgraph of G with set of vertices A. With $|G'|$ we denote the number of nodes of a graph G'.

The indeterminacy $\frac{0}{0}$ is resolved as 0. An ordered tuple $(\gamma^1, \ldots, \gamma^N)$ is denoted by $(\gamma^i)_i$ and the ordered tuple $(\gamma^1, \ldots, \gamma^{i-1}, \gamma^{i+1}, \ldots, \gamma^N)$ by γ^{-i}. By $[\cdot]$ we denote the integer part. The asymptotic notation will be also used. For real functions f and g, we write $f(x) \in O(g(x))$, if there exists a constant $c > 0$, such that $0 \leq f(x) \leq cg(x)$ for large x and $f(x) \in o(g(x))$, if for every given $c > 0$ it holds $0 \leq f(x) \leq cg(x)$, for large x. We write $f(x) \in \omega(g(x))$ if for every given $c > 0$ it holds $0 \leq cg(x) \leq f(x)$ for large x and $f(x) \in \Theta(g(x))$ if there exist constants c_1 and c_2 such that $0 \leq c_1 g(x) \leq f(x) \leq c_2 g(x)$, for large x. Finally, $f(x) \in \Omega(g(x))$ if for some constant c, it holds $f(x) \geq cg(x)$, for large x.

In what follows, we assume that all the functions involved are measurable within appropriate measurable spaces.

II. Description

We first describe the general form of the structure of interactions among the players and assign a game g^S to every interaction structure S. Then, an ensemble of interaction structures (or equivalently an ensemble of games) $\mathcal{E}$ is defined, assuming that the players have a common probabilistic description for the interaction structures S in $\mathcal{E}$.

An interaction structure consists of a set of players, their types and a graph describing their interactions. Particularly, there is a set of players $p_1, \ldots, p_N$ having types $\theta_1, \ldots \theta_N$ belonging to a set of possible types $\Theta \subset \mathbb{R}^q$. We assume that the type of each player is constant during the game. The interaction structure depends on a graph $G = (V, E)$, directed or not. Each vertex of the graph $v \in V$ corresponds to a player p_v and each edge $(v', v) \in E$ to the influence of the player $p_{v'}$ to the player p_v. The interaction structure is described in compact form by:

$$S = (\Pi, G), \tag{1}$$

where $\Pi = ((p_1, \theta_1), \ldots, (p_N, \theta_N))$. Let us denote by $\mathcal{F}^S = \sigma(S)$, the structural information.

For a given interaction structure S, a discrete time dynamic game g^S among the players is defined. Each player p_i has her own state variable x^i taking values in a subset $\mathcal{X}$ of the Euclidean space $\mathbb{R}^m$. The evolution the state variable of each player p_i is affected by the state variables and actions of her neighbours, her own control variable $u_k^i \in \mathcal{U} \subset \mathbb{R}^p$ and her own type θ_i. The dynamics for the player p_i is given by:

$$x_{k+1}^i = f^{\theta_i}\left(x_k^i, u_k^i, w_k^i, \sum_{j \in \mathcal{N}_i(G)} f_1^{\theta_i, \theta_j}(x_k^i, x_k^j, u_k^i, u_k^j), \right), \tag{2}$$

where w_k^i are mutually independent random variables with known distributions and $\mathcal{N}_i(G)$ is the neighbourhood of player p_i. The initial conditions x_0^i are random variables, possibly dependent on the interaction structure.

The cost function of player p_i has the form:

$$J_i = E\left\{ \sum_{k=0}^{T} \rho^k g^{\theta_i}\left(x_k^i, u_k^i, \sum_{j \in \mathcal{N}_i(G)} g_1^{\theta_i, \theta_j}(x_k^i, x_k^j, u_k^i, u_k^j) \right) \middle| \mathcal{F}^S \right\}, \tag{3}$$

where the time horizon T can be finite or infinite, $g \geq 0$ and $\rho \in (0, 1]$ is the discount factor.

The players do not know the interaction structure characterizing the game they are involved. Instead, they consider a statistical ensemble $\mathcal{E}$ of possible interaction structures, i.e. a collection of mental copies of the game having different interaction structures. With a slight abuse of notation, the corresponding ensemble of games is also denoted by $\mathcal{E}$. We assume that all the players consider the same ensemble and the same probability structure on that ensemble. Let us denote by $Q(\cdot)$ the distribution of the random variable S in $\mathcal{E}$.

Apart form the statistical model of the interaction structure $(\mathcal{E}, Q)$, the players have also some local information. The local information of a player p_i consists of the structural facts and the state variables of the players contained in a closed neighborhood of order n of that player. Particularly, we assume that each player knows her type and can measure her own state variable. Furthermore, she knows the types of the players, the subgraph and the state variables of the players in that neighborhood. Thus, the local information available to the player p_i at time step k is:

$$I_k^{i,n} = (G_{\bar{\mathcal{N}}_i^n(G)}, (\theta_j)_{j \in \bar{\mathcal{N}}_i^n(G)}, (x_t^j)_{j \in \bar{\mathcal{N}}_i^n(G)}^{t=0,\ldots k}), \tag{4}$$

where $\bar{\mathcal{N}}_i^n(G)$ (defined in Section I-C) denotes the closed neighborhood of order n of player p_i and $G_{\bar{\mathcal{N}}_i^n(G)}$ is the corresponding subgraph. The total information available to player p_i at time step k is $(\mathcal{E}, Q, I_k^{i,n})$.

Due to the fact that the players do not know which is the actual interaction structure S, they use strategies that can be applied in every member of the ensemble $\mathcal{E}$. The strategy of each agent can, however, depend on local information. The strategy of player p_i has the form:

$$u_k^i = \gamma_k^i(I_k^{i,n}). \tag{5}$$

We consider symmetric sets of strategies, where players with the same information (and hence type) behave in the same way. Furthermore, we focus on feedback strategies (strategies without memory) in the form:

$$u_k^i = \gamma_k(\bar{I}_k^{i,n}), \tag{6}$$

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where:

$$\bar{I}_k^{i,n} = (G_{\bar{\mathcal{N}}_i^n(G)}, (\theta_j)_{j\in\bar{\mathcal{N}}_i^n(G)}, (x_k^j)_{j\in\bar{\mathcal{N}}_i^n(G)}). \tag{7}$$

The following classes of strategies will be useful in the next section.

Strategy Classes:

(i) The class of Feedback Local Information Strategies for player i is given by:

$$\Gamma_i^{FLI} = \{\gamma^i = (\gamma_1^i, \gamma_2^i, \dots) : \gamma_k^i : \bar{I}_k^{i,n} \to \mathcal{U}, k = 1, 2, ..\}.$$

If necessary, $\Gamma_{i,n}^{FLI}$ will be used in the place of Γ_i^{FLI} to indicate the order n of the information neighborhoods.

(ii) The class of Closed Loop Perfect Information Strategies for player p_i is given by:

$$\Gamma^{CLPI} = \{\gamma^i = (\gamma_1^i, \gamma_2^i, \dots) : \gamma_k^i : I_k^{CLPI} \to \mathcal{U}, \quad k = 1, 2, \dots\},$$

where $I_k^{CLPI} = (G, (\theta_j)_{j\in V}, (x_t^j)_{j\in V}^{t\le k})$, G is the interaction graph and V is the set of all the players. □

In the following section, we shall define an approximate equilibrium notion for FLI strategies as a strategy profile which is ε-optimal within the class of CLPI strategies, with high probability.

Remark 1:

(i) The model of the ensemble of games with a commonly known probabilistic model on the ensemble is similar with the standard Bayesian game framework. However, the model of the ensemble of games is more convenient for the definition of the approximate equilibrium in the following section.

(ii) There are two types of stochasticity presented in the model. The first is due to the lack of predictability and it is described by the random variables w_k^i. The second is the uncertainty due to the lack of information (knowledge) and it is described the random variable S which contains the structural features of the game. There are some other works which divide the uncertainty into the lack of knowledge and the lack of predictability. For example, in [34] incomplete information is treated using robust optimization and the stochasticity due to the randomization of the players using the expectation.

(iii) The members of the ensemble do not necessarily have the same number of players and the ensemble is not necessarily finite. □

The model described borrows some ideas from Statistical Physics and Network Science. Statistical Physics studies systems consisting of a very large number of particles for which we are not able to measure all the initial conditions and solve the dynamic equations. Instead, the notion of a statistical ensemble of systems is used [36]. A statistical ensemble is a collection of mental copies of the system, each one of which represents a different set of initial conditions that the actual system may have. It turns out that several macroscopic properties have values close to a deterministic constant for all the systems in the ensemble, except possibly of a set of systems with very low probability. Similar results were also obtained in Network Science, such as the results about Percolation, connectivity of large Random Graphs, etc [5], [6].

In the following section, we study sets of strategies which constitute an approximate Nash equilibrium for the games corresponding to all the interaction structures of the ensemble, except possibly of a set of interaction structures with very low probability. Following some ideas from Statistical Physics and Network science, in order to study games with large number of players, we consider a sequence of ensembles of games $\mathcal{E}_\nu$ with increasing number of players and study the tail of that sequence.

III. Approximate Equilibrium and Complexity

Consider a large game in which the actions of the players depend only on local and statistical information. Due to the fact that the agents do not know in which game they participate in, it is reasonable to expect that a set of strategies in the form (6) could not typically constitute a Nash equilibrium. A Probabilistic Approximate Nash (PAN) equilibrium concept is thus defined, based on the concept of ε - Nash equilibrium. We first recall the definition of the ε - Nash equilibrium for a single game.

Definition 1: Consider a game g^S with $S \in \mathcal{E}$ and the set of dynamics and cost functions given by (2) and (3). Then a set of strategies $(\gamma^i)_i$ with $\gamma^i \in \Gamma_i^{FLI}$ constitutes an ε - Nash equilibrium, if for every player p_i it holds:

$$J_i(\gamma^i, \gamma^{-i}) - \min_{\gamma\in\Gamma^{CLPI}} \{J_i(\gamma, \gamma^{-i})\} < \varepsilon, \tag{8}$$

where the minimum is considered within the class of full information closed loop strategies. □

An approximate equilibrium concept is then defined for the ensemble of games. We are interested to characterize a set of strategies constituting an ε - Nash equilibrium for the games g^S that correspond to the most of the interaction structures in $S \in \mathcal{E}$.

Definition 2: Consider an ensemble of interaction structures $\mathcal{E}$ and the set of dynamics and cost functions as before. Then a set of strategies $(\gamma^i)_i$ with $\gamma^i \in \Gamma_i^{FLI}$ is an ε-Probabilistic Approximate Nash equilibrium (ε-PAN equilibrium) for that ensemble if:

$$P(\{S \in \mathcal{E} : (\gamma^i)_i \text{ is } \varepsilon \text{ - Nash equilibrium of } g^S\}) > 1 - \varepsilon, \tag{9}$$

i.e. $(\gamma^i)_i$ constitutes an ε - Nash equilibrium with high probability. The probability of the event in (9) can be computed using the distribution Q. □

The reason for studying sets of strategies constituting an ε-PAN equilibrium with small ε is that, with a very high probability, no player has non-negligible benefit from changing her strategy, even if she had access to all the available information, at any time step. Thus, the players do not have enough motivation to try to estimate the information they do not possess.

Remark 2: An alternative way to express inequality (9) is to use the Ky Fan metric among random variables (ex. [37]) defined as follows. The distance d between random variables X_1 and X_2 is defined as:

$$d(X_1, X_2) = \inf\{\varepsilon > 0 : P(|X_1 - X_2| > \varepsilon) \le \varepsilon\}. \tag{10}$$

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Thus, inequality (9) can be stated equivalently as:

$$d(J^i(\gamma^{-i},\gamma^i), \inf_{\gamma\in\Gamma^{CLPI}} J^i(\gamma^{-i},\gamma)) < \varepsilon, \tag{11}$$

in terms of the Ky Fan metric. □

The following Lemma 1 considers the ε-PAN equilibria of repeated games. Let us first define a repeated game that corresponds to a given static game. Consider a static game where the costs of the players are given by:

$$J_i = E\left\{ g^{\theta_i}\left(u^i, \sum_{j\in\mathcal{N}_i(G)} g_1^{\theta_i,\theta_j}(u^i,u^j)\right) \middle| \mathcal{F}^S \right\}. \tag{12}$$

The corresponding repeated game is given by:

$$\tilde{J}_i = (1-\rho)E\left\{ \sum_{k=0}^{\infty} \rho^k g^{\theta_i}\left(u_k^i, \sum_{j\in\mathcal{N}_i(G)} g_1^{\theta_i,\theta_j}(u_k^i,u_k^j)\right) \middle| \mathcal{F}^S \right\}, \tag{13}$$

with $\rho \in (0,1)$.

Lemma 1: The ε-PAN equilibrium has the following properties:

(i) An ε - PAN equilibrium of a static game remains an ε - PAN for the corresponding repeated game.

(ii) Consider a set of strategies $(\gamma^i)_i$ constituting an ε-PAN equilibrium. If the players receive more information, i.e. the information neighborhoods have order $n' > n$ and $I^{i,n'} \supset I^{i,n}$, then the set of strategies $(\gamma^i)_i$ remains an ε-PAN equilibrium. That is, the ε-PAN equilibrium is insensitive to new information.

Proof: (i) It is not difficult to see that (9) holds for $\tilde{J}$.

(ii) It holds $\gamma^i \in \Gamma_{i,n}^{FLI} \subset \Gamma_{i,n'}^{FLI}$. Furthermore, (9) holds and thus $(\gamma^i)_i$ is an ε-PAN equilibrium for the ensemble of games where the players have information $I^{i,n'}$. □

Let us now compare the ε-PAN equilibrium with the notion of Bayesian Nash equilibrium (see ex. [38]).

Remark 3:

(i) BNE and ε-PAN equilibrium can describe substantially different outcomes of the game. A game where an ε-PAN equilibrium is qualitatively different from the BNE is described in Example 1.

(ii) It is very difficult to compute a Bayesian Nash equilibrium even for simple dynamic games. In the case of LQ stochastic Dynamic Games with imperfect state feedback information, Nash equilibria have been computed only for special information patterns [39]. [40]. In the case where the structural information is also incomplete, the optimization problems are vey difficult even for single person games (optimal control problems), due to the fact that the dual control problem arises (ex. [41]).
In the case of stochastic games with incomplete information the computation of BNE is also a very difficult problem. For this class of games the notion of Empirical Evidence Equilibrium [42] was introduced based on bounded rationality assumptions for the participants.

(iii) In contrast to Bayesian Nash equilibrium, an ε - PAN set of strategies satisfies the properties (i),(ii) of Lemma 1.

(iv) In some examples, we may have PAN equilibrium, even if the players have different prior probabilities on the ensemble and each player is not sure for the exact probabilistic models of the others. (ex. Sec. IV). □

TABLE I
COSTS IN THE WEAK MONOPOLIST CASE

1\2	Fight	Accommodate
Enter	1-b, 1	-b,0
Stay Out	0,-a	0,-a

TABLE II
COSTS IN THE TOUGH MONOPOLIST CASE

1\2	Fight	Accommodate
Enter	1-b, 0	-b,1
Stay Out	0,-a	0,-a

Example 1: This example studies an ε-PAN equilibrium of the Reputation game introduced in [43]. This game involves a "big" player (monopolist) which plays successively the same game with a large number N of "small" players (entrants). The monopolist has two possible types: weak and tough. The monopolist is tough with a very small probability $\delta > 0$. In each stage game, the entrant moves first and has two possible actions: "enter" and "stay out". If the entrant enters the game, the monopolist has two options: "accommodate" and "fight". The costs for the players in each of the successive games are given in Tables I and II. We assume that $a > 1$ and that $0 < b < 1$. The only difference between a weak and a tough monopolist is that the tough monopolist prefers to fight if the entrant enters while the weak monopolist prefers to accommodate. We further assume that each one of the players is able to monitor all the previous actions.

Let us first describe an ε-PAN equilibrium of the game. Assuming that $\delta < \varepsilon$, the entrants would ignore the possibility of a tough monopolist. Thus, each one of the entrants would enter the game and the (weak) monopolist has no reason to fight them. Thus the monopoly would break.

On the other hand, if N is sufficiently large in any sequential BNE the early entrants would stay out and the monopolist would fight any early entrant who enters the game [43]. Thus, the monopoly would be maintained at least for the larger part of the game.

This example illustrates that ε-PAN equilibrium and BNE may describe very different outcomes. □

Consider a set of strategies constituting an ε-PAN equilibrium. Each player is interested and responds to a different set of players and in this sense, each player is involved in (perceives) a different game. For example, consider the game of in Figure 1. Each player is affected by her neighbors in the graph through the terms f_1, g_1. Assume also that there is an ε-PAN equilibrium assuming information neighbourhoods of order 1. Then, player p_1 acts as if he is involved in a game only with the players p_2, p_3 and p_5, the player p_5 in a game with p_1, p_2 and p_7 and so on.

If the order of the information neighborhoods of the players is small then it is probably not possible to have an ε-PAN equilibrium. Thus, we are interested in the following question:

Question 1: "Given a positive constant ε, what is the

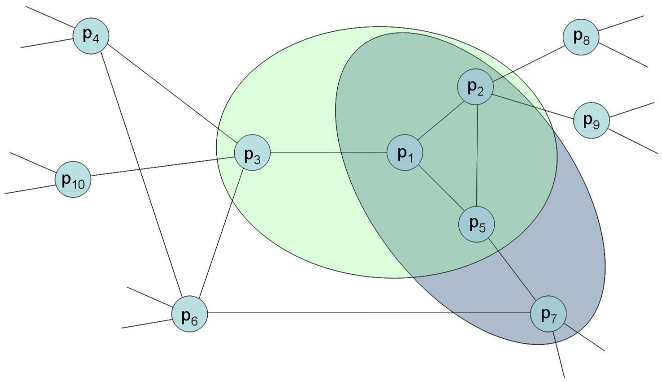

Fig. 1. The information neighborhoods of players 1 and 5 with $n = 1$.

minimum amount of information that the agents need to have in order to achieve an ε-PAN set of strategies?" □

Based on the answer to this question, the Necessary Information Complexity (NIC) function with respect to the PAN equilibrium is defined for the ensemble of games.

Definition 3:

(i) Consider an ensemble of games as described above. Let us define the following function:

$$\bar{n}(m) = \inf\{n \in \mathbb{N} : \exists \text{ set of strategies } (\gamma^i)_i \text{ with } \gamma^i \in \Gamma^{FLI}_{i,n}, \text{ in the form (6) which is } 2^{-m}\text{-PAN}\}, \quad (14)$$

and denote by $\bar{\gamma}$ the strategy that attains the minimum. The Necessary Information Complexity (NIC) function with respect to the PAN equilibrium is defined as:

$$C(m) = \max_{S \in \mathcal{E}_0, p_i \in S} \{|\bar{\mathcal{N}}_i^{\bar{n}(m)}|\}, \quad (15)$$

where $\mathcal{E}_0$ is the subset of the ensemble of games for which $\bar{\gamma}$ is a 2^{-m} - Nash equilibrium.

(ii) Consider a sequence of ensembles $\mathcal{E}_\nu$ with cost functions J_i^ν and dynamics described by $f_1^{\cdot,\cdot,\nu}$, $f_2^{\cdot,\cdot,\nu}$. Denote by $C^\nu(\cdot)$ the NIC function of the ν-th ensemble. The Asymptotic Necessary Information Complexity (ANIC) function with respect to the PAN equilibrium is given by:

$$C_a(m) = \limsup_{\nu \to \infty} C^\nu(m). \quad (16)$$

The sequence of ensembles will be called asymptotically simple if the function $C_a(m)$ is bounded and asymptotically complex if for some $m \in \mathbb{N}$ it holds $C_a(m) = \infty$. □

The function $\bar{n}(m)$ quantifies the minimum order of the information neighborhood that the players need to have for the existence of a $2^{-m} - PAN$ equilibrium. The NIC function $C(m)$ quantifies the maximum number of players in a closed neighborhood of order $\bar{n}(m)$ or equivalently the maximum number of players that is required to be observed by a single player, in order for a $2^{-m} - PAN$ equilibrium to exist.

Remark 4: A first example of asymptotically simple games is the class of Mean Field Games [1], [2], [3]. In this class, under some conditions, each player interacts with the mass of the other players which behaves asymptotically deterministically, as the number of players increases. In this case, each player needs to know only her own type and state variable, in order to have a nearly optimal behavior and thus the function C_a is bounded. □

Classes of games, where it is possible to find upper and lower bounds for the NIC and ANIC functions, are analyzed in the following sections.

IV. Games on Random Graphs

This section studies ensembles of games involving players interacting on large Erdos-Renyi random graphs. The complexity of the ensemble varies, depending on the connection probability of the random graph. In the high connectivity regime the cumulative effect of the neighbors of each player can be approximated by its mean value. Using this approximation we derive appropriate consistency conditions and characterize an an ε-PAN set of strategies, for the large number of players case.

The first subsection considers static games and the second Linear Quadratic games.

A. Static Games on Random Graphs

Let us first describe the game for a given interaction structure. The players interact on a graph G. Each player has a type $\theta_i \in [0, L]$ and the cost functions are given by:

$$J_i = g(\theta_i - u^i) + g\left(u^i - \frac{1}{|\mathcal{N}_i(G)|}\sum_{j=1}^{N} d_{ij}u^j\right), \quad (17)$$

where g is a smooth, strictly convex function with $g(0) = g'(0) = 0$ and $d_{ij} = 1$, if there is an edge between vertexes i and j and zero otherwise.

In order to describe the ensemble of games, it remains to determine a stochastic structure on the types θ_i and the graph G. We assume that θ_i are i.i.d. random variables uniformly distributed in $[0, L]$. The graph is an Erdos-Renyi random graph with connection probability c_N, i.e. each edge appears independently of the other edges with probability c_N. We further assume that the random variables d_{ij} and θ_i are mutually independent.

We first focus on strategies depending only on statistical information, assuming no knowledge about the neighbors of each player. The strategies under consideration have the form $u^i = \gamma(\theta_i)$.

A technique to derive strategies in this form is to approximate the terms in the cost function by their mean values. Specifically, we shall use the approximation:

$$\bar{u} \simeq \frac{1}{|\mathcal{N}_i(G)|}\sum_{j=1}^{N} d_{ij}u^j, \quad (18)$$

for all $i = 1, \ldots, N$.

With this approximation, the cost functions depend only on statistical information. The strategies that minimize the approximate cost functions have the form:

$$u^i = h(\theta_i, \bar{u}) = \arg\min_u \{g(u - \theta_i) + g(u - \bar{u})\}. \quad (19)$$

The function h will be shown to be well defined.

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With the strategies given by (19), the mean value of the actions should satisfy the following compatibility condition:

$$\bar{u} = \frac{1}{L}\int_0^L h(\sigma, \bar{u}) d\sigma. \tag{20}$$

The following proposition shows that if connectivity is high, then the strategies described by (19), (20) constitute an ε-PAN equilibrium, for large N. The following proposition uses the asymptotic notation defined in section I-C.

Proposition 1: Under the specified assumptions it holds:

(i) Equation (20) has a unique solution.

(ii) If $c_N \in \omega\left(\frac{\ln N}{\sqrt{N}}\right)$, then the set of strategies (19) is an ε-PAN equilibrium, for large N.

(ii) If $c_N \in \omega\left(\frac{\ln N}{\sqrt{N}}\right)$, then $C_a(m) = 1$ and the ensemble of games is asymptotically simple.

Proof: (i) The strict convexity and the lower boundedness of g imply that the function h is well defined. The function $h(\theta, \bar{u})$ can be expressed as the solution to the following equation:

$$f_\theta(u, \bar{u}) = g'(u - \bar{u}) + g'(u - \theta) = 0,$$

with respect to u. Thus it holds:

$$\min\{\theta, \bar{u}\} \le h(\theta, \bar{u}) \le \max\{\theta, \bar{u}\}. \tag{21}$$

Consider the mapping:

$$\bar{u} \to T\bar{u} = \frac{1}{L}\int_0^L h(\sigma, \bar{u}) d\sigma.$$

Inequalities in (21) imply that $TL \le L$ and $T0 \ge 0$. Thus, due to the intermediate value theorem, there is a $\bar{u}^\star$ such that $\bar{u}^\star = T\bar{u}^\star$.

The derivative of h with respect to $\bar{u}$ can be expressed, using the implicit function theorem as:

$$\begin{aligned}\frac{\partial h}{\partial \bar{u}} &= -\left(\frac{\partial f_\theta}{\partial u}\right)^{-1} \frac{\partial f_\theta}{\partial \bar{u}}\bigg|_{h(\theta,\bar{u}),\bar{u}} = \\ &= \frac{g''(h(\theta,\bar{u}) - \bar{u})}{g''(h(\theta,\bar{u}) - \bar{u}) + g''(h(\theta,\bar{u}) - \theta)} < 1.\end{aligned}$$

Thus, the solution $\bar{u}^\star$ is unique. In what follows, the unique solution of (20) will be denoted by $\bar{u}$.

(ii) The functions g and h are continuous. Using the strategies given by (19), the arguments of the functions belong to compact intervals. In those intervals, g and h are uniformly continuous. Thus, in order to show that the set of strategies given by (19) constitute an ε - PAN equilibrium, for large N, it suffices to show that for every $\varepsilon, \delta > 0$ it holds:

$$P\left(\exists i : |\bar{u} - \frac{1}{|\mathcal{N}_i(G)|}\sum_{j=1}^N d_{ij}u^j| > \delta\right) < \varepsilon,$$

for large N.

It holds:

$$\begin{aligned}\left|\bar{u} - \frac{1}{\mathcal{N}_i(G)}\sum_{j=1}^N d_{ij}u^j\right| &\le \left|\bar{u} - \frac{1}{N}\sum_{j=1}^N h(\theta_j, \bar{u})\right| + \\ &+ \left|\frac{1}{N}\sum_{j=1}^N h(\theta_j,\bar{u}) - \frac{1}{Nc_N}\sum_{j=1}^N d_{ij}u^j\right| + \\ &+ \left|\frac{1}{Nc_N}\sum_{j=1}^N d_{ij}u^j - \frac{1}{|\mathcal{N}_i(G)|}\sum_{j=1}^N d_{ij}u^j\right|.\end{aligned} \tag{22}$$

Furthermore, $\bar{u} = \int_0^L h(\sigma, \bar{u})/L d\sigma$. Due to the Glivenko-Cantelli theorem [44], the empirical distribution, $\sum_{i=1}^N \delta_{\theta_i}/N$ converges a.s. to the uniform distribution as $N \to \infty$. Thus, there exists an integer N_{01}, such that:

$$\left|\bar{u} - \sum_{j=1}^N h(\theta_j, \bar{u})/N\right| < \delta/3,$$

with probability larger than $1 - \varepsilon/3$, for $N \ge N_{01}$.

For the second term of the right hand side of (22) using (19) and Lemma 3 of Appendix A it holds:

$$\left|\frac{1}{Nc_N}\sum_{j=1}^N (c_N - d_{ij}) h(\theta_j, \bar{u})\right| < \delta/3,$$

with probability larger than $1 - \varepsilon/3$, for $N \ge N_{02}$.

Lemma 4 of Appendix A implies that the third term of (22) is less than $\delta/3$ with probability larger than $1 - \varepsilon/3$, if $N \ge N_{03}$.

This completes the proof of (ii).

(iii) An immediate consequence of (ii) □

Remark 5: The proof of Proposition 1 uses only the fact that for given i, the random variables $(d_{ij})_{j=1}^N$ are independent. Thus, the same result holds also for a more general class than the Erdos-Renyi random graph. A particular example is games on random directed graphs[1]. Furthermore, there is no need for all the players to assume the same connection probability c_N or exactly the same random graph model and there is no need for player p_i to know the connection probability that the other players assume. □

Example 2: If the function g has the form $g(z) = z^2$, the strategies given by (19) can be explicitly computed. It holds:

$$h(\theta, \bar{u}) = (\theta + \bar{u})/2. \tag{23}$$

Equation (20) implies that $\bar{u} = L/2$ and the set of strategies given by:

$$u^i = \theta_i/2 + L/4, \tag{24}$$

constitute an ε-PAN equilibrium for large N, if $c_N \in \omega\left(\frac{\ln N}{\sqrt{N}}\right)$. ■

The following proposition studies the case of low connectivity.

[1]In [45] Proposition 1, due to a miscalculation, it is stated that an ensemble of games on a random directed graph is simple if $c_N = \omega\left(\frac{\ln N}{N}\right)$. The correct is that the ensemble is asymptotically simple if $c_N = \omega\left(\frac{\ln N}{\sqrt{N}}\right)$.

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Proposition 2: Consider an integer μ. If $c_N \in o\left(\frac{1}{N^{\mu/(\mu-1)}}\right)$ then $C_a(m) \leq \mu$ and the ensemble of games is asymptotically simple.

Proof: If $c_N \in o\left(\frac{1}{N^{\mu/(\mu-1)}}\right)$ then with probability approaching 1, as $N \to \infty$, the random graph has no connected components having more than μ nodes ([46] ch. 4).

We shall show that there is a Nash equilibrium such that each player uses only the information contained in her connected component. Consider a player p_i and the connected component in the graph which contains i denoted by $\bar{G}_i$. Consider also the game with $\left|\bar{G}_i\right|$ players among the players of $\bar{G}_i$ denoted by $g^{\bar{G}_i}$, assuming that the actions of the players are restricted to belong to $[0, L]$.

Each of the games $g^{\bar{G}_i}$, due to the convexity of the function $g(\cdot)$, satisfy the conditions of Theorem 1 of [47]. Thus, it has a Nash equilibrium. This equilibrium, due to (21), is also a Nash equilibrium for the corresponding game with unrestricted strategies. Thus, with probability approaching 1 as $N \to \infty$, there exists a Nash equilibrium of the original game such that each player uses only the knowledge of her connected component. Thus, $C_a(m) \leq \mu$ and the game is asymptotically simple. □

Remark 6: The upper bound for the complexity function $C_a(m)$ increases as μ increases and $1/N^{\mu/(\mu-1)}$ approaches $1/N$. □

The following proposition deals with the intermediate connectivity case.

Proposition 3: If $c_N \in \omega\left(\frac{1}{N}\right)$ and $c_N \in o\left(\frac{\ln N}{N}\right)$ then the ensemble of games is asymptotically complex.

Proof: Due to the fact that $c_N \in \omega(1/N)$, the maximum degree, i.e the maximum number of edges connected to a node, grows unbounded with N. Thus, it suffices to show that $\bar{n}(m) \geq 1$, for some integer m. To contradict, assume that $\bar{n}(m) = 0$. For an information neighborhood of order 1, the strategies of the players have the form $u^i = \gamma(\theta_i)$.

Due to the fact that $c_N \in o(\ln N/N)$ there exists an isolated node with probability approaching 1 as $N \to \infty$ [46]. In fact, the expected number of such nodes grows unbounded with N.

For such a node the optimal cost is 0. The function $g(\cdot)$ is strictly convex and $g(0) = g'(0) = 0$. Thus, for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ with $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that $g(z) < \varepsilon$ implies $|z| < \delta$.

The assumption $\bar{n}(m) = 0$ implies that $I^i = \{\theta_i\}$ and that:

$$|\gamma(\theta) - \theta| < \delta, \tag{25}$$

if $2^{-m} < \varepsilon$.

It is not difficult to see that with probability approaching 1 as $N \to \infty$ there exists a player p_i such that $\theta_i < L/8$ and $\sum_{j \in \mathcal{N}_i(G)} \gamma(\theta_j) > L/4$. For such a player if m is large enough, a strategy satisfying (25) is not 2^{-m} optimal. This fact contradicts $\bar{n}(m) = 0$. □

The results proved are summarized in the following corollary. The situation is depicted graphically in Figure 2

Corollary 1: If $c_N \in o\left(\frac{1}{N^{\mu}(\mu-1)}\right)$ for some integer μ or if $c_N \in \omega\left(\frac{\ln N}{\sqrt{N}}\right)$ the ensemble of games is asymptotically simple. If $c_N \in \omega\left(\frac{1}{N}\right)$ and $c_N \in o\left(\frac{\ln N}{N}\right)$ then the ensemble of games is asymptotically complex. □

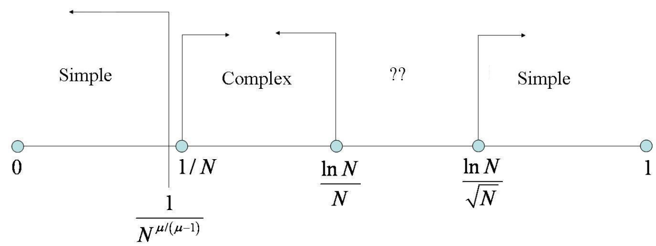

Fig. 2. The complexity of the ensemble of games for the various connectivity intervals.

B. LQ games on Random Graphs

This section describes an opinion dynamics game, involving players having some amount of stubbornness, i.e. the players tend to insist to their initial (intrinsic) opinions (ex. [48]). A large number N of players interact on a random graph $G = (V, E)$ having a connection probability c_N, i.e. each link have a probability to exist equal to c_N, independent of the existence of the other links. The state variable x_k^i of player p_i represents her opinion at time step k and the type θ_i is her initial (intrinsic) opinion, i.e. $x_0^i = \theta_i$. The random variables θ_i are i.i.d. with uniform distribution in $[0, L]$.

Each player has the ability to influence her own opinion in order to come closer to the mean value of her neighbors or closer to her initial opinion. Furthermore, the state variables are influenced by random disturbances. The dynamics is given by:

$$x_{k+1}^i = x_k^i + u_k^i + w_k^i, \tag{26}$$

where u_k^i is the control variable of player p_i and w_k^i are zero mean i.i.d. Gaussian random variables with variance σ^2.

The cost functions are given by:

$$J^i = E\left\{\sum_{k=0}^{\infty} \rho^k \left[\left(x_k^i - \frac{1}{|\mathcal{N}_i(G)|} \sum_{j \in N_i(G)} x_k^j\right)^2 + \right.\right.$$
$$\left.\left. +s\left(x_k^i - \theta_i\right)^2 + r(u_k^i)^2 \right] \middle| \mathcal{F}^s \right\}, \tag{27}$$

where $\rho \in (0, 1)$ is the discount factor, $s \geq 0$ the amount of stubbornness, i.e. how much the players are interested on their initial opinions and r a positive constant.

We then prove the existence of an ε-PAN set of strategies assuming that each player has only statistical information, under high connectivity assumptions. A set of approximate optimal control problems is first stated. For player p_i the approximate optimal control problem is:

Minimize:

$$J^{i,a} = E\left\{\sum_{k=0}^{\infty} \rho^k \left[\left(x_k^i - \bar{\theta}\right)^2 + s\left(x_k^i - \theta_i\right)^2 + r(u_k^i)^2\right]\right\}, \tag{28}$$

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where $\bar{\theta} = E[\theta_j] = L/2$,
Subject to (26).

Using the change of variables:

$$\tilde{x}_k^i = x_k^i - \theta^{i,f}, \tag{29}$$

where $\theta^{i,f} = \frac{\bar{\theta}+s\theta_i}{1+s}$, the optimal control problem becomes the following LQ problem:
Minimize:

$$J^{i,a} = E\left\{\sum_{k=0}^{\infty} \rho^k \left[(1+s)(\tilde{x}_k^i)^2 + r(u_k^i)^2]\right]\right\} + \\ + \frac{s(\bar{\theta}-\theta_i)^2}{(1+s)(1-\rho)}, \tag{30}$$

Subject to:

$$\tilde{x}_{k+1}^i = \tilde{x}_k^i + u_k^i + w_k^i. \tag{31}$$

The control law which minimizes the approximate optimal control problem is given by:

$$u_k^i = -\rho\frac{K}{\rho K + r}\tilde{x}_k^i, \tag{32}$$

where K is the positive solution of the Riccati equation:

$$K = \rho K - \frac{\rho^2 K^2}{\rho K + r} + (1+s). \tag{33}$$

The closed loop dynamic equation for player p_i is given by:

$$\tilde{x}_{k+1}^i = a\tilde{x}_k^i + w_k^i, \tag{34}$$

where $a = 1 - \rho K/(\rho K + r)$ and $a \in (0,1)$.

The following proposition identifies a class of games where the set of strategies described by (32) is ε-PAN.

Proposition 4: Assume that $c_N \in \omega(\frac{\ln N}{\sqrt{N}})$. Then:

(i) The set of strategies given by (32) constitute an ε-PAN set of strategies, for large N.
(ii) It holds $C_a(m) = 1$ and the sequence of the ensembles of games is asymptotically simple.

Proof: See Appendix B □

If the stubbornness of the players is zero, then the long time averages of the opinions of each one of the players reach a consensus.

Proposition 5: If $s = 0$ and the players use the strategies given by (32) then:

$$P(\lim_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T} x_k^i = \bar{\theta}) = 1. \tag{35}$$

Proof: See Appendix C. □

V. Static Games on Organized Structures

A. Quadratic Games on Lattices

This section studies quadratic games on μ-dimensional lattices. It is shown that the ANIC function is polynomial with degree equal to μ.

Consider $N = N_0^\mu$ players placed on a μ-dimensional lattice (hypercube). For each node a set of coordinates $(c_1, \ldots, c_\mu)$ indicates the place of the node in the lattice. Each coordinate satisfies $c_\nu \in \{1, \ldots, N_0\}$. Each player interacts with her immediate neighbors, i.e. a player with coordinates $(c_1, \ldots, c_\mu)$ interacts with every player with coordinates $(c_1, \ldots, c_\nu \pm 1, \ldots, c_\mu)$, for $\nu = 1, \ldots, \mu$. For the nodes on the faces of the hypercube, the convention $N_0 + 1 \equiv 1$ is used.

The type θ_i of each player belongs to $[-L, L]$ and the cost of a player p_i is given by:

$$J^i = a\left(u^i - \frac{1}{2^\mu}\sum_{j\in\mathcal{N}_i(G)} u^j\right)^2 + \left(u^i - \theta_i\right)^2, \tag{36}$$

where a is a positive constant.

In order to describe the ensemble of games it remains to determine a probability structure on the types of the players. We assume that θ_i are i.i.d. random variables with uniform distribution.

Let us consider the following iterative scheme:

$$z^i(t+1) = \frac{a}{(a+1)2^\mu}\sum_{j\in\mathcal{N}_i(G)} z^j(t) + \frac{1}{a+1}\theta_i, \tag{37}$$
$$z^i(0) = 0.$$

Equation (37) corresponds to the best response of player p_i if the other players use $u^j = z^j$.

The following proposition shows that the ANIC of the ensemble is at most polynomial using the fact that the mapping:

$$T : z(t) \mapsto z(t+1), \tag{38}$$

where $z(t) = [z^1(t), \ldots, z^N(t)]$, is a contraction (Lipschitz with a constant less than 1).

Proposition 6: (i) For every $\varepsilon > 0$, there exists an $n \in \mathbb{N}$ such that the set of strategies $u^i = z^i(n)$ constitute an ε-PAN equilibrium.

(ii) The ensemble of games has an ANIC satisfying $C_a(m) \in O(m^\mu)$.

Proof: (i) It holds:

$$J_i(z^i(t), z^{-i}(t)) - \min_u\{J_i(u, z^{-i}(t))\} = \\ = (a+1)(z^i(t) - z^i(t+1))^2 \tag{39}$$

The mapping $T : (\mathbb{R}^N, \|\cdot\|_\infty) \to (\mathbb{R}^N, \|\cdot\|_\infty)$ is contractive, with a Lipschitz constant $a/(a+1)$. Hence,

$$\|z(t+1) - z(t)\| \leq L\left(\frac{a}{a+1}\right)^t.$$

Therefore,

$$(a+1)L^2\left(\frac{a}{a+1}\right)^{2n} < \varepsilon, \tag{40}$$

implies that $u^i = z^i(n)$ is ε-PAN equilibrium.

(ii) At first we prove the following claim:
Claim: If the players use the strategy given by $u^i = z^i(n)$, the actions of each player depend on information contained in her closed neighborhood of order n.

For $n = 1$ the claim holds. Assume that it holds for $n-1$ and consider $z^i(n)$. The value of $z^i(n)$ according to (37) depends on θ_i and $z^j(n-1)$ for $j \in \mathcal{N}_i(G)$. But according to the induction hypothesis, each of the $z^j(n-1)$ depends on the information contained in the closed neighborhood of

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player $j \in \mathcal{N}_i(G)$ of order $n-1$. Thus, $z^i(n)$ depends on the information contained in the neighborhood of player i of order n, which concludes the proof of claim.

Using (40) with $\varepsilon = 2^{-m}$ we have:

$$\bar{n}(m) \leq \frac{m}{2\log_2\left(\frac{a+1}{a}\right)} + \frac{2\log_2 L + \log_2(a+1)}{2\log_2\left(\frac{a+1}{a}\right)}.$$

Furthermore, $|\mathcal{N}_i^{\bar{n}}| < (2\bar{n}+1)^\mu$. Thus, $C_a(m) \in O(m^\mu)$. □

A polynomial lower bound can be also derived.

Proposition 7: The asymptotic complexity function $C_a(m)$ satisfies $C_a(m) \in \Omega(m^\mu)$.

Proof: Consider a game in the ensemble and a set of actions $(u^i)_i$. Due to the contractivity of T, there exists a unique Nash equilibrium of the game. Denote by $z(\infty) = [u^{1,Nash}, \ldots u^{N,Nash}]^T$ that equilibrium.

Let us use (37) with $z^i(0) = u^i$. Due to (39), if $(u^i)_i$ is an ε-Nash equilibrium then:

$$\|z(1) - z(0)\|_\infty < \sqrt{\frac{\varepsilon}{a+1}}.$$

Thus, due to contractivity of T, it holds:

$$\begin{aligned}\|z(0) - z(\infty)\|_\infty &\leq \sum_{t=1}^{\infty} \|z(t) - z(t-1)\|_\infty \\ &= \sum_{t=1}^{\infty} \|T^t(z(1) - z(0))\|_\infty \\ &\leq \|z(1) - z(0)\|_\infty (a+1)\end{aligned}$$

Thus,

$$\max_i\{|u^i - u^{i,Nash}|\} < \sqrt{(a+1)\varepsilon}.$$

The unique Nash equilibrium can be expressed as:

$$u^{i,Nash} = \sum_{c_1',\ldots,c_\mu' \in \mathbb{Z}} b^{c_1',\ldots,c_\mu'} \theta^{c_1+c_1',\ldots,c_\mu'+c_\mu'},$$

where $c_1, \ldots, c_\mu$ are the coordinates corresponding to player p_i. It is not difficult to show that the constants b satisfy:

$$b^{c_1',\ldots,c_\mu'} > \frac{1}{a+1}\lambda^{|c_1'|+\cdots+|c_\mu'|},$$

where $\lambda = \frac{a}{(a+1)2^\mu}$.

Consider now a set of strategies in the form $u^i = \gamma(\bar{I}^{i,n-1})$. Consider a player p_i with coordinates $c_1, \ldots, c_\mu$. Then, with a probability larger than $1/2$ the player p_j with coordinates $c_1+n, \ldots, c_\mu$ has a type $|\theta_j| > L/2$. Thus, with probability larger than $1/4$, it holds $|u^i - u^{i,Nash}| > \lambda^n/(a+1)$.

Therefore, using an information neighborhood of order $n-1$, an $\lambda^{2n}/(a+1)^3$ - PAN equilibrium is not attainable. Hence:

$$\bar{n}(m) > m\frac{\ln 2}{-2\ln\lambda} - \frac{3\ln(a+1)}{-2\ln\lambda}.$$

Thus, $C_a(m) \in \Omega(m^\mu)$. □

Corollary 2: The ANIC function satisfies $C_a(m) \in \Theta(m^\mu)$.

Remark 7: The properties proved do not depend on the assumption that the hypercube has the same length N_0 in all the dimensions or on the nonlocal topological properties of the lattice. □

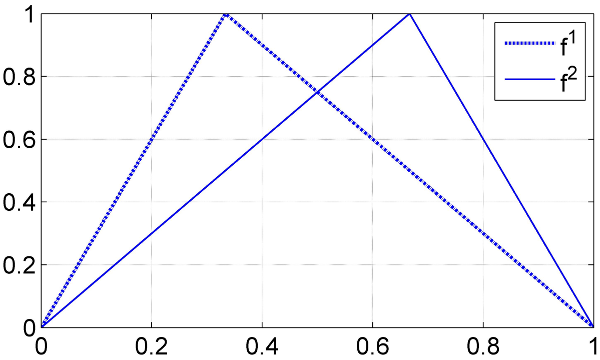

Fig. 3. The plot of f^1 and f^2

The result about the upper bound of the ANIC function can be generalized to ensembles of games on graphs with known maximum degree, using exactly the same arguments.

Proposition 8: Consider an ensemble of games with cost function (36) and interaction graphs which with probability one have maximum degree less than μ. Then, $C_a(\mu) \in O(m^\mu)$.

Proof: The proof uses essentially the same contraction arguments with Proposition 6. □

The bound of Proposition 6 is much sharper than the bound of Proposition 8, when applied to a Lattice, due to the fact that the Lattice is highly clustered.

B. A non-Quadratic Game on a Ring

In this subsection we study an example of an ensemble of games, where a PAN equilibrium can be obtained, using some form of cooperation among the players. The best response maps in this example are chaotic. Let us note that the use of chaotic maps is not unusual in the modeling of erratic behavior in economics (ex. [49]).

There are N players interacting on a ring. The type of each player has the form $\theta_i = (\xi_i, i)$ and ξ_i has two possible values 1 and 2. The cost function of each player, except player p_0, is given by:

$$J_i = (u^i - f^{\xi_i}(u^{i-1}))^2. \tag{41}$$

The functions f^1 and f^2 have the form:

$$f^i(z) = \begin{cases} z/a_i & \text{if } 0 \leq z < a_i \\ \frac{1-z}{1-a_i} & \text{if } a_i < z \leq 1 \end{cases},$$

where $a_1 = 1/3$ and $a_2 = 2/3$. The functions f^1 and f^2 are variants of the tent map [50] and their plots are shown in figure 3. For the player p_0 the cost is:

$$J_0 = (u^0 - f^{\xi_0}(u^N))^2 + (u_0 - 1)^2. \tag{42}$$

In order to describe the ensemble it remains to determine a stochastic structure on ξ_i. We assume that ξ_i are i.i.d. random variables, taking values 1 and 2 with equal probabilities.

The best response map for player p_i is given by $u^i = f^{\xi_i}(u^{i-1})$. This map is not contractive. In fact it is chaotic. However, the following proposition shows that ANIC is at most linear.

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Proposition 9: The ANIC function of the ensemble of games satisfies $C_a(m) \in O(m)$.

Proof: The proof is constructive. It is not difficult to show that, for every positive integer μ, every finite sequence $s_1, \ldots, s_\mu$ $s_j \in \{1,2\}$ and every $z \in [0,1]$ there exists a $\bar{z} \in [0,1]$ such that $|z-\bar{z}| \leq \frac{1}{2}\left(\frac{2}{3}\right)^\mu$ and $f^{s_1} \circ \cdots \circ f^{s_\mu}(\bar{z}) = 0.5$. Let us denote by $h_\mu(s_0, \ldots, s_\mu, z)$ the minimal such point $\bar{z} \in [0,1]$.

In order to construct the strategies of the players we consider distinct cases. For a player p_i, $i \notin \{0, (\mu-1)[N/\mu], \mu[N/\mu]\}$ such that $i \equiv 0(mod \mu)$:

$$u^i = h_\mu(\xi_i, \ldots, \xi_{i+\mu-1}, 0.5). \quad (43)$$

For the player $p_{(\mu-1)[N/\mu]}$:

$$u^{(\mu-1)[N/\mu]} = h_{N-(\mu-1)[N/\mu]}(\xi_i, \ldots, \xi_{i+\mu-1}, 0.5). \quad (44)$$

For the players p_i, such that $i \not\equiv 0(mod\mu), i < (\mu-1)[N/\mu]$:

$$u^i = f^{\xi_i}(f^{\xi_{i-1}}(\ldots f^{\mu[i/\mu]}(0.5))). \quad (45)$$

For the players $p_{(\mu-1)[N/\mu]+1}, \ldots, p_{N-1}$:

$$u^i = f^{\xi_i}(f^{\xi_{i-1}}(\ldots f^{(\mu-1)[N/\mu]}(0.5))). \quad (46)$$

For the player p_0:

$$u^i = h_\mu(\xi_0, \ldots, \xi_{\mu-1}, 0.75). \quad (47)$$

It is not difficult to see that the set of strategies (43)- (47) constitute an $\frac{1}{4}\left(\frac{4}{9}\right)^\mu$-PAN set of strategies. Thus, $C_a(m) \leq m$ and the proof is complete. □

Remark 8: The game has a lot of Nash equilibria. In a Nash equilibrium, the following equations hold:

$$u^0 = \frac{1}{2} + \frac{f^{\xi_0}(f^{\xi_{N-1}}(\ldots(f^{\xi_1}(u^0))))}{2} \quad (48)$$

$$u^i = f^{\xi_i}(u^{i-1}). \quad (49)$$

Equation (48) has approximately N/2 solutions. The strategies of the players in every Nash equilibrium are given by (49). Thus, a full knowledge of the information is needed. The ε-PAN set of strategies, described in the proof of Proposition 9 is not close to any one of the Nash equilibria. □

Remark 9: The ε-PAN set of strategies is in some sense cooperative. Particularly, a player p_i, $i = l\mu$ can improve her own performance based on her own information. However, these agents help the others have a predictable best response with local information only and thus to behave optimally. If such a player changes her action to the optimal response, then she would expect that the other players would also use their best responses. Due to the chaoticity of the maps f^1, f^2, we would expect that for a long amount of time the best responses would not converge. This would make the situation worse for these players.

The application of the best response maps is illustrated in the following example. □

Example 3: In this example the repeated version of the game is studied, assuming that the players use a best response dynamic rule and that $N = 20$. The actions of the players p_1, p_4, p_9 and p_{14} are shown in Figure 4. This example shows that the dynamic rule based on best response fails to converge. □

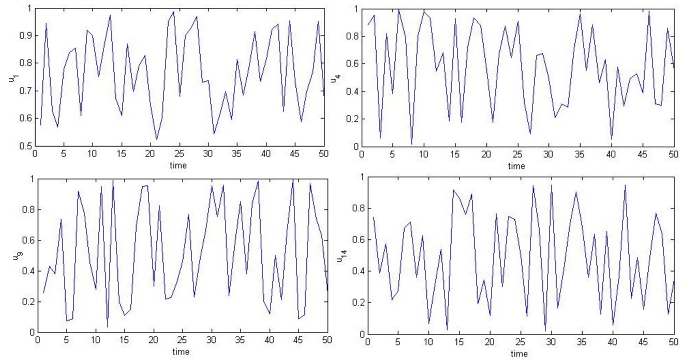

Fig. 4. The evolution of the actions of players p_1, p_4, p_9 and p_{14}, in Example 3.

VI. A Numerical Study of a LQ Game on a Ring

In this section we study numerically LQ games on large rings. We assume that there are N players placed on a ring, such that each player p_i interacts with the players p_{i+1} and p_{i-1}. The convention $N+1 \equiv 1$ is used as before. Each one of the players has her own dynamic equation given by:

$$x_{k+1}^i = x_k^i + \lambda(x_k^{i-1} + x_k^{i+1}) + u_k^i, \quad (50)$$

where λ is a constant describing the coupling through the state equation and x_0^i are zero mean i.i.d. random variables with variance equal to 1. The cost functions are given by:

$$J_i = \sum_{k=1}^{\infty}\left[(x_k^i)^2 + s\left(x_k^i - \frac{x_k^{i+1} + x_k^{i-1}}{2}\right)^2 + r(u_k^i)^2\right], \quad (51)$$

where s denotes the coupling through the cost functions.

Assume that the players have information neighborhoods of order n. A very simple technique is used to obtain approximate equilibrium policies:

Step 1: Each player p_i considers a reduced ring game with $2n+1$ players, i.e. the players $p_{i-n}, \ldots p_{i+n}$. That is, she assumes the existence of an edge between players p_{i-n} and p_{i+n}. Figure 5 shows the reduced game that player p_1 perceives in case where $n = 2$.

Step 2: Player p_i computes a Nash equilibrium of the reduced game. Let us denote by $\gamma^{i,n}$ the Nash strategy of the player i in the reduced ($2n+1$-player) game.

Step 3: Apply $\gamma^{i,n}$ in the original N players game.

We then compare the value of the cost function when the players use the strategies $\gamma^{i,n}$ with the best response within the class of CLPI strategies. Figures 6, 7 and 8 illustrate the difference $J^i(\gamma^{i,n}, (\gamma^{j,n})_{j\neq i}) - \min_\gamma J^i(\gamma, (\gamma^{j,n})_{j\neq i})$ in logarithmic scale for different values of λ and s. The simulation was performed with $r = 1$ and $N = 101$.

Remark 10: Figures 6-8 illustrate that the complexity is approximately at most linear. The reason for the low complexity in the current example is that the Nash strategies of the reduced order games $\gamma^{i,n}$ assign low gains to distant players. Note that none of the reasons for low complexity existing in the previous examples, is present. That is, this low complexity is not due to a law of large numbers, the best

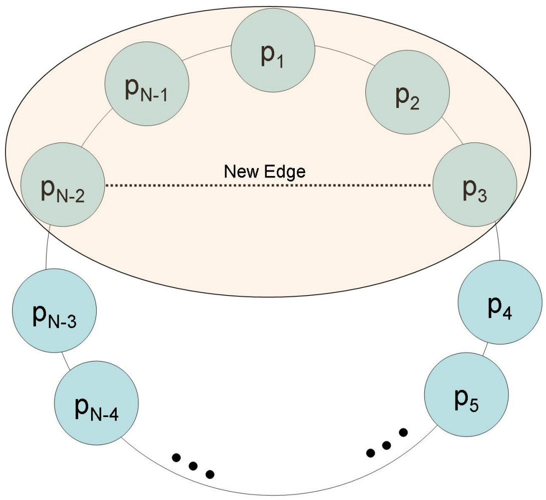

Fig. 5. Player p_1 considers only the players $p_2, p_3, p_{N-1}, p_{N-2}$ and himself and assumes the existence of a new edge between players p_3 and p_{N-2}.

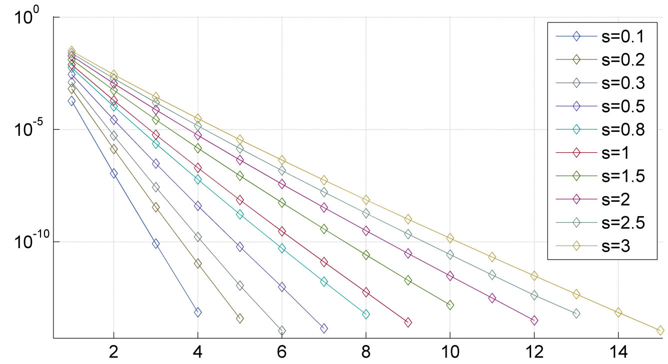

Fig. 6. The difference $J^i(\gamma^{i,n}, (\gamma^{j,n})_{j\neq i}) - \min_\gamma J^i(\gamma, (\gamma^{j,n})_{j\neq i})$, as a function of n, for $\lambda = 0$.

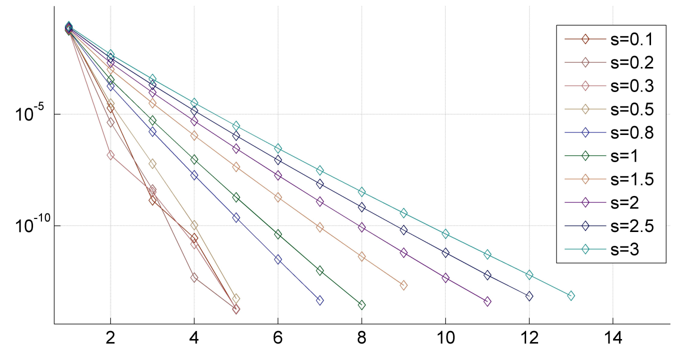

Fig. 7. The difference $J^i(\gamma^{i,n}, (\gamma^{j,n})_{j\neq i}) - \min_\gamma J^i(\gamma, (\gamma^{j,n})_{j\neq i})$ as a function of n, for $\lambda = 0.2$.

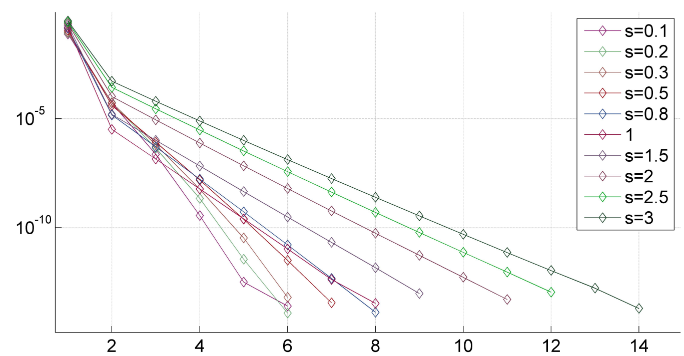

Fig. 8. The difference $J^i(\gamma^{i,n}, (\gamma^{j,n})_{j\neq i}) - \min_\gamma J^i(\gamma, (\gamma^{j,n})_{j\neq i})$ as a function of n, for $\lambda = -0.2$.

response maps are not contractive and there is no cooperation among the players. $\square$

VII. Conclusion

In this article we presented a Probabilistic Approximate Nash equilibrium notion for the study of static and dynamic games on Large Networks of interacting agents. We further introduced the NIC and ANIC complexity functions to quantify the minimum amount of information required for the existence of a PAN equilibrium. Several special cases of static and dynamic games on Erdos-Renyi random graphs and Lattices were studied. In the cases analyzed, we derived conditions for low complexity and identified classes of games with high informational complexity. The basic techniques to prove low complexity and find PAN equilibria were concentration of probability, contractivity of the best response maps, cooperation among the players and reduction to simpler games.

Future work includes the study of games on Small World and Scale Free networks, as well as the study of games with special cost structures (ex. supermodularity).

Appendix

A. Some Probability Inequalities

The following results will be repeatedly used throughout the proof of Propositions 1 and 4.

Theorem 1 (Bernstein Inequality): Let $X_1, \dots, X_N$ be zero mean, independent random variables such that $|X_i| \leq M$. Denoting by $\bar{\sigma}^2 = \frac{1}{N}\sum_{i=1}^N Var\{X_i\}$, for for all positive t it holds:

$$P\left(\left|\frac{1}{N}\sum_{i=1}^N X_i\right| \geq t\right) \leq 2\exp\left(-\frac{Nt^2}{2\bar{\sigma}^2 + 2Mt/3}\right) \quad (52)$$

Lemma 2: For a finite or infinite sequence of events $A_1, A_2, \dots$ it holds:

$$P(\bigcap_{i=1,2,\dots} A_i) \geq 1 - \sum_{i=1,2,\dots} P(A_i^c).$$

Theorem 1, called Bernstein inequality, falls into the class of concentration inequalities [51]. Lemma 2 has an immediate proof.

Lemma 3: Let X_{ij}, $i, j \in \mathbb{N}$ be a set of zero mean random variables absolutely bounded by a constant M. Assume that $X_{i1}, X_{i2}, \dots$ are mutually independent. Assume also that c_N satisfies $c_N \in \omega(\sqrt{\ln N/N})$ and consider an $\varepsilon_1 > 0$. Then, it holds:

$$P\left(\left|\frac{1}{Nc_N}\sum_{j=1}^N X_{ij}\right| < \varepsilon_1, \text{ for every } i = 1, 2, \dots\right) > 1 - \varepsilon_1,$$

for large N.

Proof: Applying Bernstein inequality with $t = \varepsilon_1 c_N$, we

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obtain:

$$P\left(\left|\frac{1}{Nc_N}\sum_{j=1}^{N}X_{ij}\right|>\varepsilon_1,\right)\leq 2\exp\left(-\frac{Nc_N^2\varepsilon_1^2}{2M^2+2Mc_N\varepsilon_1/3}\right)$$

Applying Lemma (2) we have:

$$P\left(\left|\frac{1}{Nc_N}\sum_{j=1}^{N}X_{ij}\right|<\varepsilon_1,\text{ for every } i=1,2,\dots\right)\geq$$
$$\geq 1-2N\exp\left(-\frac{Nc_N^2\varepsilon_1^2}{2M^2+2Mc_N\varepsilon_1/3}\right)$$

The fact that $c_N^2 \gg \ln N/N$ completes the proof □

Lemma 4: Consider an Erdos-Renyi random graph with connection probability c_N. Let d_{ij} be a random variable with $d_{ij}=1$ if there is an edge between i and j and $d_{ij}=0$ otherwise. Then, if $c_N \in \omega(\ln N/\sqrt{N})$, for every given $\delta_1,\varepsilon_1>0$ it holds:

$$P\left(\left|\frac{Nc_N}{|\mathcal{N}_i(G)|}-1\right|<\delta_1,\text{ for every } i\right)>1-\varepsilon_1,$$

for large N.

Proof: For every $\delta_1>0$ there is an $\delta_2=\delta_2(\delta_1)>0$ such that: $\left|\frac{|\mathcal{N}_i(G)|}{Nc_N}-1\right|<\delta_2$ implies $\left|\frac{Nc_N}{|\mathcal{N}_i(G)|}-1\right|<\delta_1$. Furthermore,

$$\frac{|\mathcal{N}_i(G)|}{Nc_N}-1=\frac{1}{Nc_N}\sum_{j=1}^{N}(d_{ij}-c_N).$$

Applying Lemma 3 with $X_{ij}=d_{ij}-c_N$ we conclude to the desired result. □

B. Proof of Proposition 4

(i) It holds:

$$J^i\leq J^{i,a}+E\left\{\sum_{k=0}^{\infty}\rho^k\left[\left(\bar{\theta}-\frac{1}{|N_i(G)|}\sum_{j\in N_i(G)}x_k^j\right)^2\right]\middle|\mathcal{F}^s\right\} \quad (53)$$

Using the triangle inequality and simple manipulations, the second term of the right hand side of (53) is be bounded above by the sum of the following terms:

$$E\left[\sum_{k=0}^{\infty}\rho^k\left[\bar{\theta}-\frac{1}{N}\sum_{j=1}^{N}(\theta^{j,f}+a^k\tilde{x}_0^j)\right]^2\middle|\mathcal{F}^s\right] \quad (54)$$

$$E\left[\sum_{k=0}^{\infty}\rho^k\left[\frac{1}{N}\sum_{j=1}^{N}x_k^j-\frac{1}{N}\sum_{j=1}^{N}(\theta^{j,f}+a^k\tilde{x}_0^j)\right]^2\middle|\mathcal{F}^s\right] \quad (55)$$

$$E\left[\sum_{k=0}^{\infty}\rho^k\left[\frac{1}{Nc_N}\sum_{j=1}^{N}(d_{ij}-c_N)x_k^j\right]^2\middle|\mathcal{F}^s\right] \quad (56)$$

$$E\left[\sum_{k=0}^{\infty}\rho^k\left[\left(\frac{Nc_N}{|\mathcal{N}_i(G)|}-1\right)\frac{1}{N}\sum_{j=1}^{N}x_k^j\right]^2\middle|\mathcal{F}^s\right] \quad (57)$$

We shall show that the expressions (54) - (57) are small for all $i=1,\dots,N$ with high probability, if N is large enough. In the expressions (54) - (57), the expectation and the summation over k are interchangeable due to Bepo Levi theorem [44]. The terms (54) and (55) are common among the players.

1) The term (54)*:* It holds:

$$\bar{\theta}-\frac{1}{N}\sum_{j=1}^{N}(\theta^{j,f}+a^k\tilde{x}_0^j)=\frac{s+a^k}{1+s}\left[\bar{\theta}-\frac{1}{N}\sum_{j=1}^{N}\theta_j\right]$$

Thus the term (54) is bounded by:

$$\frac{1}{1-\rho}\left[\bar{\theta}-\frac{1}{N}\sum_{j=1}^{N}\theta_j\right]^2,$$

which due to the weak law of large numbers ([44]) is smaller than $\bar{\varepsilon}$ with probability larger than $1-\bar{\varepsilon}$.

2) The term (55)*:* Fixing k, it holds:

$$X_k=\frac{1}{N}\sum_{j=1}^{N}(x_k^j-\theta^{j,f}-a^k\tilde{x}_0^j)=\frac{1}{N}\sum_{j=1}^{N}\sum_{t=0}^{k}a^t w_{k-t-1}^j$$

Due to independence we have $E[X_k^2]\leq\frac{\sigma^2}{N(1-a)}$. Thus,

$$\sum_{k=0}^{\infty}E[X_k^2]\leq\frac{\sigma^2}{N(1-a)(1-\rho)}.$$

Hence, term (55) is less than $\bar{\varepsilon}$ for large N.

3) The term (56)*:* It holds:

$$x_k^j=a^kx_0^j+\theta^{j,f}+\sum_{t=0}^{k-1}a^tw_{k-t-1}^j. \quad (58)$$

Denoting by $X^{ij}=\frac{d_{ij}-c_N}{Nc_N}$, $Y_k^j=a^k(\theta_j-\bar{\theta})-\frac{\bar{\theta}+s\theta_j}{1+s}$ and $\xi_k^j=\sum_{t=0}^{k-1}a^tw_{k-t-1}^j$, for the term (56) we have:

$$\sum_{k=0}^{\infty}\rho^kE\left[\left[\frac{1}{Nc_N}\sum_{j=1}^{N}(d_{ij}-c_N)x_k^j\right]^2\middle|\mathcal{F}^s\right]=$$
$$=\sum_{k=0}^{\infty}\rho^kE\left[\left[\sum_{j=1}^{N}X^{ij}(Y_k^j+\xi_k^j)\right]^2\middle|\mathcal{F}^s\right] \quad (59)$$

The random variables X^{ij} and Y_k^j are $\mathcal{F}^s$ measurable and ξ_k^j are zero mean and independent for every given fixed k. Thus,

$$E\left[\left[\sum_{j=1}^{N}X^{ij}(Y_k^j+\xi_k^j)\right]^2\middle|\mathcal{F}^s\right]=$$
$$=\left[\sum_{j=1}^{N}X^{ij}Y_k^j\right]^2+\sum_{j=1}^{N}\left[(X^{i,j})^2E\left[(\xi_k^j)^2\middle|\mathcal{F}^s\right]\right]\leq$$
$$\leq\left[\sum_{j=1}^{N}X^{ij}Y_k^j\right]^2+\frac{\sigma^2}{1-a^2}\sum_{j=1}^{N}\left[(X^{i,j})^2\right] \quad (60)$$

Given a fixed k, applying Lemma 3 to the set of random variables $Nc_N X^{ij} Y_k^j$ with $\varepsilon_1 = (1-\rho)\bar{\varepsilon}/2$ we have:

$$P\left(\left[\sum_{j=1}^{N} X^{ij} Y_k^j\right]^2 < ((1-\rho)\bar{\varepsilon}/2)^2 < (1-\rho)\bar{\varepsilon}/2\right) > \\ > 1-(1-\rho)\bar{\varepsilon}/2 > 1-\bar{\varepsilon}/2.$$

Using the fact that $|Nc_N X^{ij}| \leq 1$ and $c_N \in \omega(1/\sqrt{N})$, the last term of the right hand side of the inequality (60) is smaller than $\bar{\varepsilon}/2$ with probability 1 for large N.

Thus, due to (59), the term (56) is less than $\bar{\varepsilon}$ with probability larger than $1-\bar{\varepsilon}$.

4) The term (57)*:* Using (58) we have:

$$E\left[\sum_{k=0}^{\infty} \rho^k \left[\frac{1}{N}\sum_{j=1}^{N} x_k^j\right]^2 \middle| \mathcal{F}^s\right] \leq \frac{1}{1-\rho}\left[4L^2 + \frac{\sigma^2}{N(1-a^2)}\right].$$

The right hand side of the inequality is less than $5L^2/(1-\rho)$, for large N.

Using Lemma 4 and the fact that $\left(\frac{Nc_N}{|\mathcal{N}_i(G)|} - 1\right)^2$ is $\mathcal{F}^s$ measurable, we conclude that the term (57) is less than $\bar{\varepsilon}$ with probability larger than $1-\bar{\varepsilon}$, for large N.

The choice $\bar{\varepsilon} = \varepsilon/4$ completes the proof.

(ii) Immediate. ■

C. Proof of Proposition 5

For $i = 1, \ldots, N$, it holds $\theta^{i,f} = \bar{\theta}$. The random variable $\sum_{k=0}^{T} a^k x_0^i / T$ converges to 0 almost surely. Thus, due to (58), it remains to show that the sequence of random variables:

$$X_T^i = \frac{1}{T}\sum_{k=1}^{T}\sum_{t=0}^{k-1} a^t w_{k-t-1}^i$$

converges almost surely to 0. The random variable X_T^i can be written as:

$$X_T^i = \frac{1}{T}\sum_{\nu=0}^{T-1}(1 + a + \cdots + a^{T-1-\nu}) w_\nu^i.$$

Hence, X_T^i is zero mean and Gaussian satisfying $Var(X_T^i) < \frac{\sigma^2}{(1-a)}\frac{1}{T} = \sigma_T^2$. Thus,

$$P(|X_T^i| > 1/l) < \exp\left(-1/(l\sigma_T)^2\right).$$

Denote by $B_{T,l} = \{\omega \in \Omega : |X_T^i| > 1/l\}$. It is not difficult to see that $\sum_{T=1}^{\infty} P(B_{T,l}) < \infty$. Therefore, using the 1st Borel-Cantelli Lemma [44], we have: $P(\limsup_T B_{T,l}) = 0$. Hence,

$$P\left(\{\omega : X_T^i \to 0\}^c\right) = P\left(\bigcup_{l=1}^{\infty} \limsup_T B_{T,l}\right) = 0.$$

Thus, $X_T^i \to 0$ almost surely and the proof is completed. ■

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