On Stability and LQ Control of MJLS With a Markov Chain With General State Space

Ioannis Kordonis and George P. Papavassilopoulos

Abstract—**We study the mean square stability and the LQ control of discrete time Markov Jump Linear Systems where the Markov chain has a general state space. The mean square stability is characterized by the spectral radius of an operator describing the evolution of the second moment of the state vector. Two equivalent tests for the mean square stability are obtained based on the existence of a positive definite solution to a Lyapunov equation and a uniformity result respectively. An algorithm for testing the mean square stability is also developed based on the uniformity result. The finite and infinite horizon LQ problems are considered and their solutions are characterized by appropriate Riccati integral equations. An application to Networked Control Systems (NCS) is finally presented and a simple example is studied via simulation.**

Index Terms—**Markov jump linear systems, stochastic optimal control, stochastic stability.**

I. Introduction

MARKOV jump linear systems (MJLS) are linear dynamic models where the matrices describing the evolution of the state vector, depend on the state of a Markov chain. The existing results on the stability and LQ control of MJLS are dealing with Markov chains having a discrete state space, i.e., finite or countably infinite. However, in several applications, the natural choice for the state space of the Markov chain is not discrete. Some examples of Markov chains with continuous state space are given in [1, Ch. 1 and 2]. In this technical note, we extend the stability analysis and the LQ control of discrete time MJLS to a more general state space Markov chain case, including the continuous state space case.

An example of such applications is the study of systems with dependent random communication delays, such as Networked Control Systems [2], [3]. The amount of time delay stands for the state variable of the Markov chain. A natural choice for the Markov chain state space is an uncountable subset of the real numbers, such as a closed interval. Another example could be a dynamic linear economic model, having coefficients depending on the price of some asset traded in a stock market. The price of a stock market is usually modeled as a geometric Brownian motion [4]. The state space of this Markov chain, is the positive real numbers. Examples of MJLS with continuous state space also arise in the study of gain scheduling control of nonlinear systems with a Markovian desired trajectory or a Markovian measurable disturbance. More generally, several systems have been modeled as linear uncertain systems, where the matrices describing the evolution of the state vector, belong to a compact polyhedron [5]. Assuming that there exists a Markovian model for the uncertainty, design problems involving MJLS with continuous state space Markov chain will be obtained and will lead to less conservative stability conditions. Another example, which also motivates the current work, comes from the optimal control problems, arising in the mean field approximation of LQ games involving a large number of randomly entering players [6], where the players are considered to belong to a continuum.

The first work studying the mean square stability of MJLS with finite state space Markov chain was [7]. Other contributions include [8] where the stochastic additive case was studied and [9]–[11] where necessary and sufficient conditions involving Lyapunov equations were derived. The mean square stability for MJLS with a countably infinite Markov chain was studied in [12], using an operator theoretic point of view. Some problems related to the mean square stability of MJLS with general state space were studied in [13], [14] under some ergodicity assumptions. The relation among several notions of stochastic stability for MJLS was studied in [11].

The first work studying LQ control problems related to MJLS was in a continuous time setting [15]. A lot of work has been done on the Linear Quadratic control of the discrete time MJLS as well. The finite horizon LQ control problem for the finite state space Markov chain case was solved in [16] and its infinite horizon counterpart in [17]. The existence of a solution was first studied using controllability notions in [18] and testable conditions were derived in [9]. A related work with infinite horizon ergodic criterion and safety constraints is [19]. Filtering problems for MJLS are studied in [20] and a review of several results is given in the books [21] and [22]. The LQ control problem, for a system involving a Markov chain with countably infinite state space was studied in [12].

The contribution of this work is twofold. The first part, is the study of the mean square stability of MJLS when the Markov chain has a general state space. The mean square exponential stability notion is characterized by the spectral radius of a certain operator. Then, testable equivalent conditions are derived based on the operator theoretic result. The second part of the contribution of this work is the extension of the solution of finite and infinite horizon LQ problems to MJLS with general state space. The basic difference between the current work and the literature is that the techniques applied for the stability analysis of MJLS with discrete state space could, not be directly extended to the continuous or general state space. In a comparison to older results, a more general class of models could be analyzed. Examples of models of Markov chains with general state spaces could be found in [1]

The technical note uses the following notation. The probability is denoted by $Pr(\cdot)$ and the expectation by $E[\cdot]$. The value of the Markov chain is denoted by y_k and its state space is the metric space D. Denote by $\mathcal{B}(D)$ the σ-algebra of Borelian subsets of D. The evolution y_k is described by the notion of stochastic kernel, i.e., a function $\bar{K}(\cdot,\cdot) : D \times \mathcal{B}(D) \rightarrow [0,1]$ such that $Pr(y_{k+1} \in B | y_k = y) = \bar{K}(y, B)$. A matrix function $Q : D \rightarrow \mathbf{R}^{n\times n}$ will be called strictly positive definite, if there exists a positive constant c such that $Q(y) > cI$, for any $y \in D$. Finally, the spectral radius of an operator T is denoted by $r(T)$.

II. Problem Description

The system under consideration is the following:

$$x_{k+1} = A(y_k)x_k + B(y_k)u_k + w_k \tag{1}$$

where $x_k \in \mathbf{R}^n$ is the state vector, $u_k \in \mathbf{R}^m$ is the control input, $A(\cdot)$ and $B(\cdot)$ are Borel measurable, bounded matrix functions of appro-

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The authors are with the National Technical University of Athens, School of Electrical and Computer Engineering, Athens 157 80, Greece (e-mail: jkordonis1920@yahoo.com; yorgos@netmode.ntua.gr).

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priate dimensions and w_k are zero mean i.i.d. random variables with finite second moments.

Two types of problems are considered. The first type is the stability problem and it is stated as follows:

Stability problem: "Under which conditions the free system

$$x_{k+1} = A(y_k)x_k \tag{2}$$

is stable". The notions of stochastic stability that we study in the current work are given in the following definition.

Definition 1: The system given by (2) is:

(i) Mean square stable, if $E[x_k^T x_k] \to 0$ for any x_0, y_0 random variables, such that $E[x_0^T x_0] < \infty$.

(ii) Mean square exponentially stable, if for any x_0, y_0 random variables, such that $E[x_0^T x_0] < \infty$, there exist constants $r \in (0,1)$ and $M > 0$ such that it holds $E[x_k^T x_k] < Mr^k$, for any positive integer k.

(iii) Stochastically mean square stable, if for any x_0, y_0 random variables, such that $E[x_0^T x_0] < \infty$, it holds $\sum_{k=0}^{\infty} x_k^T x_k < \infty$.

The second type of problems considered is the LQ control problems. These problems are stated in a finite or an infinite horizon setting as follows:

Finite Horizon LQ Control Problem: "Find the control law $u_k = \gamma(x_k, y_k, k)$, that minimizes the LQ criterion

$$E\left[x_{N+1}^T Q_{N+1}(y_{N+1})x_{N+1} + \sum_{k=0}^{N}\left(x_k^T Q_k(y_k)x_k + u_k^T R_k(y_k)u_k\right)\right]\text{''}. \tag{3}$$

Infinite Horizon LQ Control Problem: "Find the control law $u_k = \gamma(x_k, y_k)$ (if any), that minimizes the LQ criterion:

$$E\left[\sum_{k=0}^{\infty} a^k \left(x_k^T Q(y_k)x_k + u_k^T R(y_k)u_k\right)\right]\text{''}. \tag{4}$$

The standard assumptions are made on the matrices involved in the cost functions. More specifically, we assume that Q_k and Q are positive semidefinite, bounded matrix functions and R_k and R are strictly positive definite bounded matrix functions. For the discount factor it holds $a \in (0,1)$.

III. Stability Analysis

In order to examine the mean square stability of a system in the form (2), let us introduce the following quantity:

$$P_k(C) = E[x_k x_k^T \chi_{y_k \in C}] \tag{5}$$

for C any Borelian subset of D. For any k, $P_k(\cdot)$ is a set function $P_k : \mathcal{B}(D) \to \mathbf{R}^{n\times n}$. It will be shown that $P_k(\cdot)$ is a symmetric matrix of signed measures.

The stability analysis is based on the evolution of the quantity $P_k(\cdot)$. Thus, we introduce the space in which $P_k(\cdot)$ belongs. Denote by X' the space of signed measures on $(D, \mathcal{B}(D))$ and by $|\cdot|$ the total variation norm. Then, the space of symmetric matrices of signed measures is defined as $X = \prod_{j=1}^{n(n+1)/2} X'$. Let us introduce on X, the norm $||\cdot||$, where $||[\mu_{ij}]|| = \sum_{i=1}^{n}\sum_{j=1}^{i} |\mu_{ij}|$, i.e., the sum of total variations. With this norm, X becomes a Banach space.

The evolution of $P_k(\cdot)$ is described using a linear operator $T = T_{A,\bar{K}} : X \to X$. It will be shown that T has the form

$$(TP)(B) = \sum_{l=1}^{n}\sum_{m=1}^{n}\int_D \bar{A}_{l,m}\bar{K}(y,B)P^{l,m}(dy) \tag{6}$$

where $P^{l,m}$ is the l,m element of P and the formulae for the matrices $\bar{A}_{l,m}(y)$ are given in the proof of Theorem 1. The stability properties for a system in the form (2) depend on the spectral radius $r(T)$ ([23]) of the operator T as, shown in the following theorem.

Theorem 1: The quantity $P_k(\cdot)$ is a symmetric matrix of signed measures and its evolution is given by (6). Furthermore, the following hold:

(i) The system is mean square exponentially stable if $r(T) < 1$.

(ii) If $r(T) > 1$ then the system is not mean square stable.

(iii) The system is mean square stochastically stable if and only if $r(T) < 1$.

(iv) The system is mean square exponentially stable then $r(T) < 1$.

Proof: See Appendix. ■

The conditions of Theorem 1 involve the spectral radius of an infinite dimensional operator and thus, they are not easy to check. Based on Theorem 1, we are going to study further properties of exponentially stable systems that will allow us to obtain conditions that are easier to check.

The next proposition studies the uniformity of the exponential mean square stability on the initial conditions. Particularly, it is shown that in an exponential mean square stabile system, $E[x_k^T x_k]$ is going to be small in a finite number of steps irrespectively of the initial condition y_0. It is also shown that the converse is true.

Proposition 1: The following are equivalent:

(i) The system is exponentially mean square stable.

(ii) There exist a positive constant $M > 0$ and $a \in (0,1)$ such that $E[x_k^T x_k] < Ma^k E[x_0^T x_0]$ for any x_0, y_0 random variables.

(iii) There exist an $a \in (0,1)$ and a positive integer k_0 such that $E[x_{k_0}^T x_{k_0}] < a x_0^T x_0$, for any x_0, y_0 non-random initial conditions.

Proof: See Appendix. ■

Condition (ii) of Proposition 1 has a stronger formulation than the definition of the mean square exponential stability, because the constants M and a are independent of the initial conditions, i.e., the exponential convergence to 0 is uniform on the initial conditions. Part (iii) of Proposition 1 shows that the mean square exponential stability is equivalent to the fact that the function $V(x) = x^T x$ is a "k_0-step Lyapunov function" for (2). Furthermore, Proposition 1, shows that a system which is not mean square exponentially stable could not be uniformly mean square stable.

This result also leads to a computational test for mean square exponential stability. The following Algorithm uses recursive computations to decide if $E[x_k^T x_k] < a x_0^T x_0$, for any x_0, y_0 non-random initial conditions.

Algorithm 1: Stability Test

S1 Set $L_{New}(y) = I$ and $Cnt = 1$.

S2 Set $L(y) = L_{New}(y)$.

S3 Compute:

$$L_{New}(y) = \int A^T(y')L(y')A(y')K(y,dy'). \tag{7}$$

S4 If $L_{New} < I$ for any $y \in D$ then return "The system is exponentially mean square stable" and halt.

S5 Set $Cnt = Cnt + 1$

S6 If $Cnt > MaxCnt$ then halt. Else go to Step 2.

Formula (7), computes recursively the matrix $E[A^T(y_0)\dots A^T(y_{k_0-1})A(y_{k_0-1})\dots A(y_0)]$ and thus Algorithm 1, is valid due to Proposition 1 (iii).

An alternative way to deal with the stability problem is to study the system using "one step quadratic Lyapunov functions" of the form $V(x,y) = x^T M(y)x$. In Proposition 2 the exponential mean square stability is proved to be equivalent to the existence of a positive definite solution to a Lyapunov equation.

Proposition 2: Consider a strictly positive definite matrix function $Q(y)$. The following are equivalent:

(i) The system is exponentially mean square stable

(ii) There exists a bounded, strictly positive definite matrix function $M(y)$ that satisfies the Lyapunov equation:

$$A^T(y)E[M(y_1)|y_0 = y]A(y) - M(y) = -Q(Y). \tag{8}$$

Proof: See the Appendix. ■

Let us note that if the stochastic kernel is continuous, i.e., it could be described using densities, the Lyapunov equation (8) becomes a linear vector integral equation of Fredholm type. Thus, in several cases(8) could be solved numerically.

Remark 1: The techniques applied to study the mean square stability of MJLS with discrete state space, could not be applied to a MJLS with general state space. More precisely, the quantity involved in the stability analysis of MJLS with discrete state space is, in several cases, identically zero when applied to MJLS with general state space. Thus, it is not appropriate for stability analysis.

Another interesting notion is stabilizability, i.e., the existence of a stabilizing control law. It refers to a system under control in the form:

$$x_{k+1} = A(y_k)x_k + B(y_k)u_k. \tag{9}$$

Let us define stabilizability:

Definition 2: The system under control (9) is stabilizable, if there exists a bounded matrix function $L(y)$ such that the closed-loop system given by:

$$x_{k+1} = [A(y_k) + L(y_k)B(y_k)]x_k \tag{10}$$

is mean square exponentially stable. In this case, the pair $(A(\cdot), B(\cdot))$ will be called stabilizable.

IV. Optimal Control Problems

At first, the finite horizon linear quadratic control problem is studied. The system under control is slightly more general than the system given by (1)

$$x_{k+1} = A_k(y_k)x_k + B_k(y_k)u_k + w_k \tag{11}$$

i.e., time varying matrices A and B are allowed. The problem under consideration is to find a control law $u_t = \gamma(x_t, y_t, t)$ that minimizes the cost function given by (3). The solution to the finite horizon linear quadratic control problem is given recursively by the following equations:

$$K_{N+1}(y_{N+1}) = Q_{N+1}(y_{N+1}) \tag{12}$$

$$\begin{aligned}\Lambda_{k+1}(y_k) &= E[K_{k+1}(y_{k+1}|y_k)] \\ &= \int_D K_{k+1}(y')\bar{K}(y_k, dy')\end{aligned} \tag{13}$$

$$\begin{aligned}K_k &= Q_k + A_k^T[\Lambda_{k+1} - \Lambda_{k+1}B_k\cdot \\ &\cdot (R + B_k^T\Lambda_{k+1}B_k)^{-1}B_k^T\Lambda_{k+1}]A_k\end{aligned} \tag{14}$$

$$L_k = -(R + B^T\Lambda_{k+1}B_k)^{-1}B_k^T\Lambda_{k+1}A_k \tag{15}$$

$$u_k = L_k(y_k)x_k. \tag{16}$$

Proposition 3: Consider the system given by (11) and the cost criterion (3). Then, the control law computed recursively using the equations (12)–(16) is optimal.

Proof: Application of dynamic programming. ■

Let us now study the infinite horizon linear quadratic control problem, i.e., minimize (4) subject to (1). The solution of this problem depends on the following Riccati integral equation:

$$\begin{aligned}K(y) = Q(y) + A^T(y)\Bigg[&a\Lambda(y) - a\Lambda(y)B(y)\cdot \\ &\cdot\left(\frac{R(y)}{a} + B^T(y)\Lambda(y)B(y)\right)^{-1} B(y)^T\Lambda(y)\Bigg]A(y)\end{aligned} \tag{17}$$

where

$$\Lambda(y) = E[K(y_{k+1})|y_k = y] = \int_D K(y')\bar{K}(y, dy'). \tag{18}$$

The following Theorem 2 characterizes the optimal control policy in terms of the solution of the Riccati equation (17). Before stating Theorem 2, let us denote by $J_\mu(x, y)$ the value of the cost function (4) when $u_k = \mu(x_k, y_k, k)$ and $x_0 = x$, $y_0 = y$. Let us also denote by $J^*(x, y)$, the optimal value of the cost function (4).

Theorem 2: Consider the system given by equation (1) and the cost function (4). Then:

(i) Assume that there exists a policy μ that makes the criterion (4), finite i.e., $J_\mu(x, y) < \infty$ for any x, y. Then, optimal cost has the form $J^*(x, y) = x^TK(y)x + c(y)$, where $K(y)$ satisfies the Riccati equation (17). Furthermore, the optimal control is given by

$$\begin{aligned}u_k = L(y_k)x_k = &-\left(B^T(y_k)\Lambda(y_k)B(y_k) + \frac{R(y_k)}{a}\right)^{-1}\cdot \\ &\cdot B^T(y_k)\Lambda(y_k)A(y_k)x_k.\end{aligned} \tag{19}$$

(ii) Conversely, assume that a bounded function $K(y)$ satisfies the Riccati equation (17). Assume that the undisturbed closed-loop system given by:

$$x_{k+1} = (A(y_k) + L(y_k)B(y_k))x_k \tag{20}$$

is mean square exponentially stable. Then the policy given by (19) is optimal.

Proof: The proof shares many ideas with [21] or [12]. The basic difference is the proof of the finiteness of the cost when the controller given by (19) is used. That proof uses essentially the results of Section III. The differences are, however, of a technical character and thus the detailed proof is omitted. ■

Theorem 2 characterizes the optimal control law when $a \in (0, 1)$. However, (17)–(19) provide also the optimal controller when $a = 1$ and $w_k = 0$.

Remark 2:

(i) The existence of a policy μ that makes $J_\mu(x, y)$ finite for any x, y, is equivalent to the stabilizability of the pair $(\sqrt{a}A(\cdot), B(\cdot)))$

(ii) Equation (17), is a new form of Riccati equation. Specifically, it is a nonlinear vector integral equation. The solution of equation (17) could be approximated using the value iteration method (ex. [24]).

(iii) If the matrices $A(\cdot)$, $B(\cdot)$, $Q(\cdot)$, $R(\cdot)$ are continuous and the stochastic kernel is strongly Feller [1], then any solution of (17) is continuous.

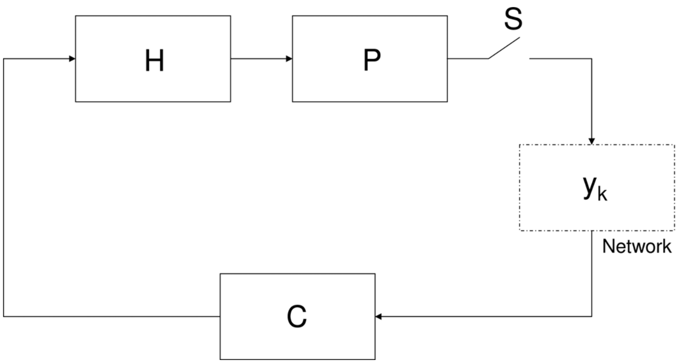

Fig. 1. Networked Control System.

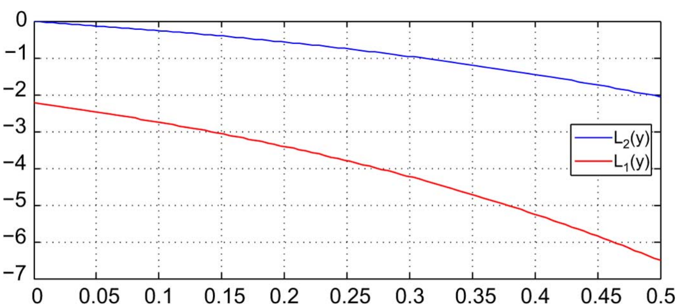

Fig. 2. Gains $L_1(y)$ and $L_2(y)$.

V. An Application

In this section, we study a simple example of application of MJLS with general state space Markov chain on systems with random delays. Examples of such systems include Networked Control Systems (NCS) (ex. [2], [3], [25] and [26]) and distributed optimization algorithms (ex. [8]). Particularly, we study a simple model of NCS with dependent delays. Furthermore, a simple numerical example is given, illustrating the solvability of the equations derived in Sections III and IV.

Consider a NCS as in Fig. 1 ([26] or [25]). The continuous time plant P is controlled by a controller C. The sampler S works at a constant rate and the time intervals among the sampling times have length T. The information is transmitted from the sampler to the controller through a communication channel, which introduces a random delay. Let us denote the delay of the transmission of the k-th measurement as y_k. In order to keep the model as simple as possible, we consider time delays only from the sampler to controller and we assume that the time delay introduced by the channel, is less than the sampling time, i.e., $y_k \in [0, T]$. A Markovian model for the time delay is introduced, i.e., there exists a stochastic kernel $\bar{K}(\cdot,\cdot)$ such that $\bar{K}(y, B) = Pr(y_k + 1 \in B | y_k = y)$. Finally, assume that the zero order hold H is event triggered, i.e., it holds the old value of the control until the new value comes.

The system under control P, is linear and its equation is given by

$$\dot{x}_c = A_c x_c + B_c u_c. \tag{21}$$

We will study the discretization of the system (21) on the time steps $t = kT$, for $k = 0, 1, \ldots$. Let us, thus, define $x_k = x_c(kT)$ and $u_k = u_c(kT + y_k)$, i.e., the control value obtained using the measurement of x_k. The system (21) has an input $u_c(t) = u_{k-1}$, on the time interval $t \in [kT, kT + y_k)$ and $u_c(t) = u_k$ on the interval $t \in [kT + y_k, (k+1)T)$. Thus, in order to describe the evolution of x_k, we use the augmented state vector $\tilde{x}_k = [x_k^T \; u_k^T]^T$. The evolution of $\tilde{x}_k$ is given by

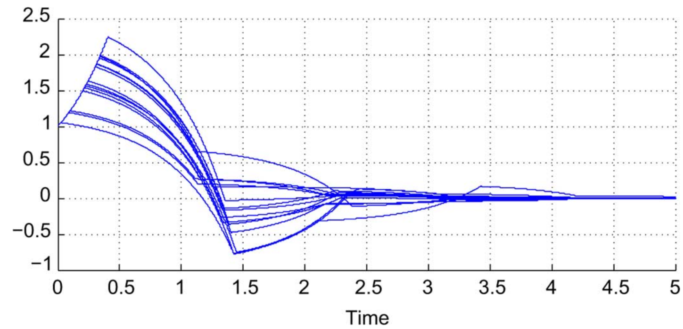

Fig. 3. Several sample paths of the closed-loop system.

$$\tilde{x}_{k+1} = A_d(y_k)\tilde{x}_k + B_d(y_k)u_k \tag{22}$$

where

$$A_d(y_k) = \begin{bmatrix} e^{A_cT} & \int_0^{y_k} e^{A_c(T-\tau)}B_c d\tau \\ 0 & 0 \end{bmatrix},$$
$$B_d(y_k) = \begin{bmatrix} \int_0^{T-y_k} e^{A_c(T-y_k-\tau)}B_c d\tau \\ I \end{bmatrix}. \tag{23}$$

Thus, the problem is reduced to the design of a controller for the MJLS (22) and the techniques of Sections III and IV could be applied. In the following example, a controller is designed for a simple system under control P.

1) Example 1: Consider the plant P described by

$$\dot{x}_c = 2x_c + u_c.$$

Assume that $T = 1$, the maximum delay is 0.5 and the stochastic kernel is described using the density function

$$f_y(z) = \begin{cases} \frac{4z}{y} & \text{if } 0 \le z < y \le 0.5 \\ 4 & \text{if } 0 \le y = z \le 0.5 \\ \frac{4(1-2z)}{(1-2y)} & \text{if } 0 \le y < z \le 0.5 \end{cases}$$

i.e., $\bar{K}(y, B) = \int_B f_y(z)dz$. The matrices of the discretized system (22) are given by

$$A_d = \begin{bmatrix} e^2 & \frac{e^2(1-e^{-2y})}{2} \\ 0 & 0 \end{bmatrix},$$
$$B_d = \begin{bmatrix} \frac{(e^{2-2y}-1)}{2} \\ 1 \end{bmatrix}.$$

A LQ control law is designed. The matrices describing the quadratic criterion (2) are given by $Q(y_k) = diag(3, 0)$ and $R(y_k) = 1$. For the matrix functions $A(y), B(y), Q(y)$ and $R(y)$, the Riccati integral equation (17) is solved using the value iteration method [24]. The components of the gain vector $L(y_k) = [\, L_1(y_k) \quad L_2(y_k) \,]$ are plotted in Fig. 2.

The closed-loop system is simulated and several sample paths are presented in Fig. 3.

Remark 3:

(i) Example 1 was considered, in order to illustrate that equations of Section IV could be used to design LQ control laws for NCS with dependent time delays described by a Markov chain with continuous state space. We would like also to point out that this model for the delays is more general than the models used in the literature.

(ii) The model used is as simple as possible. Thus, it can be generalized in several directions. For example the hypothesis $y_k \le T$ could be dropped, a time delay from the controller to the ZOH can be considered or packet losses could be studied.

VI. Conclusion

The class of MJLS with general state space was studied. The mean square exponential stability is characterized by the spectral radius of an infinite dimensional operator and proved to be uniform. An algorithm for testing stability was derived based on the uniformity result. The mean square exponential stability is also proved to be equivalent to the existence of a positive definite solution of a vector integral equation of Fredholdm type: the Lyapunov integral equation. The solution to the LQ control problem is characterized by a Riccati integral equation. The results derived, were used to design a controller to NCS with dependent random delays. The model of the random delays is more general than those used in the literature.

Appendix

Proof of Theorem 1: Let us first show that P_k is a matrix of signed measures. It holds $P_k(\emptyset) = 0$. In order to show that each element of the matrix P_k is a signed measure, it suffices to show σ-additivity. For a sequence of disjoint sets $(A_m)_{m=1}^{\infty}$, σ-additivity could be shown using the functions $f_{i,j}^{k,m} : \Omega \to \mathbf{R}$ with $f_{i,j}^{k,m} = e_i^T x_k x_k^T e_j \chi_{y \in \bigcup_{p=1}^{m} A_p}$ and dominated convergence theorem.

In order to derive the formula for T, we make the following computations:

$$\begin{aligned} P_{k+1}(B) &= E[A(y_k)\phi(y_k)A^T(y_k)\chi_{y_{k+1}\in B}] \\ &= \int_D A(y)\phi(y)A^T(y)\bar{K}(y,B)\mu_k(dy) \end{aligned}$$

where, $\phi(y_k)$ is a version of $E[x_k x_k^T|y_k]$ and μ_k is the distribution of y_k. Let $\phi_{l,m}$ and $P_k^{l,m}$ be the l, m, elements of ϕ and P_k respectively. Let also $\bar{A}_{l,m}(y)$ be functions such that $A(y)\phi(y)A^T(y) = \sum_{l=1}^{n}\sum_{l=1}^{n} \bar{A}_{l,m}(y)\phi_{l,m}(y)$. Thus, since $\phi_{l,m}$ is the Radon—Nikodym derivative $dP_k^{l,m}/d\mu_k$, it holds:

$$P_{k+1}(B) = \sum_{l=1}^{n}\sum_{m=1}^{n}\int_D A_{l,m}\bar{K}(y,B)P^{l,m}(dy)$$

which completes the proof of (6).

Assuming that $r(T) < 1$ spectral formula implies $\|P_k(\cdot)\| \to 0$ exponentially. Thus, the inequality $E[x_k^T x_k] \le \|P_k(\cdot)\|$ completes the proof of (i).

To prove (ii) let us assume that $r(T) > 1$. Then there exists an initial value $\bar{P}$ such that $T^k\bar{P} \to \infty$ as $k \to \infty$. Let $\bar{P}^+, \bar{P}^-$ be the Hahn decomposition of $\bar{P}$. Without loss of generality $T^k\bar{P}^+ \to \infty$ as $k \to \infty$. It is not difficult to show that there exist x_0, y_0 random variables such that $\bar{P}^+(B) = E[x_0 x_0^T \chi_{y_0\in B}]$. The proof of (ii) is completed using the following fact:

Fact 1: It holds:

$$\|P_k(\cdot)\| \le 2nE[x_k^T x_k].$$

To prove (iii) let us observe that the "only if" part of (iii) is a direct consequence of [27] Lemma 1 and the "if part" is a consequence of (i).

The proof of (iv) follows directly from (iii).

Proof of Proposition 1: We first show that (i) implies (ii). Using Fact 1, it is not difficult to show that $E[x_k^T x_k] \le 2nE[x_0^T x_0]\|T^k\|$. The spectral formula implies that there exist an integer k_0 and a positive constant ϵ such that $\|T^{k_0}\| < 1-\epsilon$. Thus, using Euclidian division of k by k_0, we conclude to the desired result:

$$E[x_k^T x_k] < Ma^k E[x_0^T x_0]$$

with $M = 2n \max_{k=0,\ldots,k_0} \|T^k\|/(1-\epsilon)$ and $a = (1-\epsilon)^{1/k_0}$.

The fact that (ii) implies (iii), is obvious.

It remains to show that (iii) implies (i). Let us introduce the following quantities:

$$\Phi_i(y) = E[A^T(y_0)\ldots A^T(y_{i\cdot k_0})A(y_{i\cdot k_0})A^T(y_0)]$$

i,e, the expectation of the product of $2ik_0$ matrices. It could be shown inductively that $\Phi_i(y) < a^i I$. Let N be a positive integer such that $2na^N = \bar{a} < 1$. Then for every x_0, y_0 random variables it holds: $E[x_{k_0\cdot N}^T x_{k_0\cdot N}] \le a^N E[x_0^T x_0]$ and using Fact 1 we obtain:

$$\|P_{k_0\cdot N}(\cdot)\| \le 2nE[x_{k_0\cdot N}^T x_{k_0\cdot N}] \le \bar{a}E[x_0^T x_0] \le \bar{a}\|P_0(\cdot)\|.$$

Thus, $r(T) < \bar{a}^{1/(iN)} < 1$ which completes the proof.

Proof of Proposition 2: We first show that (i) implies (ii). Let us consider a sequence of matrix functions $M_{N-k}(y)$ given by:

$$x_k^T M_{N-k} x_k = E\left[\sum_{t=k}^{N} x_t^T Q(y_t)x_t | x_t, y_t\right]$$

and their limit $M(y) = \lim_{N\to\infty} M_{N-k}(y)$. The matrix function Q is bounded, thus there exists a positive constant c such that $Q(y) < cI$. It holds:

$$x_0^T M_N x_0 \le cE\left[\sum_{t=0}^{N} x_t^T x_t\right] \le \frac{c\bar{M}}{1-a}x_0^T x_0$$

where $\bar{M}$ and a satisfy the Proposition 1 (ii). Thus $M(y)$ is bounded. Furthermore, it holds:

$$x_0^T M_N(y_0)x_0 - E[x_1 M_{N-1}x_1|x_0,y_0] = x_0 Q(y_0)x_0.$$

Taking limits, we conclude to (8).

It remains to show that (ii) implies (i). Following the same steps as [9, Theorem 2.1], we conclude that:

$$E[x_k^T x_k] \le \frac{c_1}{c_2}a^k E[x_0^T x_0]$$

where c_1 and c_2 are positive constants such that $c_1 I < Q(y) < c_2 I$ and

$$a = 1 - \min_{y\in D}\{\frac{\lambda_{min}(Q(y))}{\lambda_{max}(Q(y))}\} \in (0,1).$$

Thus Proposition 1 (iii) completes the proof.

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Formation Control and Network Localization via Orientation Alignment

Kwang-Kyo Oh, *Member, IEEE*, and Hyo-Sung Ahn, *Member, IEEE*

***Abstract*—We propose a formation control strategy based on inter-agent displacements for single-integrator modeled agents in the plane. Since the orientations of the local reference frames of the agents are not aligned with each other due to the absence of a common sense of orientation, the proposed strategy consists of an orientation alignment law and a formation control law. Under the proposed strategy, if the interaction graph is uniformly connected and all the initial orientation angles belong to an interval with length less than π, the orientations are exponentially aligned and the formation exponentially converges to the desired formation. We also show that the proposed strategy can be utilized for network localization as a dual problem.**

***Index Terms*—Formation control, orientation alignment, single-integrator.**

I. Introduction

Formation control of mobile agents based on local and partial measurements has recently attracted a considerable amount of research interest. In the literature, one may find two dominant formation control problem formulations, which might be called displacement- and distance-based approaches, depending on whether a common sense of orientation is available to agents.

In displacement-based approaches [1]–[3], based on a common sense of orientation, agents measure the relative positions (displacements) of their neighbors with respect to a common reference frame as depicted in Fig. 1(a). Then the agents directly control the relative positions to control their formation. It has been known that consensus-based control laws ensure global asymptotic convergence of the formation to the desired formation [1]–[3].

In distance-based approaches [4]–[6], agents measure the relative positions of their neighbors only with respect to their own local reference frames whose orientations are not necessarily aligned with each other due to the absence of a common sense of orientation. Since the orientations of the local reference frames are not aligned as depicted in Fig. 1(b), the desired formation cannot be specified by desired relative positions in general, and thus the agents cannot control the relative positions directly. Rather than the relative positions, the agents adjust the norms of the relative positions to control their formation. Though distance and angle measurements are available to the agents, formation control problems are complicated in these approaches because of misaligned orientations [4]–[6].

Though displacement-based approaches are effective in the sense that global asymptotic convergence is ensured, they require agents to carry some direction sensor such as a compass. While distance-based

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K.-K. Oh is with the Digital Convergence Research Center, Korea Electronics Technology Institute, Gwangju 500-480, Korea (e-mail: kwangkyo.oh@gmail.com).

H.-S. Ahn is with the School of Mechatronics, Gwangju Institute of Science and Technology (GIST), Gwangju 500-712, Korea (e-mail: hyosung@ieee.org; hyosung@gist.ac.kr).

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