GAMES WITH LONG TERM AND SHORT TERM PLAYERS

Section 1

In many dynamic games we have players of different time horizons. For example a bank is a long-term player whereas individual customers have a much shorter time horizon. In addition these shorter time horizon players are active during different and partially overlapping time intervals.

Examples of dynamic games where the players have different time horizons abandon in practice: a bank and its customers, a big electricity company and the smaller producers, a university and its students. The last example may very well serve as a generic paradigm. The university has an infinitive time horizon and the students have short, say 5-year horizons, overlapping since a freshman, a sophomore, a junior e.t.c. although they act each year simultaneously, they have different maturity and thus different control strategies.

In many applications up to now the “smaller” short term players are aggregated as a single long-term player. In the present paper we intend to address explicitly the impact of the short term versus the long-term characters of the players, as well as the overlapping of the intervals of play of the short term players. We study a deterministic version of the problem in discrete time. The Nash equilibrium is employed. We consider the LQ case and since we are interested in strategies that survive in a stochastic framework (Refs. 2, 5) we use the principle of optimality to derive the solutions. We derive the associated Ricatti equations (closed loop case). We present several examples in the LQ set-up and demonstrate several interesting features pertaining to the impact of the long term-short term interplay on the strategies and costs of the players.

Our paper consists of five sections. The first section is the introduction. In the second section we formulate our problem, we define the state equation and costs and we derive the Ricatti equations with which we characterise the Nash strategies for the short-term and the long-term players. In the third section we specialise to the scalar case which we are going to solve numerically for several parameter values using Matlab. In the fourth section we present our numerical results for several values of the parameters and we discuss the results. In the fifth section we consider again the scalar version of the problem where all the small players are concatenated in one player with the same time duration as the long-term player. We derive the associated Nash equilibrium. Our aim is to compare the solution derived now with the one we derived earlier where we took explicitly into account the short-term time overlapping character of the small players. The sixth is a short section with conclusions.
The state equation is:

\[ x_{k+1} = Ax_k + Bu_k + B_1u_{k1} + B_2u_{k2} + B_3u_{3k} + B_4u_{4k} + B_5u_{5k} , \quad k = 0,1,2,... \]  

(1)

where \( x_k \) is the state, \( u_k \) is the control of the long term player (university), \( u_{ik} \) is the control of the \( i \)-th year student \((i=1-5)\) and \( A,B \) are given matrices of appropriate dimensions.

The quadratic costs of the major player (\( J \)) and the minor players (\( J_i \)) who act in the interval between \( l \) and \( (l+5) \) are:

\[
J = \sum_{k=0}^{\infty} (x_k^T Q x_k + u_k^T R u_k), J_i = \sum_{k=0}^{\infty} (x_k^T Q x_k + u_k^T R u_k) + x_k^T Q f x_k
\]

(2)

The \( Q ' s \) are symmetric real non negative matrices and the \( R ' s \) are symmetric positive defined matrices which are known. In our case we consider \( A, B, Q, R \) constant. (References 1-4 contain the appropriate material needed for deriving the Nash solution of linear quadratic problems.) To clarify how we derive the solution we think as follows. Consider a student who starts his studies at time \( k=30 \) with control \( u_{1,30}=L_1x_{30} \) when he is a first-year student. Next year his control is \( u_{2,31}=L_2x_{31} \), the next \( u_{3,32}=L_3x_{32}, u_{4,33}=L_4x_{33} \) and the fifth and last year \( u_{5,34}=L_5x_{34} \). Notice that the \( L_1, L_2, L_3, L_4, L_5 \) are independent of the year this student started his studies.

As regards the long-term player for his optimal reaction we consider the state equation:

\[ x_{k+1} = Ax_k + Bu_k + (BL_1 + BL_2 + BL_3 + BL_4 + BL_5)x_k \]

and we use the Ricatti equation:

\[ K = A_0^T(K - KB_0(B_0^T KB_0 + R)^{-1} B_0^T K)A_0 + Q_0 \]

(4)

Then the long term player’s optimal reaction is:

\[ u^*_k = L_0x_k \]

(5)

and his optimal cost is

\[ J^* = x_0^T K x_0 \]

(6)

To derive the equations that provide the \( L_i ' s \) of the student we think as follows. We will examine for example a University and students who enter the University and their studies last for 5 years. Consider the student who enters the calendar year 30 (\( k=30 \)). He sees the following system (eq. 7-13) where in this first equation (eq. 7) he acts as first year student and the other-year students act with the fixed laws \( L_2x_{30}, L_3x_{30}, L_4x_{30}, L_5x_{30} \).

\[ x_{k+1} = (A + B_1L_0 + BL_2 + BL_3 + BL_4 + BL_5)x_k + Bu_{1k} = A_1x_k + Bu_{1k} \]

(7)

Similarly when he is a second year student he sees the following system

\[ x_{k+2} = (A + B_1L_0 + BL_4 + BL_3 + BL_4 + BL_5)x_{k+1} + Bu_{2,k+1} = A_2x_{k+1} + Bu_{2,k+1} \]

(8)

and the other-year students act with the fixed laws \( L_1, L_3, L_4, L_5 \) and so on. Thus the whole system of equations that the student who entered the calendar year \( k=30 \) and his studies last five years sees, is:
\[
\begin{align*}
    x_{k+1} &= (A + B_0 L_0 + B L_2 + B L_3 + B L_4 + BL_5) x_k + B u_{k} = A x_k + Bu_k, \\
    x_{k+2} &= (A + B_0 L_0 + B L_1 + B L_2 + B L_4 + BL_5) x_{k+1} + B u_{2,k+1} = A x_{k+1} + Bu_{2,k+1}, \\
    x_{k+3} &= (A + B_0 L_0 + B L_1 + B L_2 + B L_3 + B L_5) x_{k+2} + B u_{3,k+2} = A x_{k+2} + Bu_{3,k+2}, \\
    x_{k+4} &= (A + B_0 L_0 + B L_1 + B L_2 + B L_3 + B L_5) x_{k+3} + B u_{4,k+3} = A x_{k+3} + Bu_{4,k+3}, \\
    x_{k+5} &= (A + B_0 L_0 + B L_1 + B L_2 + B L_3 + B L_4) x_{k+4} + B u_{5,k+4} = A x_{k+4} + Bu_{5,k+4}
\end{align*}
\]

For this system of equations and the cost

\[
J_{30} = \sum_{k=0}^{4} (A^T x_{k+30} + B x_{k+30} + R x_{k+30} + Q x_{k+30} + A^T u_{k+30} + B^T u_{k+30} + R u_{k+30} + Q u_{k+30}) + x_{30}^T Q x_{30}
\]

we derive the optimal policy by employing the Ricatti equations. The Li’s are given by the following system of equations.

\[
\begin{align*}
    L_1 &= -(B^T K_2 B + R)^{-1} B^T K_2 A_1 \\
    K_1 &= A_1^T (K_2 - K_2 B (B^T K_2 B + R)^{-1} B^T K_2) A_1 + Q_f \\
    L_2 &= -(B^T K_3 B + R)^{-1} B^T K_3 A_2 \\
    K_2 &= A_2^T (K_3 - K_3 B (B^T K_3 B + R)^{-1} B^T K_3) A_2 + Q_f \\
    L_3 &= -(B^T K_4 B + R)^{-1} B^T K_4 A_3 \\
    K_3 &= A_3^T (K_4 - K_4 B (B^T K_4 B + R)^{-1} B^T K_4) A_3 + Q_f \\
    L_4 &= -(B^T K_5 B + R)^{-1} B^T K_5 A_4 \\
    K_4 &= A_4^T (K_5 - K_5 B (B^T K_5 B + R)^{-1} B^T K_5) A_4 + Q_f \\
    L_5 &= -(B^T K_6 B + R)^{-1} B^T K_6 A_5 \\
    K_5 &= A_5^T (K_6 - K_6 B (B^T K_6 B + R)^{-1} B^T K_6) A_5 + Q_f \\
    K_6 &= Q_f
\end{align*}
\]

The total cost of a student who entered the University at year 30 is:

\[
J_{30}^* = x_{30}^T K_1 x_{30}
\]

Notice that we consider linear no memory strategies. We know that there may exist other solutions, which are not necessarily linear and may have memory. We know nonetheless (Selten and Ref.2) that these solutions disappear in the presence of noise.

**Section 3**

Here we consider the scalar case and study the costs of the players by changing the parameters.
In this case we consider the matrices A, B, Q, R as constant scalars $\alpha, b, q, r$. We take the R’s to be 1. So the system of the matrix equations becomes:
\[ x_{k+1} = ax_k + u_k + L_1 x_k + L_2 x_k + L_3 x_k + L_4 x_k + L_5 x_k \]  
(27)

\[ x_{k+1} = (a + L_1 + L_2 + L_3 + L_4 + L_5)x_k + u_k = \bar{a}x_k + u_k \]  
(28)

\[ \bar{a} = a + L_1 + L_2 + L_3 + L_4 + L_5 \]  
(29)

\[ u_k^* = L_0x_k \]  
(30)

\[ L_0 = -(K + I)^{-1}K \bar{a} \]  
(31)

\[ K = a(K - K(K + I)^{-1}K)a + q_0 \]  
(32)

\[ J = \sum_{k=0}^{\infty} (q_0 x_k^2 + u_k^2) \]  
(33)

\[ x_{k+1} = (a + L_0 + L_2 + L_3 + L_4 + L_5)x_k + u_{ik} = a_i x_k + u_{ik} \]  
(34)

\[ x_{k+2} = (a + L_0 + L_1 + L_3 + L_4 + L_5)x_{k+1} + u_{2,i+1} = a_2 x_{k+1} + u_{2,i+1} \]  
(35)

\[ x_{k+3} = (a + L_0 + L_1 + L_2 + L_3 + L_5)x_{k+2} + u_{3,i+2} = a_3 x_{k+2} + u_{3,i+2} \]  
(36)

\[ x_{k+4} = (a + L_0 + L_1 + L_2 + L_3 + L_5)x_{k+3} + u_{4,i+3} = a_4 x_{k+3} + u_{4,i+3} \]  
(37)

\[ x_{k+5} = (a + L_0 + L_1 + L_2 + L_3 + L_4)x_{k+4} + u_{5,i+4} = a_5 x_{k+4} + u_{5,i+4} \]  
(38)

\[ L_1 = -(K_2 + I)^{-1}K_2 a_i \]  
(39)

\[ K_i = a_i(K_2 - K_2(K_2 + I)^{-1}K_2)a_i + q_f \]  
(40)

\[ L_2 = -(K_3 + I)^{-1}K_3 a_2 \]  
(41)

\[ K_2 = a_2(K_3 - K_3(K_3 + I)^{-1}K_3)a_2 + q_f \]  
(42)

\[ L_3 = -(K_4 + I)^{-1}K_4 a_3 \]  
(43)

\[ K_3 = a_3(K_4 - K_4(K_4 + I)^{-1}K_4)a_3 + q_f \]  
(44)

\[ L_4 = -(K_5 + I)^{-1}K_5 a_4 \]  
(45)

\[ K_4 = a_4(K_5 - K_5(K_5 + I)^{-1}K_5)a_4 + q_f \]  
(46)

\[ L_5 = -(K_6 + I)^{-1}K_6 a_5 \]  
(47)

\[ K_5 = a_5(K_6 - K_6(K_6 + I)^{-1}K_6)a_5 + q_f \]  
(48)

\[ K_6 = q_f \]  
(49)

(The K’s and L’s are scalar’s)

After some transformations we created the following scalar equations (five for the short term players and one for the long term player) where the \(x_i\)’s, stand for the K’i’s, i=0,1..5:

\[ x_5 = q_f \]  
(50)

\[ x_4 = q_f + A^2(x_5 + x_5^2) \]  
(51)

\[ x_3 = q_f + A^2(x_4 + x_4^2) \]  
(52)

\[ x_2 = q_f + A^2(x_3 + x_3^2) \]  
(53)

\[ x_1 = q_f + A^2(x_2 + x_2^2) \]  
(54)

\[ x_0 = q_0 + A^2(x_0 + x_0^2) \]  
(55)
\[ A = \frac{a}{1 + S + x_0} \quad (56) \]
\[ S = x_1 + x_2 + x_3 + x_4 + x_5 \quad (57) \]
\[ x_0 = \frac{a}{A} - 1 - S \quad (58) \]

Substituting \( x_0 \) from (58) into (55) we obtain the equivalent

\[ F(A) = (\alpha - A - AS)(1 - A^2) - Aq - A(\alpha - A - AS)^2 = 0 \quad (59) \]

Since \( S \) can be calculated explicitly as a function of \( A \) by using recursively (50)-(54) we conclude that \( F(A) \) is a function of \( A, a, q_0, q_f \) which has to be solved for its roots. Solving \( F(A)=0 \) we find the \( A \) and immediately the \( x_i \)'s from (50)-(54), (58). If \( F(A)=0 \) has many solutions \( A \), then there exist many different \( x_i \)'s, i.e. the game has many solutions. Of course we seek solutions \( A \) of \( F(A)=0 \) which are smaller than 1 in magnitude so that the close-loop system is stable. (Notice that \( A \) is the closed loop matrix).

Notice also that in order that (55) has a real solution \( x_0 \) it must hold:

\[ A < \frac{1}{\sqrt{q_0} + \sqrt{q_0 + 1}} \quad (60) \]

It must be \( x_1 > x_2 > x_3 > x_4 > x_5 \), and all the \( x_i \)'s positive. Also \( A \) and \( \alpha \) have the same sign (see (50)) and that is why in the numerical examples we take \( \alpha \) positive.

**Section 4**

Our next step is, by using the Matlab to solve these equations for several values of \( \alpha, q_0, q_f \).

We present some results of the numerical experiments in Table 1.

For each triplet of values of \( \alpha, q_0, q_f \) we solve first the equation \( F(A)=0 \) (59). Knowing the solution \( A \) of (59) we use this value to calculate the \( x_i \)'s. Notice that \( F(0)=\alpha > 0 \), \( F(1)<0 \) and thus \( F(A)=0 \) always has a solution in \((0,1)\). Since multiple solutions of \( F(A)=0 \) implies many solutions of the game, we provide some plots of \( F(A) \), see Appendix A. It appears in our experiments that \( F(A)=0 \) has a unique solution. It would be interesting to verify that this holds for any value of \( \alpha, q_0, q_f \) and thus to be able to conclude that the scalar case has always a unique solution.

We experimented with several values of \( \alpha, q_0, q_f \) and we tried to combine cases with \( \alpha \) stable \((0<\alpha<1)\), \( \alpha \) unstable \((\alpha>1)\), \( \alpha \) small / large, \( q_0 \) small/large, \( q_f \) small / large.
Our purpose in this section is to compare the solutions obtained in sections 2,3 with the solution that would become almost equal. So we conclude that in unstable system and with small values for qf,q we notice that when qf>qo then Jf>Jo and they have small prices Jf>Jo. While the qf, qo get bigger prices and still holds qf=qo then Ji gets closer to Jo until they become almost equal. So we conclude that in unstable system and with small values for qf,qo the cost of University is much bigger than students’ and the long term player is essentially more sensitive. In a more stable system (α=0.1) we notice that the players interchange roles and the students are more sensitive with bigger cost.

Section 5

Our purpose in this section is to compare the solutions obtained in sections 2,3 with the solution that would result if we were to group all the small duration players (students) together. In this case we would have two long term players. We work out only the scalar case.

By considering all the students (short term players) as one player with a cost equal to the sum of the costs of all the students, and a state equation

\[ x_{k+1} = ax_k + u_k + 5v_k \]

where 5vk is the control of this concatenated player, we have a classical linear quadratic infinite time game with two players. We take the R’s of the costs equal to 1 and α>0. We can derive the associated Ricatti equations for the Nash solution. After some transformations it turns out that we have to solve the following system of equations.

\[
\begin{align*}
x_1 &= 25q_f + A^2(x_1 + x_1^2) \\
x_0 &= q_o + A^2(x_0 + x_0^2) \\
A &= \frac{a}{1 + x_0 + x_1}
\end{align*}
\]

(61) (62) (63)

A is the closed loop matrix and in order to have real solutions \( x_1, x_0 \) and A stable it must be:

\[
A < \frac{1}{\sqrt{q_o} + \sqrt{q_o} + 1}, A < \frac{1}{\sqrt{25q_f} + \sqrt{25q_f} + 1}, 0 < A < 1
\]

(64)
The cost of the university is proportional to \( x_0 \) and the cost of the concatenated student player is proportional to \( x_1 \).

We present some numerical results of solving (61)-(63) for several values of the parameters

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Table 2

By comparing the values of \( x_0, x_1 \) (for the same parameter values) of section 4 (Table 1) and section 5 (Table 2) several conclusions can be drawn about the validity and usefulness of concatenating the many small players into one.

By observing the values \( x_0, x_1 \) at Table 2 we draw some conclusions about the costs of the University \((J_o)\) and the cost of the concatenated player \((J_f)\) which cost are proportional to \( x_0 \) and \( x_1 \) respectively. When the system is unstable \((\alpha = 10)\) we notice that when \( q_f > q_o \) then \( J_f > J_o \) (the same with Table 1 at the model of section 4) and \( q_f < q_o \) then \( J_f > J_o \) which is opposite with what we notice at section 4. When \( q_f = q_o \) and they have small prices then \( J_f >> J_o \) (the same with section 4).

**Section 6**

In our future search we intend to use the Stackelberg equilibrium model for the players. Of interest is also the case where the time duration of the short time players is a random variable taking values between 1 and 5 or greater. Similarly we can consider cases where the appearance of a small duration player at each instant of time is itself a random event.

**References**

Appendix A

PLOTS \( F(A) - A \) for Section 4

Plot 24

Plot 25

Plot 28

Plot 34