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GAMES WITH LONG TERM AND SHORT TERM PLAYERS

Section 1

In many dynamic games we have players of different time horizons. For example a bank is a long-term player whereas individual customers have a much shorter time horizon. In addition these shorter time horizon players are active during different and partially overlapping time intervals.
Examples of dynamic games where the players have different time horizons abandon in practice: a bank and its customers, a big electricity company and the smaller producers, a university and its students.
The last example may very well serve as a generic paradigm. The university has an infinitive time horizon and the students have short, say 5-year horizons, overlapping since a freshman, a sophomore, a junior e.t.c. although they act each year simultaneously, they have different maturity and thus different control strategies.
In many applications up to now the “smaller” short term players are aggregated as a single long-term player. In the present paper we intend to address explicitly the impact of the short term versus the long-term characters of the players, as well as the overlapping of the intervals of play of the short term players. We study a deterministic version of the problem in discrete time. The Nash equilibrium is employed. We consider the LQ case and since we are interested in strategies that survive in a stochastic framework (Refs. 2 , 5) we use the principle of optimality to derive the solutions. We derive the associated Ricatti equations (closed loop case). We present several examples in the LQ set-up and demonstrate several interesting features pertaining to the impact of the long term-short term interplay on the strategies and costs of the players.

Our paper consists of five sections. The first section is the introduction. In the second section we formulate our problem, we define the state equation and costs and we derive the Ricatti equations with which we characterise the Nash strategies for the short-term and the long-term players. In the third section we specialise to the scalar case which we are going to solve numerically for several parameter values using Matlab. In the fourth section we present our numerical results for several values of the parameters and we discuss the results. In the fifth section we consider again the scalar version of the problem where all the small players are concatenated in one player with the same time duration as the long-term player. We derive the associated Nash equilibrium. Our aim is to compare the solution derived now with the one we derived earlier where we took explicitly into account the short-term time overlapping character of the small players. The sixth is a short section with conclusions.

Section2

The state equation is:

$$x_{k+1} = Ax_k + B_0 u_k + B_1 u_{1k} + B_2 u_{2k} + B_3 u_{3k} + B_4 u_{4k} + B_5 u_{5k}, \quad k = 0,1,2,... \qquad (1)$$

where x_K is the state ,u_k is the control of the long term player (university), u_{ik} is the control of the i-th year student (i=1-5) and A,B_i are given matrices of appropriate dimensions.
The quadratic costs of the major player (J) and the minor players (J_l) who act in the interval between l and (l+5) are:

$$J = \sum_0^{\infty}(x_k^T Q_0 x_k + u_k^T R_0 u_k), J_l = \sum_{k=0}^{4}(x_{k+l+1}^T Q_f x_{k+l+1} + u_{(k+1)(l+k)}^T R_f u_{(k+1)(l+k)}) + x_l^T Q_f x_l \qquad (2)$$

The Q's are symmetric real non negative matrices and the R's are symmetric positive defined matrices which are known. In our case we consider A, B, Q, R constant. (References 1-4 contain the appropriate material needed for deriving the Nash solution of linear quadratic problems.) To clarify how we derive the solution we think as follows. Consider a student who starts his studies at time k=30 with control $u_{1,30}=L_1x_{30}$ when he is a first-year student. Next year his control is $u_{2,31}=L_2x_{31}$, the next $u_{3,32}=L_3x_{32}$, $u_{4,33}=L_4x_{33}$and the fifth and last year $u_{5,34}=L_5x_{34}$. Notice that the L_1, L_2, L_3, L_4, L_5 are independent of the year this student started his studies.

As regards the long-term player for his optimal reaction we consider the state equation:

$$x_{k+1} = Ax_k + B_o u_k + (BL_1 + BL_2 + BL_3 + BL_4 + BL_5)x_k \qquad (3)$$

and we use the Ricatti equation:

$$K = A_0^T (K - KB_0 (B_0^T KB_0 + R)^{-1} B_0^T K) A_0 + Q_0 \qquad (4)$$

Then the long term player's optimal reaction is :

$$u_k^* = u^* = L_0 x, \quad L_0 = -(B_0^T KB_0 + R)^{-1} B_0^T KA \qquad (5)$$

and his optimal cost is

$$J^* = x_0^T Kx_0. \qquad (6)$$

To derive the equations that provide the L_i's of the student we think as follows. We will examine for example a University and students who enter the University and their studies last for 5 years. Consider the student who enters the calendar year 30 (k=30). He sees the following system (eq. 7-13) where in this first equation (eq. 7) he acts as first year student and the other-year students act with the fixed laws L_2x_{30}, L_3x_{30}, L_4x_{30}, L_5x_{30}.

$$x_{k+1} = (A + B_0 L_0 + BL_2 + BL_3 + BL_4 + BL_5)x_k + Bu_{1k} = A_1 x_k + Bu_{1,k} \qquad (7)$$

Similarly when he is a second year student he sees the following system

$$x_{k+2} = (A + B_0 L_0 + BL_1 + BL_3 + BL_4 + BL_5)x_{k+1} + Bu_{2,k+1} = A_2 x_{k+1} + Bu_{2,k+1} \qquad (8)$$

and the other-year students act with the fixed laws L_1,L_3,L_4,L_5 and so on.
Thus the whole system of equations that the student who entered the calendar year k=30 and his studies last five years sees, is:

$$x_{k+1} = (A + B_0L_0 + BL_2 + BL_3 + BL_4 + BL_5)x_k + Bu_{1k} = A_1x_k + Bu_{1,k} \quad (9)$$
$$x_{k+2} = (A + B_0L_0 + BL_1 + BL_3 + BL_4 + BL_5)x_{k+1} + Bu_{2,k+1} = A_2x_{k+1} + Bu_{2,k+1} \quad (10)$$
$$x_{k+3} = (A + B_0L_0 + BL_1 + BL_2 + BL_4 + BL_5)x_{k+2} + Bu_{3,k+2} = A_3x_{k+2} + Bu_{3,k+2} \quad (11)$$
$$x_{k+4} = (A + B_0L_0 + BL_1 + BL_2 + BL_3 + BL_5)x_{k+3} + Bu_{4,k+3} = A_4x_{k+3} + Bu_{4,k+3} \quad (12)$$
$$x_{k+5} = (A + B_0L_0 + BL_1 + BL_2 + BL_3 + BL_4)x_{k+4} + Bu_{5,k+4} = A_5x_{k+4} + Bu_{5,k+4} \quad (13)$$

For this system of equations and the cost

$$J_{30} = \sum_{k=0}^{4}(x_{k+30+1}^T Q_f x_{k+30+1} + u_{(k+1)(30+k)}^T R_f u_{(k+1)(30+k)}) + x_{30}^T Q_f x_{30} \quad (14)$$

we derive the optimal policy by employing the Ricatti equations. The Li's are given by the following system of equations.

$$L_1 = -(B^T K_2 B + R)^{-1} B^T K_2 A_1 \quad (15)$$
$$K_1 = A_1^T (K_2 - K_2 B(B^T K_2 B + R)^{-1} B^T K_2) A_1 + Q_f \quad (16)$$
$$L_2 = -(B^T K_3 B + R)^{-1} B^T K_3 A_2 \quad (17)$$
$$K_2 = A_2^T (K_3 - K_3 B(B^T K_3 B + R)^{-1} B^T K_3) A_2 + Q_f \quad (18)$$
$$L_3 = -(B^T K_4 B + R)^{-1} B^T K_4 A_3 \quad (19)$$
$$K_3 = A_3^T (K_4 - K_4 B(B^T K_4 B + R)^{-1} B^T K_4) A_3 + Q_f \quad (20)$$
$$L_4 = -(B^T K_5 B + R)^{-1} B^T K_5 A_4 \quad (21)$$
$$K_4 = A_4^T (K_5 - K_5 B(B^T K_5 B + R)^{-1} B^T K_5) A_4 + Q_f \quad (22)$$
$$L_5 = -(B^T K_6 B + R)^{-1} B^T K_6 A_5 \quad (23)$$
$$K_5 = A_5^T (K_6 - K_6 B(B^T K_6 B + R)^{-1} B^T K_6) A_5 + Q_f \quad (24)$$
$$K_6 = Q_f \quad (25)$$

The total cost of a student who entered the University at year 30 is :

$$J_{30}^{\ *} = x_{30}^T K_1 x_{30} \quad (26)$$

Notice that we consider linear no memory strategies. We know that there may exist other solutions, which are not necessarily linear and may have memory. We know nonetheless (Selten and Ref.2) that these solutions disappear in the presence of noise.

Section 3

Here we consider the scalar case and study the costs of the players by changing the parameters.
In this case we consider the matrices A, B, Q, R as constant scalars α,b,q,r. We take the R's to be 1. So the system of the matrix equations becomes:

$$x_{k+1} = ax_k + u_k + L_1x_k + L_2x_k + L_3x_k + L_4x_k + L_5x_k \quad (27)$$

$$x_{k+1} = (a + L_1 + L_2 + L_3 + L_4 + L_5)x_k + u_k = \bar{a}\,x_k + u_k \quad (28)$$

$$\bar{a} = a + L_1 + L_2 + L_3 + L_4 + L_5 \quad (29)$$

$$u_k^* = L_0 x_k \quad (30)$$

$$L_0 = -(K + I)^{-1} K\,\bar{a} \quad (31)$$

$$K = \bar{a}^T (K - K(K + I)^{-1} K)\bar{a} + q_0 \quad (32)$$

$$J = \sum_{k=0}^{\infty} (q_0 x_k^2 + u_k^2) \quad (33)$$

$$x_{k+1} = (a + L_0 + L_2 + L_3 + L_4 + L_5)x_k + u_{1k} = a_1x_k + u_{1k} \quad (34)$$

$$x_{k+2} = (a + L_0 + L_1 + L_3 + L_4 + L_5)x_{k+1} + u_{2,k+1} = a_2x_{k+1} + u_{2,k+1} \quad (35)$$

$$x_{k+3} = (a + L_0 + L_1 + L_2 + L_4 + L_5)x_{k+2} + u_{3,k+2} = a_3x_{k+2} + u_{3,k+2} \quad (36)$$

$$x_{k+4} = (a + L_0 + L_1 + L_2 + L_3 + L_5)x_{k+3} + u_{4,k+3} = a_4x_{k+3} + u_{4,k+3} \quad (37)$$

$$x_{k+5} = (a + L_0 + L_1 + L_2 + L_3 + L_4)x_{k+4} + u_{5,k+4} = a_5x_{k+4} + u_{5,k+4} \quad (38)$$

$$L_1 = -(K_2 + I)^{-1} K_2 a_1 \quad (39)$$

$$K_1 = a_1(K_2 - K_2(K_2 + I)^{-1} K_2)a_1 + q_f \quad (40)$$

$$L_2 = -(K_3 + I)^{-1} K_3 a_2 \quad (41)$$

$$K_2 = a_2(K_3 - K_3(K_3 + I)^{-1} K_3)a_2 + q_f \quad (42)$$

$$L_3 = -(K_4 + I)^{-1} K_4 a_3 \quad (43)$$

$$K_3 = a_3(K_4 - K_4(K_4 + I)^{-1} K_4)a_3 + q_f \quad (44)$$

$$L_4 = -(K_5 + I)^{-1} K_5 a_4 \quad (45)$$

$$K_4 = a_4(K_5 - K_5(K_5 + I)^{-1} K_5)a_4 + q_f \quad (46)$$

$$L_5 = -(K_6 + I)^{-1} K_6 a_5 \quad (47)$$

$$K_5 = a_5(K_6 - K_6(K_6 + I)^{-1} K_6)a_5 + q_f \quad (48)$$

$$K_6 = q_f \quad (49)$$

(The K's and L's are scalar's)

After some transformations we created the following scalar equations (five for the short term players and one for the long term player) where the x_i's , stand for the K_i's, i=0,1..5:

$$x_5 = q_f \quad (50)$$

$$x_4 = q_f + A^2(x_5 + x_5{}^2) \quad (51)$$

$$x_3 = q_f + A^2(x_4 + x_4{}^2) \quad (52)$$

$$x_2 = q_f + A^2(x_3 + x_3{}^2) \quad (53)$$

$$x_1 = q_f + A^2(x_2 + x_2{}^2) \quad (54)$$

$$x_0 = q_0 + A^2(x_0 + x_0{}^2) \quad (55)$$

$$A = \frac{a}{1 + S + x_0} \quad (56)$$

$$S = x_1 + x_2 + x_3 + x_4 + x_5 \quad (57)$$

$$x_0 = \frac{a}{A} - 1 - S \quad (58)$$

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Substituting x_0 from (58) into (55) we obtain the equivalent

$$F(A) = (\alpha - \mathrm{A} - \mathrm{A}S)(1 - \mathrm{A}^2) - \mathrm{A}q_p - \mathrm{A}(\alpha - \mathrm{A} - \mathrm{A}S)^2 = 0 \quad (59)$$

Since S can be calculated explicitly as a function of A by using recursively (50)-(54) we conclude that F(A) is a function of A, a , q_0, q_f which has to be solved for its roots. Solving F(A)=0 we find the A and immediately the x_i's from (50)-(54), (58). If F(A)=0 has many solutions A, then there exist many different x_i's, i.e. the game has many solutions. Of course we seek solutions A of F(A)=0 which are smaller than 1 in magnitude so that the close-loop system is stable. (Notice that A is the closed loop matrix).
Notice also that in order that (55) has a real solution x_0 it must hold:

$$A < \frac{1}{\sqrt{q_0} + \sqrt{q_0 + 1}} \quad (60)$$

It must be $x_1 > x_2 > x_3 > x_4 > x_5$, and all the x_i's positive. Also A and α have the same sign (see (50)) and that is why in the numerical examples we take α positive.

<u>Section 4</u>

Our next step is, by using the Matlab to solve these equations for several values of α, q_0, q_f.
We present some results of the numerical experiments in Table 1.
For each triplet of values of α, q_0, q_f. we solve first the equation F(A)=0 (59). Knowing the solution A of (59) we use this value to calculate the x_i's. Notice that F(0)=α > 0 , F(1)<0 and thus F(A)=0 always has a solution in (0,1). Since multiple solutions of F(A)=0 implies many solutions of the game, we provide some plots of F(A) , see Appendix A. It appears in our experiments that F(A)=0 has a unique solution. It would be interesting to verify that this holds for any value of α, q_0, q_f and thus to be able to conclude that the scalar case has always a unique solution.
We experimented with several values of α, q_0, q_f and we tried to combine cases with α stable (0<α<1) , α unstable (α>1) , α small / large , q_0 small/large , q_f small / large.

N/N	q_f	q_0	α	x_0	x_1
1	0.2	0.3	2	0.940643857	0.355739695
2	0.4	5	3	10.95606453	0.427787782
3	0.2	0.2	10	107.2815088	0.202033206
4	5	0.1	2	0.147308367	5.17751777
5	0.1	10	3	18.0845003	0.102653147
6	0.1	10	9	89.13006363	0.101097654
7	4	4	0.2	4.636156441	4.001217467
8	3	2	1	2.376367131	3.035724933
9	1	5	0.2	5.10851722	1.000648631
10	5	1	0.2	0.308854969	5.001733902
11	1	5	3	8.112666758	1.098785773
12	5	1	3	0.812698284	5.386418711
13	10	1	10	1.293726313	14.84245578
14	1	10	10	99.18945565	1.018555487
15	1	10	0.1	8.285322242	1.000098014
16	10	1	0.1	1.629990457	10.00039713
17	5	1	0.1	0.314056467	5.000433269
18	5	1	3	1.113122694	5.378089324
19	5	1	10	0.615899776	10.59389327
20	10	1	0.1	4.554129883	10.00035642
21	10	1	3	1.119064198	10.36973035
22	10	1	10	1.293726313	14.84245578
23	1	5	0.1	3.99919982	1.00020006
24	1	5	3	8.112666758	1.098785773
25	1	5	10	93.91815071	1.020622924
26	1	10	0.1	8.285322242	1.000098014
27	1	10	3	14.65925325	1.04426562
28	1	10	10	99.18945565	1.018555487
29	5	5	0.1	0.314056467	5.000433269
30	5	5	3	5.139082174	5.287513421
31	5	5	10	17.69422576	7.179090223
32	0.5	0.5	0.1	0.49812324	0.500469337
33	0.5	0.5	3	1.032002046	0.936738724
34	0.5	0.5	10	93.55503582	0.508129951
35	10	10	0.1	1.629990457	10.00039713
36	10	10	3	9.126166344	10.27832769
37	10	10	10	12.34529519	13.34944245

Table 1

By observing the values x_0,x_1 we draw some conclusions about the costs of the University(J_o) and the cost of the student(J_l) which cost are proportional to x_0 and x_1 respectively. When the system is unstable (α=10) we notice that when $q_f>q_o$ then $J_f>J_o$ and respectively when $q_f<q_o$ then $J_f<J_o$. When $q_f=q_o$ and they have small prices $J_f>>J_o$. While the q_f, q_o get bigger prices and still holds $q_f=q_o$ then J_f gets closer to J_p until they become almost equal. So we conclude that in unstable system and with small values for q_f,q_p the cost of University is much bigger than students' and the long term player is essentially more sensitive.
In a more stable system (α=0.1) we notice that the players interchange roles and the students are more sensitive with bigger cost.

Section 5

Our purpose in this section is to compare the solutions obtained in sections 2,3 with the solution that would result if we were to group all the small duration players (students) together. In this case we would have two long term players. We work out only the scalar case.
By considering all the students (short term players) as one player with a cost equal to the sum of the costs of all the students, and a state equation
$x_{k+1} = ax_k + u_k + 5v_k$
where $5v_k$ is the control of this concatenated player , we have a classical linear quadratic infinite time game with two players. We take the R's of the costs equal to 1 and α>0. We can derive the associated Ricatti equations for the Nash solution. After some transformations it turns out that we have to solve the following system of equations.

$$x_1 = 25q_f + A^2(x_1 + x_1^2) \quad (61)$$

$$x_0 = q_0 + A^2(x_0 + x_0^2) \quad (62)$$

$$A = \frac{a}{1 + x_0 + x_1} \quad (63)$$

A is the closed loop matrix and in order to have real solutions x_1, x_0 and A stable it must be:

$$A < \frac{1}{\sqrt{q_0} + \sqrt{q_0 + 1}}, A < \frac{1}{\sqrt{25q_f} + \sqrt{25q_f + 1}}, 0 < A < 1 \quad (64)$$

The cost of the university is proportional to x_0 and the cost of the concatenated student player is proportional to x_1.

We present some numerical results of solving (61)-(63) for several values of the parameters

N/N	q_f	q_0	α	x_0	x_1	A
1	5	1	0.1	1.00000124	125.009765	0.000787341
2	5	1	3	1.00097744	133.8020305	0.023620231
3	5	1	10	1.003950318	223.6711045	0.044311499
4	10	1	0.1	1.000000315	250.0098813	0.00039681
5	10	1	3	1.000264549	258.8967572	0.01149879
6	10	1	10	1.001625954	349.1463592	0.028478022
7	1	5	0.1	5.000312068	25.00676429	0.00322507
8	1	5	3	5.204657329	31.47982683	0.079608361
9	1	5	10	5.217728109	115.8495711	0.081922022
10	1	10	0.1	10.00084863	25.00501577	0.002777325
11	1	10	3	10.64962595	29.80879337	0.072361659
12	1	10	10	10.92680096	106.6542415	0.084330512
13	5	5	0.1	5.000017479	125.0091778	0.000763305
14	5	5	3	5.013982614	133.301559	0.02153385
15	5	5	10	5.059925384	220.1440748	0.02153385
16	0.5	0.5	0.1	0.500038222	12.50861048	0.007138447
17	0.5	0.5	3	0.51421074	20.69211184	0.135096659
18	0.5	0.5	10	0.506052038	110.713038	0.089111398
19	10	10	0.1	10.00001615	250.0092116	0.000 383128
20	10	10	3	10.01368411	258.3110144	0.011138971
21	10	10	10	10.08866021	344.1268369	0.028151925

Table 2

By comparing the values of x_0, x_1 (for the same parameter values) of section 4 (Table 1) and section 5 (Table 2) several conclusions can be drawn about the validity and usefulness of concatenating the many small players into one.
By observing the values x_0,x_1 at Table 2 we draw some conclusions about the costs of the University(J_o) and the cost of the concatenated player (J_f) which cost are proportional to x_0 and x_1 respectively. When the system is unstable (α=10) we notice that when $q_f>q_o$ then $J_f>J_o$ (the same with Table 1 at the model of section 4) and $q_f<q_o$ then $J_f>J_o$ which is opposite with what we notice at section 4. When $q_f=q_o$ and they have small prices then $J_f>>J_o$ (the same with section 4).

Section 6

In our future search we intend to use the Stackelberg equilibrium model for the players. Of interest is also the case where the time duration of the short time players is a random variable taking values between 1 and 5 or greater. Similarly we can consider cases where the appearance of a small duration player at each instant of time is itself a random event.

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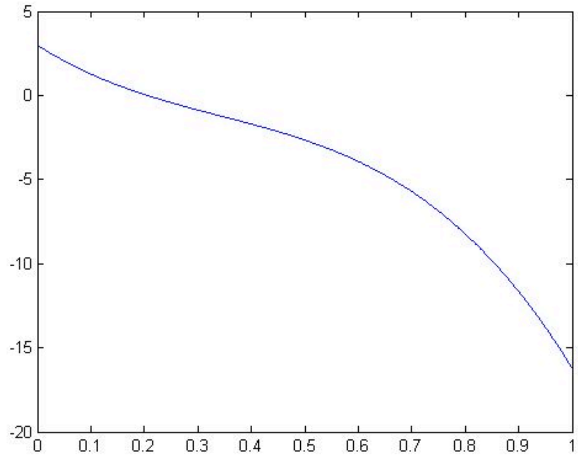
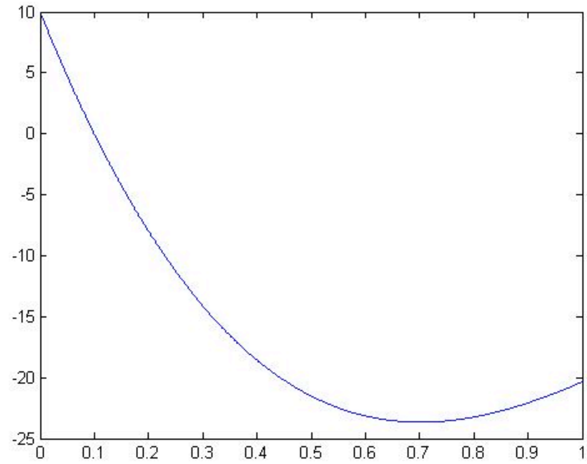
Appendix A

PLOTS F(A) – A for Section 4

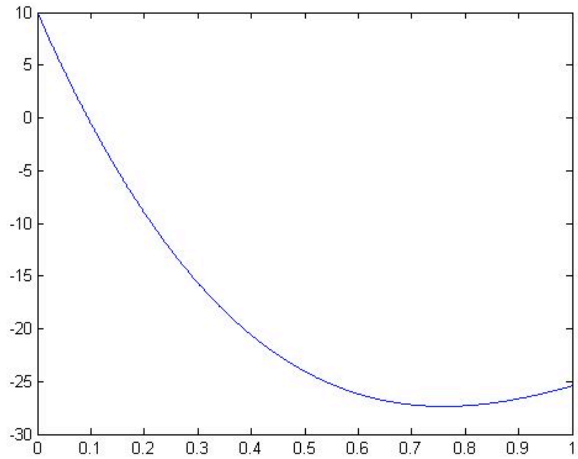
Plot 24



Plot 25



Plot 28



Plot 34