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A Nash Game with Long-term and Short-term Players

George P. Papavassilopoulos, Hisham Abou-Kandil, Marc Jungers

Abstract— **We formulate and study a game where there is a player who is involved for a long time interval and several small players who stay in the game for short time intervals. The long-term player plays open loop whereas the short-term players play memoryless closed loop or open loop. This is motivated by the fact that the long-term player is a player who usually represents a state or institutional authority that has to commit himself to long-term plans and regulations that are announced in advance and remain unchanged for a long time, whereas the short-term players not having such an institutional role can change policies arbitrarily often. We study this game for Nash strategies in a Linear Quadratic discrete time deterministic set-up. For the memoryless closed loop strategies we confine ourselves to strategies linear in the state. The derived associated Riccati-type equations are of a novel character and are of interest as such. Comparisons with the case where all players play memoryless closed loop or open loop are carried out.**

Index Terms— **Nash strategy; Linear Quadratic games; Open Loop strategy; Memoryless Closed Loop strategy; Different and Overlapping Time Horizons; Overlapping Generations.**

I. Introduction

In this paper, we formulate and study a game where there is a player who is involved for a long time interval and several small players who stay in the game for short time intervals. Such dynamic game models could be related to overlapping generation's models in economics [2], [19]. The long-term player has a strategy with an open loop information structure whereas the short duration players apply a memoryless closed loop strategies, *i.e.* they satisfy the Dynamic Programming Principle, (see [1], [3], [7], [22], [23]). This is motivated by the fact that the long-term player is a player who usually represents a state or institutional authority that has to commit himself to long-term plans and regulations that are announced in advance and remain unchanged for a long period, whereas the short-term players not having such an institutional role can change policies arbitrarily often. The case where the short-term players also play an open loop type strategy is also examined.

This game is studied in a Linear Quadratic, Deterministic, and Discrete Time setup, where the short-term players use linear feedback strategies and all the players are in Nash equilibrium, and where they all have the same cost structure independently of the time period they join the game. The short-term players are assumed to stay in the game for the same fixed period of time (taken to be of length five without any loss of generality) whereas the long-term players time horizon is taken to be infinite. In a previous work we examined the case where all players were playing Linear Memoryless Closed Loop [12]. Our present choice of strategies is motivated as follows: In many applications the long-term player is usually committed to a particular policy, which is preannounced, and adhered to for long time durations. Not revising his policy very often is thought of as a way of providing a stable environment within which the short-term players make and adapt their decisions and occasionally drop out of the game at will, a behavior which is not expected from or permitted to the long-term player. One can think of the long-term player as a Government agency or a Bank. Such institutions are long-term players who stay in business for a very long time whereas most of their customers stay for relatively shorter time periods. Thus it would be of interest to derive the Nash solution with this type of strategies for the players and compare the resulting costs to those ensued in the case where all players play Linear Memoryless Closed Loop *i.e.* when even the long-term institutional player is allowed often changes in his policy. Such type of games with overlapping horizons have also potential applications in traffic systems and energy market [11].

An important feature of the solutions derived is that they lead to Riccati type equations for calculating the gains, which are interlaced in time *i.e.* their evolution depends on present and past values of the gains. Future versions of this game may consider additional features, such as the random entry time and exit of the small players, continuous time analogues, etc.

Related work has been reported in [14], [15], [18]. The present work is an extension of [14], where the same framework was considered with all the players playing Linear Memoryless Closed Loop strategies that satisfy the Dynamic Programming Principle whereas here we consider that the long-term player plays Open Loop. This leads to a different set of conditions in the form of coupled Riccati equations, which is of a novel character. It is interesting to compare the solution presented here with the solution of [12]. This is done in an example where the difference between an institutional player who commits himself for a long time (*i.e.* plays Open Loop) and an institutional player who updates his strategy all the time (*i.e.* plays Memoryless-Closed Loop) is examined.

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The paper is organized as follows: In Section II, the basic model is presented while the Nash solutions are derived in Section III. Section IV provides algorithms to solve the obtained coupled Riccati-type equations. Numerical examples are given in Section V and concluding remarks make up Section VI.

II. Problem Statement

Let us consider the following state evolution equation

$$x_{k+1} = Ax_k + Bu_k + \sum_{i=1}^{M} B_i u_k^i, \quad k \in \mathbb{N}, \tag{1}$$

where $x_k \in \mathbb{R}^n$, is the state, $u_k \in \mathbb{R}^m$ represents the control of the long-term player and $u_k^i \in \mathbb{R}^{m_i}$ represent the controls of the short-term players ($i \in \{1, \cdots, M\}$) who acts at the time instants $k \in \mathbb{N}$. In Equation (1), u_k^i is the control of the player who entered the game at time $k+1-i$ and he will stay for $k-M+i$. A short-term player coexists not only with the long-term player but also with other short-term players similar to him who entered the system before or after him and whose time horizons overlap partly with his. The costs associated with these players are:

$$J = \frac{1}{2} \sum_{k \in \mathbb{N}} \left(x_k^T Q x_k + u_k^T R u_k \right), \tag{2}$$

$$\begin{aligned} J_{\mathrm{s}}[k+1, k+M] = & \frac{1}{2} x_{k+M+1}^T Q_{M+1} x_{k+M+1} \\ & + \frac{1}{2} \sum_{j=1}^{M} \left(x_{k+j}^T Q_j x_{k+j} + (u_{k+j}^j)^T R_j u_{k+j}^j \right) \end{aligned} \tag{3}$$

The functional J in (2) is the cost of the long-term player and $J_{\mathrm{s}}[k+1, k+M]$ in (3) is the cost of the short-term player who enters at time $k+1$ and acts until the time instant $k+M$. For the matrices involved we have ($i \in \{1, \cdots, M\}$) $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $B_i \in \mathbb{R}^{n \times m_i}$. The matrices $Q \in \mathbb{R}^{n \times n}$ and $Q_i \in \mathbb{R}^{n \times n}$ are symmetric and positive semidefinite. The matrices $R \in \mathbb{R}^{m \times m}$ and $R_i \in \mathbb{R}^{m_i \times m_i}$ are symmetric and positive definite.

We will consider the Nash solution with all the short-term players playing Closed Loop (memoryless) whereas the long-term player plays Open Loop. We will restrict the Closed Loop solutions to be linear in the state. We do that since although nonlinear solutions are known to exist in the deterministic Linear Quadratic framework, it is also known that in the presence of stochastic disturbances the nonlinear solutions are not sustainable. Also, due to the symmetry of the short-term players it is natural to seek solutions for them which are also symmetric, *i.e.* the five-tuple of gains used by any short-term player, composed of his gains during his first, second, third, fourth and fifth actions, is the same five-tuple that will be used by any other short-term player independently of the time he starts his short-term -of length five-career. We will also consider the Nash solution with all the short-term players and the long-term player all play Open Loop. Clearly the formulae and the methodology extend trivially to the case where the horizon of the short time players is not five but any arbitrary number of stages. Actually the study of the behavior of the solution as the time horizon of the short-term players increases is of interest in answering important questions about the modeling characteristics of the games introduced here.

III. Solution

A. Open Loop – Closed Loop case

In order to derive the Nash equilibrium delineated above we proceed as follows. Let us first consider the short-term player who acts during the time interval $[k, k+M-1]$. He sees the following system of equations ($\ell \in \{0, \cdots, M-1\}$):

$$x_{k+\ell+1} = Ax_{k+\ell} + B_{1+\ell} u_{k+\ell}^{1+\ell} + Bu_{k+\ell} + \sum_{j \in \{1, \cdots, M\}, j \neq (1+\ell)} B_j u_{k+\ell}^j, \tag{4}$$

where $u_{k+\ell}^{1+\ell}$ are his controls ($\ell \in \{0, \cdots, M-1\}$). The inputs $u_{k+\ell}$ ($\ell \in \{0, \cdots, M-1\}$) are the Open Loop controls of the Long-term player, and the rest are the Closed Loop controls of the other short-term players. Applying the discrete time minimum principle, see [4], we obtain the following system:

$$\begin{aligned} H_j^k = & \frac{1}{2} (x_{k+j-1}^T Q_j x_{k+j-1} + (u_{k+j-1}^j)^T R_j u_{k+j-1}^j) \\ & + (p_{j+1}^k)^T \big(Ax_{k+j-1} + Bu_{k+j-1} + B_j u_{k+j-1}^j \\ & + \sum_{i \in \{1, \cdots, M\}, i \neq j} B_i L_i x_{k+j-1} \big). \end{aligned} \tag{5}$$

The control u_{k+j-1}^j aims at minimizing H_j^k and is given by

$$u_{k+j-1}^j = -R_j^{-1} B_j p_{j+1}^k. \tag{6}$$

Moreover we have the relations, for $j \in \{1, \cdots, M\}$:

$$p_j^k = \frac{\partial H_j^k}{\partial x_{k+j-1}} = Q_j x_{k+j-1} + \big(A + \sum_{i \in \{1, \cdots, M\}, i \neq j} B_i L_i \big)^T p_{j+1}^k, \tag{7}$$

and finally

$$p_{M+1}^k = Q_{M+1} x_{k+M}. \tag{8}$$

Notice that since the long-term player plays Open Loop, his control does not contribute to $\frac{\partial H_{i+1}^k}{\partial x_{k+i}}$, $i \in \{0, \cdots, M-1\}$, whereas the controls of the other short-term players do.

It should be noticed that since we consider that the short-term players play the same M–tuple of gains independently of when they start, we have the relations, $\ell \in \{0, \cdots, M-1\}$:

$$u_{k+\ell}^{1+\ell} = -R_{1+\ell}^{-1} B_{1+\ell} p_{2+\ell}^{k+\ell} = L_{1+\ell} x_{k+\ell}, \tag{9}$$

that is

$$u_k^j = L_j x_k, \quad \forall j \in \{1, \cdots, M\}. \tag{10}$$

Let us now consider the long-term player. He sees the state equation:

$$x_{k+1} = Ax_k + Bu_k + \sum_{j \in \{1, \cdots, M\}} B_j u_k^j, \quad k \in \mathbb{N}, \tag{11}$$

or equivalently:

$$x_{k+1} = (A + \sum_{j \in \{1, \cdots, M\}} B_j L_j) x_k + Bu_k, \quad k \in \mathbb{N}. \tag{12}$$

Applying the discrete time minimum principle for the infinite time invariant Linear Quadratic problem, yields :

$$H_k^0 = \frac{1}{2}\big(x_k^T Q x_k + (u_k)^T R u_k\big) + \overline{p}_{k+1}^T\big((A + \sum_{j\in\{1,\cdots,M\}} B_j L_j)x_k + Bu_k\big). \quad (13)$$

u_k minimizing H_k^0 is given by

$$u_k = -R^{-1}B^T\overline{p}_{k+1}, \quad (14)$$

and we have

$$\overline{p}_k = \frac{\partial H_k^0}{\partial x_k} = Qx_k + (A + \sum_{j\in\{1,\cdots,M\}} B_j L_j)^T\overline{p}_{k+1}, \quad (15)$$

with

$$\lim_{k\to+\infty}\overline{p}_k = 0. \quad (16)$$

To sum-up we have the following system of equations $\ell \in \{0;\cdots,M-1\}$:

$$L_{1+\ell}x_{k+\ell} = -R_{1+\ell}^{-1}B_{1+\ell}^T p_{2+\ell}^k; \quad (17)$$

$$p_{1+\ell}^k = Q_{1+\ell}x_{k+\ell} + \big(A + \sum_{i\in\{1,\cdots,M\}, i\neq(1+\ell)} B_i L_i\big)^T p_{2+\ell}^k \quad (18)$$

and Equation (8), in addition of Equations (14) and (15).

Let us set

$$p_{1+\ell}^k = K_{1+\ell}x_{k+\ell}, \quad (19)$$
$$\overline{p}_k = Kx_k, \quad (20)$$

one can easily see that equivalently we have to solve the system $j \in \{1,\cdots,M\}$:

$$K_j = Q_j + \big(A + \sum_{i\in\{1,\cdots,M\}, i\neq j} B_i L_i\big)^T K_{j+1}A_{\text{cl}} \quad (21)$$

and

$$K_{M+1} = Q_{M+1}. \quad (22)$$

In addition,

$$K = Q + \big(A + \sum_{i\in\{1,\cdots,M\}} B_i L_i\big)^T K A_{\text{cl}}, \quad (23)$$

with $j \in \{1,\cdots,M\}$:

$$L_j = -R_j^{-1}B_j^T K_{j+1}A_{\text{cl}}, \quad (24)$$

$$L = -R^{-1}B^T K A_{\text{cl}}, \quad (25)$$

$$A_{\text{cl}} = A + BL + \sum_{j\in\{1,\cdots,M\}} B_j L_j. \quad (26)$$

By underlining the matrix A_{cl}, we finally get the system $j \in \{1,\cdots,M\}$

$$K_j = Q_j + A_{\text{cl}}^T(K_{j+1} + KSK_{j+1} + K_{j+1}S_jK_{j+1})A_{\text{cl}}, \quad (27)$$
$$K = Q + A_{\text{cl}}^T(K + KSK)A_{\text{cl}}, \quad (28)$$
$$A = (I_n + SK + \sum_{j\in\{1,\cdots,M\}} S_j K_{j+1})A_{\text{cl}}. \quad (29)$$

and

$$K_{M+1} = Q_{M+1}, \quad (30)$$

where $S = B^T R^{-1} B$ and $S_j = B_j^T R_j^{-1} B_j$, $j \in \{1,\cdots,M\}$.

In order to satisfy the condition (16), it suffices that the closed loop matrix A_{cl} has its eigenvalues strictly inside the unit disk.

Proposition 1: If the system of Equations (27)–(30) admits solutions K, $\{K_j\}_{j\in\{1,\cdots,M+1\}}$ which are positive semidefinite and the matrix A_{cl} is asymptotically stable, then the strategies (10) for the short-term players and the strategy (14) for the long-term player, are in a Nash equilibrium where the short-term players play Memoryless Closed Loop and the long-term player plays Open Loop. (Notice that the long-term player's strategy is not Memoryless Closed Loop and the formula

$$u_k = -R^{-1}B^T K x_k \quad (31)$$

indicates only its value realization). The optimal cost of the long-term player is

$$J^* = \frac{1}{2}x_0^T K x_0, \quad (32)$$

and for the short time player who enters at time k is

$$J_s^*[k, k+M-1] = \frac{1}{2}x_k^T K_1 x_k. \quad (33)$$

□

Notice that the formulae are very similar to those obtained for the case where all players play Memoryless Closed Loop considered in [12]. The extra terms appearing in the recursion for K_i are the only but substantial difference. They are of reminiscent of the Riccati type equations appearing in the theory of the classical LQ Nash Games, (see [1], [3], [7], [22], [23]), but have some different features worth pointing out, such as the noncausal character.

B. Open Loop – Open Loop case

For the case where not only the long-term horizon player plays Open Loop, but the small horizon players also play Open Loop, then instead of Equation (7), we have to use $j \in \{1,\cdots,M\}$:

$$p_j^k = \frac{\partial H_j^k}{\partial x_{k+j-1}} = Q_j x_{k+j-1} + A^T p_{j+1}^k. \quad (34)$$

Thus we end up with the system:

$$L_{1+\ell}x_{k+\ell} = -R_{1+\ell}^{-1}B_{1+\ell}^T p_{2+\ell}^k \quad (35)$$

$$p_j^k = Q_j x_k + A^T p_{j+1}^k, \quad (36)$$

and

$$p_{M+1}^k = Q_{M+1}x_{k+M}, \quad (37)$$

in addition of

$$u_k = -R^{-1}B^T\overline{p}_{k+1}, \quad (38)$$

$$\overline{p}_k = Qx_k + (A + \sum_{j\in\{1,\cdots,N\}} B_j L_j)^T\overline{p}_{k+1}, \quad (39)$$

$$\lim_{k\to+\infty}\overline{p}_k = 0. \quad (40)$$

Let us set:

$$p_{j+1}^k = K_{j+1}x_{k+j-1}, \quad \forall j \in \{0,\cdots,M\}, \quad (41)$$

and

$$\overline{p}_k = Kx_k. \tag{42}$$

One can easily see that equivalently we have to solve the system:

$$K_j = Q_j + A^T K_{j+1} A_{\text{cl}}, \quad \forall j \in \{1, \cdots, M\} \tag{43}$$

$$K_{M+1} = Q_{M+1}, \tag{44}$$

$$K = Q + \big(A + \sum_{j \in \{1,\cdots,M\}} B_j L_j\big)^T K A_{\text{cl}}, \tag{45}$$

where the following relations hold

$$L_j = -R_j^{-1} B_j^T K_{j+1} A_{\text{cl}}, \tag{46}$$

$$L = -R^{-1} B^T K A_{\text{cl}} \tag{47}$$

$$A_{\text{cl}} = A + BL + \sum_{j \in \{1,\cdots,M\}} B_j L_j. \tag{48}$$

Substituting in the above equations the relation

$$A + \sum_{j \in \{1,\cdots,M\}} B_j L_j = A_{\text{cl}} - BL, \tag{49}$$

we finally get the system

$$\begin{aligned} K_j &= Q_j + A^T K_{j+1} A_{\text{cl}}, \ \forall j \in \{1, \cdots, M\}, && (50)\\ K_{M+1} &= Q_{M+1}, && (51)\\ K &= Q + A_{\text{cl}}^T (K + KSK) A_{\text{cl}}, && (52)\\ A &= (I_n + SK + \sum_{j \in \{1,\cdots,M\}} S_j K_{j+1}) A_{\text{cl}}. && (53) \end{aligned}$$

In order to satisfy the condition (40), it suffices that the closed loop matrix A_{cl} has its eigenvalues strictly inside the unit disk.

Proposition 2: If the system of Equations (50)–(53) admits solutions K and $\{K_j\}_{j \in \{1,\cdots,M\}}$ which are positive semidefinite and the matrix A_{cl} is asymptotically stable, then the strategies (10) for the short-term players and (38) for the long-term player, are in a Nash equilibrium where the short-term players and the long-term player play Open Loop. (Notice that the long-term player's strategy is not Memoryless Closed Loop and the formulae

$$u_{k+\ell}^{1+\ell} = -R_{1+\ell}^{-1} B_{1+\ell}^T K_{2+\ell} x_{1+\ell}, \ \forall \ell \in \{0, \cdots, M-1\} \tag{54}$$

and

$$u_k = -R^{-1} B^T K x_k \tag{55}$$

indicate only their value realizations).The optimal cost of the long-term player is

$$J^* = \frac{1}{2} x_0^T K x_0, \tag{56}$$

and for the short time player who enters at time k is

$$J_{\text{s}}^*[k, k+M-1] = \frac{1}{2} x_k^T K_1 x_k. \tag{57}$$

□

The following section provides algorithms to solve the systems of equations which are at the heart of the two propositions.

IV. Algorithms

It is noteworthy that is the two cases, the system of Equations (27)–(30) and (50)–(53) may be viewed as new coupled algebraic and difference Riccati equations. They cannot be integrated backward in time because the equations depends on the algebraic solution K. They cannot be solved directly by algebraic methods [5], [16], [17]. Thus only numerical methods could be used to solve such coupled equations. The following algorithms are based on a fixed point argument and are new modified versions of algorithms described in [8], [10], [12], [13].

The solution K and K_ℓ, ($\ell \in \{1, \cdots, M+1\}$) of the equations (27)–(30) are symmetric and positive semidefinite due to the symmetry of these equations and the definiteness of the weighting matrices of the criteria J and $J_{\text{s}}[k+1, k+M]$. A way to solve the equations (27)–(30) is to consider Algorithm 1.

Algorithm 1 (Open-loop/Closed-loop):

- Define an error level ε.
- Initialize $K^{(0)}$ and $K_\ell^{(0)}$ ($\ell \in \{1, \cdots, M+1\}$). For instance, consider trivial matrices.
- At every step $c \in \mathbb{N}^*$, set

$$A_{\text{cl}}^{(c-1)} = \left(I_n + SK^{(c-1)} + \sum_{\ell \in \{1,\cdots,M\}} S_\ell K_\ell^{(c-1)} \right)^{-1} A \tag{58}$$

and apply the iterative scheme

$$\begin{aligned} K^{(c)} &= Q + (A_{\text{cl}}^{(c-1)})^T \times \\ &\quad (K^{(c-1)} S K^{(c-1)} + K^{(c-1)}) A_{\text{cl}}^{(c-1)} && (59)\\ K_{M+1}^{(c)} &= Q_{M+1} && (60)\\ K_{\ell-1}^{(c-1)} &= (A_{\text{cl}}^{(c-1)})^T \Big(K^{(c-1)} S K_\ell^{(c-1)} \\ &\quad + K_\ell^{(c-1)} S_\ell K_\ell^{(c-1)} + K_\ell^{(c-1)} \Big) \times \\ &\quad A_{\text{cl}}^{(c-1)} + Q_{\ell-1}, \ \forall \ell \in \{2, \cdots, M+1\} && (61) \end{aligned}$$

- Compute $\varepsilon^{(c)}$ defined by

$$\varepsilon_K^{(c)} = \| K^{(c)} - Q - (A_{\text{cl}}^{(c)})^T (K^{(c)} S K^{(c)} + K^{(c)}) A_{\text{cl}}^{(c)} \|_2 \tag{62}$$

$$\begin{aligned} \varepsilon_{K,\ell}^{(c)} &= \| K_{\ell-1}^{(c)} - Q_\ell \\ &- (A_{\text{cl}}^{(c)})^T (K_{j+1}^{(c)} + K^{(c)} S K_{j+1}^{(c)} + K_{j+1}^{(c)} S_j K_{j+1}^{(c)}) A_{\text{cl}}^{(c)} \|_2 \end{aligned} \tag{63}$$

and

$$\varepsilon^{(c)} = \max \left(\varepsilon_K^{(c)}, \{ \varepsilon_{K,\ell}^{(c)} \}_{\ell \in \{1,\cdots,M+1\}} \right). \tag{64}$$

- Stop the algorithm when

$$\varepsilon^{(c)} < \varepsilon. \tag{65}$$

□

Algorithm 2 is dedicated to solve the system of equations (50)–(53).

Algorithm 2 (Open-loop/Open-loop):

- Define an error level ε.

- Initialize $K^{(0)}$ and $K_\ell^{(0)}$ ($\ell \in \{1,\cdots,M+1\}$). For instance, Identity matrices or the solution of the decoupled equations could be used.
- At every step $c \in \mathbb{N}^*$, set

$$A_{\rm cl}^{(c-1)} = \left(I_n + SK^{(c-1)} + \sum_{\ell\in\{1,\cdots,M\}} S_\ell K_\ell^{(c-1)}\right)^{-1} A \tag{66}$$

and apply the iterative scheme

$$\begin{aligned} K^{(c)} &= Q + (A_{\rm cl}^{(c-1)})^T \times \\ &\quad (K^{(c-1)}SK^{(c-1)} + K^{(c-1)})A_{\rm cl}^{(c-1)} \quad (67) \\ K_{M+1}^{(c)} &= Q_{M+1} \quad (68) \\ K_{\ell-1}^{(c-1)} &= A^T K_\ell^{(c-1)} A_{\rm cl}^{(c-1)} \times \\ &\quad +Q_{\ell-1},\ \forall \ell \in \{2,\cdots,M+1\}. \quad (69) \end{aligned}$$

- Compute $\varepsilon^{(c)}$ defined by

$$\varepsilon_K^{(c)} = \|K^{(c)} - Q - (A_{\rm cl}^{(c)})^T (K^{(c)}SK^{(c)} + K^{(c)})A_{\rm cl}^{(c)}\|_2 \tag{70}$$

$$\varepsilon_{K,\ell}^{(c)} = \|K_{\ell-1}^{(c)} - A^T K_{j+1}^{(c)} A_{\rm cl}^{(c)} - Q_\ell\|_2 \tag{71}$$

and

$$\varepsilon^{(c)} = \max\left(\varepsilon_K^{(c)}, \{\varepsilon_{K,\ell}^{(c)}\}_{\ell\in\{1,\cdots,M+1\}}\right). \tag{72}$$

- Stop the algorithm when

$$\varepsilon^{(c)} < \varepsilon. \tag{73}$$

□

The following section will illustrate the efficiency of our approach.

V. Examples. Comparisons of Solutions

Let us consider the following example, where $M = 5$, $n = 2$, $\varepsilon = 10^{-6}$,

$$A = \begin{bmatrix} 0.5 & 0.2 \\ 0 & 0.7 \end{bmatrix};\ B = \begin{bmatrix} 1.5 \\ 1 \end{bmatrix};$$

$$B_\ell = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix},\ (\forall \ell \in \{1,\cdots,M\});$$

$$Q = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix};\ R = 1;$$

$$Q_{M+1} = Q_\ell = 10I_n, R_\ell = 1\ (\forall \ell \in \{1,\cdots,M\});$$

By applying Algorithm 1, the breaking condition is reached at the 12th iteration. The evolution of the error is depicted in Figure 1. We thus obtain the numerical results:

$$K^{(12)} = \begin{bmatrix} 1.2787 & 1.2998 \\ 1.2998 & 5.7608 \end{bmatrix} > 0_2;$$

and

$$K_1^{(12)} = \begin{bmatrix} 10.1542 & -0.4770 \\ -0.2825 & 11.0307 \end{bmatrix}.$$

In addition, we have $A_{\rm cl}^{(12)} = \begin{bmatrix} 0.0698 & -0.1407 \\ -0.0927 & 0.2196 \end{bmatrix}$, with $\lambda(A_{\rm cl}^{(12)}) = \{0.0082, 0.2813\}$. That is $A_{\rm cl}$ is stable.

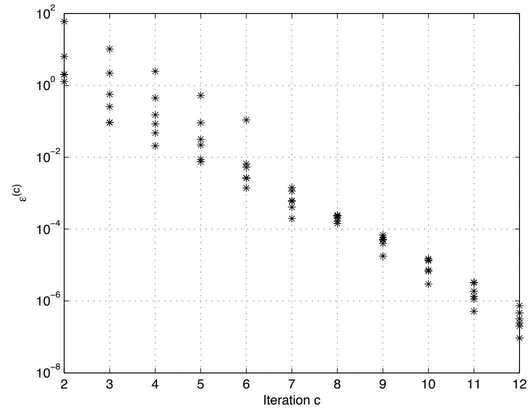

Fig. 1. Errors $\varepsilon_K^{(c)}$ and $\{\varepsilon_{K,\ell}^{(c)}\}_{\ell\in\{1,\cdots,M\}}$ in function of the iteration c, for Algorithm 1.

By considering the same numerical example, we apply Algorithm 2. The breaking condition is reached at the 11th iteration. The evolution of the error is depicted in Figure 2. The numerical results are as follows

$$K^{(11)} = \begin{bmatrix} 1.2823 & 1.2912 \\ 1.2912 & 5.7814 \end{bmatrix} > 0_2;$$

and

$$K_1^{(11)} = \begin{bmatrix} 10.3933 & -0.8043 \\ -0.6164 & 11.5015 \end{bmatrix}.$$

In addition, we have $A_{\rm cl}^{(11)} = \begin{bmatrix} 0.0685 & -0.1378 \\ -0.0924 & 0.2191 \end{bmatrix}$, with $\lambda(A_{\rm cl}^{(12)}) = \{0.0081, 0.2795\}$. That is $A_{\rm cl}$ is stable.

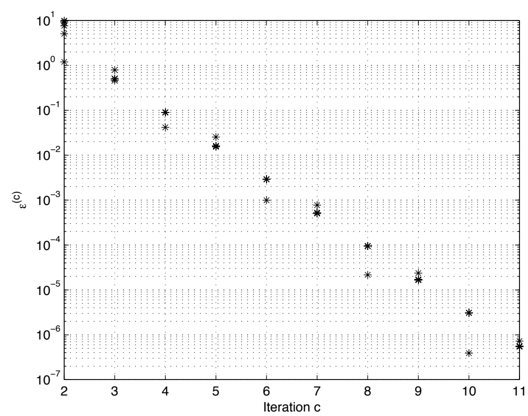

Fig. 2. Errors $\varepsilon_K^{(c)}$ and $\{\varepsilon_{K,\ell}^{(c)}\}_{\ell\in\{1,\cdots,M\}}$ in function of the iteration c, for Algorithm 2.

It is important to point out that the two algorithms present, at least on these numerical examples, a logarithmic convergence rate. In addition, the results obtained by the two algorithms are distinct, even if they are close.

The cost function $J_{\rm s}[k, k+M]$ for the two functions are shown in Figure 3. With the semilog scale, the plots seem to be the same. Nevertheless by drawing their difference in Figure 4, we can emphasize that they are distinct. The error convergences to zero due to the stability of the closed-loop matrix A_{cl}. It should be noted also that the sign of this error depends on the time for this example. A framework is thus not always better than the other.

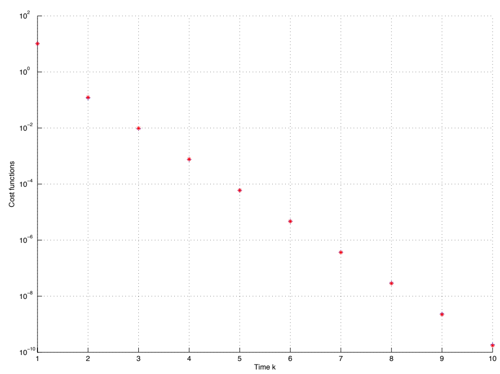

Fig. 3. Cost functions $J_{\mathrm{s}}[k, k+M]$ in function of time k.

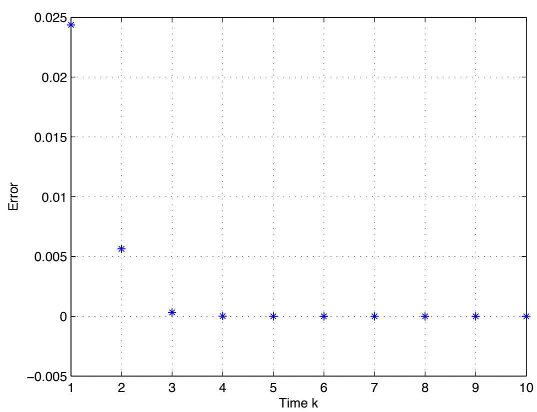

Fig. 4. Error between the cost functions $J_{\mathrm{s}}[k, k+M]$ in function of time k.

VI. Conclusions

Several interesting questions are worthy of further investigations, such as the continuous time analogues in the Linear Quadratic set-up, [18], the further theoretical study of the new types of Riccati-type equations and the development of good computational algorithms for solving them. Considering that the long-term player can be easily thought of as a Leader the study of the Open Loop and Memoryless Closed Loop solutions for the Stackelberg equilibrium concept [1], [3], [6], [7], [9], [20], [21], is also a worthy avenue of exploration. Another interesting question is to consider that the long-term player plays for a large but finite length of time, derive the solutions and take the limit of the solutions as the time period of the long-term player goes to infinity. Does this limit exist and how is it related to the infinite time solutions examined here? (Notice that in Dynamic Games the solutions of finite time problems do not relate in simple ways to the solutions of their infinite time analogues. Similar remarks hold for the interplay of existence questions between finite and infinite time analogues). Stochastic analogues pertaining in particular to random entries, and exits of the short time players is another interesting topic, see [15] for some early results.

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