LQ Nash Games with Players Participating in the Game for Random Time Intervals

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Abstract

In this paper we study Discrete Time Linear Quadratic Games involving players staying in the game for random time intervals. Two types of problems are considered. In the first problem all players enter the game at time 0. At any time step, a player involved in the game until that time step, leaves the game with a probability depending on the number of players that participate the game that moment. In the second problem we consider an overlapping generation game with random entrance and exit. Particularly at each time step a random number of players enters the game. Each player could stay in the game for a maximum number of steps. However the duration of its presence in the game is a random variable that depends on the number of players that participate in the game. In both problems we characterize the Nash equilibria consisting of Symmetric Linear Feedback Strategies. The case of a large number of players is studied via a Mean Field approximation. Numerical examples are also given.

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Structure

- Related Topics
- Description
- Nash Equilibria
- Large Population Case
- Special Cases
- Conclusion and Future Work

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Related Topics

Most of the Dynamic Game models: time interval that each player participates in the game as well as number of players is known a-priori.

- Games with Population Uncertainty (Poisson Games)
- Games with Random Horizon
  - Repeated game setting
  - Dynamic game setting
- Games with Large Number of Players (NCE)
- LQ Games with Random Entrance (Padova)

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Description: Participation in the Game

- Initially M players $\quad I_0 = \{p_1, \ldots, p_M\}$
- Participation is described by the vector: $\Xi_k = \left[\left(\xi_k^1\right) \ \ldots \ \left(\xi_k^M\right) \right]^T$

  where: $\xi_k^i = 1$ if player i participates in the game at time step k
- Evolution:

  The configuration $\Xi_{k+1} = \Xi'$ is allowed after the configuration $\Xi_k = \Xi$ if every player that belongs to Ξ', also belong to Ξ.

  The evolution is described by a set of parameters: $\pi_{k,m,l}$

$$\Pr\left(\Xi_{k+1} = \Xi' \mid \Xi_k = \Xi\right) = \begin{cases} \pi_{k,r\Xi,r\Xi'} & \text{if } \Xi' \text{ is allowed} \\ 0 & \text{if } \Xi' \text{ is not allowed} \end{cases}$$

- Normalization

$$\sum_{l=0}^{j} \binom{j}{l} \pi_{k,j,l} = 1$$

Description: Dynamics and Costs

- Dynamics for the player i:

$$x_{k+1}^i = Ax_k^i + Bu_k^i + F\left(y_k\right)z_k$$

where: $z_k = \sum_{i=1}^{M} \xi_k^i x_k^i / M \qquad y_k = \frac{1}{M}\sum_{i=1}^{M} \xi_k^i$

x_0^i i.i.d. random variables with finite second moments

- Cost Functions:

$$J^i = E\left\{\left[\left(x_N^i\right)^T \quad z_N^T\right] Q_N\left(y_N\right)\left[\left(x_N^i\right)^T \quad z_N^T\right]^T \xi_N^i + \right.$$

$$\left. + \sum_{k=0}^{N-1} \xi_k^i \left(\left[\left(x_k^i\right)^T \quad z_k^T\right] Q_k\left(y_k\right)\left[\left(x_k^i\right)^T \quad z_k^T\right]^T + \left(u_k^i\right)^T R_k\left(y_k\right) u_k^i\right)\right\}$$

Description: Measurements

- Player i has access to x_k^i , z_k and y_k

- The players use Symmetric Linear Feedback Strategies (SLFS):

$$u_k^i = L_k^1\left(y_k\right)z_k + L_k^2\left(y_k\right)x_k^i$$

- It will be shown that Nash equilibria within the class of SLFS are Nash equilibria within the class of Feedback Strategies.

Structure

- Related Topics
- Description
- **Nash Equilibria**
- Large Population Case
- Special Cases
- Conclusion and Future Work

Nash Equilibria

- We find Nash Equilibria using Dynamic Programming simultaneously for all the players.
- Define recursively the "cost to go" quantities:

$$J_k^i\left(X_k,\Xi_k\right)=\underset{u_t^j=\gamma_t^j\left(X_k,\Xi_k,t\right)}{E}\left\{\left[\left(x_N^i\right)^T \quad z_N^T\right]Q\left(y_N\right)\left[\left(x_N^i\right)^T \quad z_N^T\right]^T \xi_N^i + \right.$$

$$\left.+\sum_{t=k}^{N-1}\xi_t^i\left(\left[\left(x_t^i\right)^T \quad z_t^T\right]Q\left(y_t\right)\left[\left(x_t^i\right)^T \quad z_t^T\right]^T+\left(u_t^i\right)^T R\left(y_t\right)u_t^i\right)\right\}$$

- It will be shown that, under certain conditions, it holds:

$$J_k^i\left(X_k,\Xi_k\right)=\xi_k^i\left[\left(x_k^i\right)^T \quad z_k^T\right]\begin{bmatrix}\Lambda_k^{xx}\left(y_k\right) & \Lambda_k^{xz}\left(y_k\right)\\ \Lambda_k^{zx}\left(y_k\right) & \Lambda_k^{zz}\left(y_k\right)\end{bmatrix}\begin{bmatrix}x_k^i\\ z_k\end{bmatrix}+\xi_k^i\sum_{j=1}^{M}\xi_k^j\left(x_k^j\right)^T\Sigma_k\left(y_k\right)x_k^j$$

Nash Equilibria

- Applying principle of optimality for player i :

$$J_k^i\left(X_k,\Xi_k\right)=\min_{u_k} E\left\{\xi_{k+1}^i\left[\left(x_{k+1}^i\right)^T \quad z_{k+1}^T\right]\begin{bmatrix}\Lambda_{k+1}^{xx}\left(y_{k+1}\right) & \Lambda_{k+1}^{xz}\left(y_{k+1}\right)\\ \Lambda_{k+1}^{zx}\left(y_{k+1}\right) & \Lambda_{k}^{zz}\left(y_{k+1}\right)\end{bmatrix}\begin{bmatrix}x_{k+1}^i\\ z_{k+1}\end{bmatrix}+\right.$$

$$\left.+\xi_{k+1}^i\sum_{j=1}^{M}\xi_{k+1}^j\left(x_{k+1}^j\right)^T\Sigma_{k+1}\left(y_{k+1}\right)x_{k+1}^j+\xi_k^i\left(\left[\left(x_k^i\right)^T \quad z_k^T\right]Q\left(y_k\right)\left[\left(x_k^i\right)^T \quad z_k^T\right]^T+\left(u_k^i\right)^T R\left(y_k\right)u_k^i\right)\mid X_k,\Xi_k\right\}$$

where:

$$z_{k+1}=\frac{1}{M}\sum_{j=1}^{M}\xi_{k+1}^j\left(Ax_k^j+Bu_k^j+F\left(y_k\right)z_k\right)$$

Thus we need to compute quantities of the form:

$$E\left[\xi_{k+1}^i\Gamma_{k+1}\left(y_{k+1}\right)\mid\Xi_k\right] \quad E\left[\xi_{k+1}^i\xi_{k+1}^j\Gamma_{k+1}\left(y_{k+1}\right)\mid\Xi_k\right] \quad E\left[\xi_{k+1}^j\xi_{k+1}^{j'}\xi_{k+1}^i\Gamma_{k+1}\left(y_{k+1}\right)\mid\Xi_k\right]$$

Nash Equilibria

- Let us define:

$$\tilde{\Gamma}_{k+1}(y_k) = \sum_{l=1}^{My_k} \pi_{k,My_k,l} \binom{My_k - 1}{l-1} \Gamma_{k+1}(l/M)$$

$$\bar{\Gamma}_{k+1}(y_k) = \sum_{l=2}^{My_k} \pi_{k,My_k,l} \binom{My_k - 2}{l-2} \Gamma_{k+1}(l/M)$$

$$\bar{\bar{\Gamma}}_{k+1}(y_k) = \sum_{l=3}^{My_k} \pi_{k,My_k,l} \binom{My_k - 3}{l-3} \Gamma_{k+1}(l/M)$$

Thus it holds:

$$E\left[\xi_{k+1}^i \Gamma(y_{k+1}) \mid \Xi_k\right] = \xi_k^i \tilde{\Gamma}_{k+1}(y_k)$$

$$E\left[\xi_{k+1}^i \xi_{k+1}^j \Gamma_{k+1}(y_{k+1}) \mid \Xi_k\right] = \xi_k^i \xi_k^j \bar{\Gamma}_{k+1}(y_k) + \delta_{ij} \xi_k^i \left(\tilde{\Gamma}_{k+1}(y_k) - \bar{\Gamma}_{k+1}(y_k)\right)$$

$$E\left[\xi_{k+1}^j \xi_{k+1}^{j'} \xi_{k+1}^i \Lambda_{k+1}^{zz}(y_{k+1}) \mid \Xi_k\right] = \Big[\bar{\bar{\Gamma}}_{k+1}(y_k) + \left(\bar{\Gamma}_{k+1}(y_k) - \bar{\bar{\Gamma}}_{k+1}(y_k)\right)\left(\delta_{j,j'} + \delta_{i,j'} + \delta_{i,j}\right) + \\ + \left(\tilde{\Gamma}_{k+1}(y_k) - \bar{\Gamma}_{k+1}(y_k)\right)\delta_{i,j,j'}\Big] \xi_k^j \xi_k^{j'} \xi_k^i$$

Nash Equilibria

- Introducing these formulas to $J_k^i\left(X_k, \Xi_k\right)$ we obtain:

$$J_k^i\left(X_k, \Xi_k\right) = \xi_k^i \left[\left(x_k^i\right)^T \quad z_k^T \quad \left(u_k^i\right)^T \quad v_k^T \right] S_k\left(y_k\right) \left[\left(x_k^i\right)^T \quad z_k^T \quad \left(u_k^i\right)^T \quad v_k^T \right]^T +$$

$$+ \xi_k^i \sum_{j=1}^{M} \left[\xi_k^j \left(Ax_k^j + Bu_k^j\right)^T \hat{\Sigma}\left(y_k\right)\left(Ax_k^j + Bu_k^j\right) \right]$$

where:

$$v_k = \frac{1}{M} \sum_{j=1}^{M} \xi_k^j u_k^j$$

and

$$S_k\left(y_k\right) = \begin{bmatrix} S_k^{xx} & S_k^{xz} & S_k^{xu} & S_k^{xv} \\ S_k^{zx} & S_k^{zz} & S_k^{zu} & S_k^{zv} \\ S_k^{ux} & S_k^{uz} & S_k^{uu} & S_k^{uv} \\ S_k^{vx} & S_k^{vz} & S_k^{vu} & S_k^{vv} \end{bmatrix}$$

Nash Equilibria

$$S_k^{xx} = A^T \Big\{ \tilde{\Sigma}_{k+1}(y_k) - \bar{\Sigma}_{k+1}(y_k) + \tilde{\Lambda}_{k+1}^{xx}(y_k) + \frac{1}{M}\left(\tilde{\Lambda}_{k+1}^{xz}(y_k) - \bar{\Lambda}_{k+1}^{xz}(y_k)\right) +$$

$$\frac{1}{M}\left(\tilde{\Lambda}_{k+1}^{zx}(y_k) - \bar{\Lambda}_{k+1}^{zx}(y_k)\right) + \frac{1}{M^2}\left(\tilde{\Lambda}_{k+1}^{zz}(y_k) - \bar{\Lambda}_{k+1}^{zz}(y_k)\right)\Big\} A$$

$$S_k^{xz} = A^T \Big\{ \left(\tilde{\Sigma}_{k+1}(y_k) - \bar{\Sigma}_{k+1}(y_k)\right)F(y_k) + \tilde{\Lambda}_{k+1}^{xx}(y_k)F(y_k) + \bar{\Lambda}_{k+1}^{xz}(y_k)\left[A + y_k F(y_k)\right] + \frac{1}{M}\left[\tilde{\Lambda}_{k+1}^{xz}(y_k) - \bar{\Lambda}_{k+1}^{xz}(y_k)\right]F(y_k) +$$

$$\frac{1}{M}\left(\tilde{\Lambda}_{k+1}^{zx}(y_k) - \bar{\Lambda}_{k+1}^{zx}(y_k)\right)F(y_k) + \left(\bar{\Lambda}_{k+1}^{zz}(y_k) - \bar{\bar{\Lambda}}_{k+1}^{zz}(y_k)\right)\left(A + y_k F(y_k)\right)/M + \left(\tilde{\Lambda}_{k+1}^{zz}(y_k) - \bar{\Lambda}_{k+1}^{zz}(y_k)\right)F(y_k)/M^2 \Big\}$$

$$S_k^{xu}(y_k) = A^T \Big\{ \left(\tilde{\Sigma}_{k+1}(y_k) - \bar{\Sigma}_{k+1}(y_k)\right) + \tilde{\Lambda}_{k+1}^{xx}(y_k) + \frac{1}{M}\left[\tilde{\Lambda}_{k+1}^{xz}(y_k) - \bar{\Lambda}_{k+1}^{xz}(y_k)\right] + \frac{1}{M}\left[\tilde{\Lambda}_{k+1}^{zx}(y_k) - \bar{\Lambda}_{k+1}^{zx}(y_k)\right] +$$

$$+ A^T\left(\tilde{\Lambda}_{k+1}^{zz}(y_k) - \bar{\Lambda}_{k+1}^{zz}(y_k)\right)/M^2 \Big\} B$$

$$S_k^{xv} = A^T \left[\bar{\Lambda}_{k+1}^{xz}(y_k) + \left(\bar{\Lambda}_{k+1}^{zz}(y_k) - \bar{\bar{\Lambda}}_{k+1}^{zz}(y_k)\right)/M\right] B$$

$$S_k^{zz} = MF^T(y_k)\bar{\Sigma}_{k+1}(y_k)A + My_kF^T(y_k)\bar{\Sigma}_{k+1}(y_k)F(y_k) + MA\bar{\Sigma}_{k+1}(y_k)F(y_k) + F^T(y_k)\left(\tilde{\Sigma}_{k+1}(y_k) - \bar{\Sigma}_{k+1}(y_k)\right)F(y_k) +$$

$$+ F^T(y_k)\tilde{\Lambda}_{k+1}^{xx}(y_k)F(y_k) + F^T(y_k)\bar{\Lambda}_{k+1}^{xz}(y_k)\left[A + y_kF(y_k)\right] + \frac{1}{M}F^T(y_k)\left[\tilde{\Lambda}_{k+1}^{xz}(y_k) - \bar{\Lambda}_{k+1}^{xz}(y_k)\right]F(y_k)$$

$$+ \left[A^T + y_kF^T(y_k)\right]\bar{\Lambda}_{k+1}^{zx}(y_k)F(y_k) + \frac{1}{M}F^T(y_k)\left[\tilde{\Lambda}_{k+1}^{zx}(y_k) - \bar{\Lambda}_{k+1}^{zx}(y_k)\right]F(y_k) +$$

$$+ \left(A + y_kF(y_k)\right)^T \bar{\bar{\Lambda}}_{k+1}^{zz}(y_{k+1})\left(A + y_kF(y_k)\right) + F^T(y_k)\left(\bar{\Lambda}_{k+1}^{zz}(y_k) - \bar{\bar{\Lambda}}_{k+1}^{zz}(y_k)\right)A/M +$$

$$+ A^T\left(\bar{\Lambda}_{k+1}^{zz}(y_k) - \bar{\bar{\Lambda}}_{k+1}^{zz}(y_k)\right)F(y_k)z_k/M + y_kF^T(y_k)\left(\bar{\Lambda}_{k+1}^{zz}(y_k) - \bar{\bar{\Lambda}}_{k+1}^{zz}(y_k)\right)F(y_k)/M +$$

$$+ F^T(y_k)\left(\bar{\Lambda}_{k+1}^{zz}(y_k) - \bar{\bar{\Lambda}}_{k+1}^{zz}(y_k)\right)\left(A + y_kF(y_k)\right)/M + F^T(y_k)\left(\tilde{\Lambda}_{k+1}^{zz}(y_k) - \bar{\Lambda}_{k+1}^{zz}(y_k)\right)F(y_k)/M^2$$

Nash Equilibria

$$S_k^{zu} = F^T(y_k)\left(\tilde{\Sigma}_{k+1}(y_k) - \bar{\Sigma}_{k+1}(y_k)\right)B + F^T(y_k)\tilde{\Lambda}_{k+1}^{zx}(y_k)B + \frac{1}{M}F^T(y_k)\left[\tilde{\Lambda}_{k+1}^{xz}(y_k) - \bar{\Lambda}_{k+1}^{xz}(y_k)\right]B +$$

$$+\left(A^T + y_k F^T(y_k)\right)^T \bar{\Lambda}_{k+1}^{zx}(y_k)B + \frac{1}{M}F^T(y_k)\left[\tilde{\Lambda}_{k+1}^{zx}(y_k) - \bar{\Lambda}_{k+1}^{zx}(y_k)\right]B +$$

$$+\left(A + y_k F(y_k)\right)^T\left(\bar{\Lambda}_{k+1}^{zz}(y_k) - \bar{\bar{\Lambda}}_{k+1}^{zz}(y_k)\right)B/M + F^T(y_k)\left(\tilde{\Lambda}_{k+1}^{zz}(y_k) - \bar{\Lambda}_{k+1}^{zz}(y_k)\right)B/M^2$$

$$S_k^{zv} = MF^T(y_k)\bar{\Sigma}_{k+1}(y_k)B + F^T(y_k)\bar{\Lambda}_{k+1}^{xz}(y_k)B + \left(A + y_k F(y_k)\right)^T\bar{\bar{\Lambda}}_{k+1}^{zz}(y_{k+1})B$$

$$+F^T(y_k)\left(\bar{\Lambda}_{k+1}^{zz}(y_k) - \bar{\bar{\Lambda}}_{k+1}^{zz}(y_k)\right)B/M + F^T(y_k)\left(\bar{\Lambda}_{k+1}^{zz}(y_k) - \bar{\bar{\Lambda}}_{k+1}^{zz}(y_k)\right)B/M$$

$$S_k^{uu} = B^T\left\{\left(\tilde{\Sigma}_{k+1}(y_k) - \bar{\Sigma}_{k+1}(y_k)\right) + \tilde{\Lambda}_{k+1}^{zx}(y_k) + \frac{1}{M}\left[\tilde{\Lambda}_{k+1}^{xz}(y_k) - \bar{\Lambda}_{k+1}^{xz}(y_k)\right] + \frac{1}{M}\left[\tilde{\Lambda}_{k+1}^{zx}(y_k) - \bar{\Lambda}_{k+1}^{zx}(y_k)\right] +\right.$$

$$\left.+\left(\tilde{\Lambda}_{k+1}^{zz}(y_k) - \bar{\Lambda}_{k+1}^{zz}(y_k)\right)\frac{1}{M^2}\right\}B$$

$$S_k^{uv} = \frac{1}{M}B^T\left[\bar{\Lambda}_{k+1}^{xz}(y_k) + \left(\bar{\Lambda}_{k+1}^{zz}(y_k) - \bar{\bar{\Lambda}}_{k+1}^{zz}(y_k)\right)\right]B$$

$$S_k^{vv} = B^T\bar{\bar{\Lambda}}_{k+1}^{zz}(y_{k+1})B$$

$$\hat{\Sigma}_k(y_k) = \bar{\Sigma}_{k+1}(y_k) + \left(\bar{\Lambda}_{k+1}^{zz}(y_k) - \bar{\bar{\Lambda}}_{k+1}^{zz}(y_k)\right)/M^2$$

Nash Equilibria

- $J_k^i\left(X_k,\Xi_k\right)$ are quadratic (strictly convex) functions of u_k^i and thus they are minimized when:

$$\frac{\partial J_k^i\left(X_k,\Xi_k\right)}{\partial u_k^i}=\xi_k^i\left[S_k^1\left(y_k\right)u_k^i+S_k^2\left(y_k\right)v_k+S_k^3\left(y_k\right)x_k^i+S_k^4\left(y_k\right)z_k\right]=0$$

where:

$$S_k^1\left(y_k\right)=2S_k^{uu}+\frac{1}{M}\left(S_k^{uv}\right)^T+\frac{1}{M}S_k^{vu}+2B^T\hat{\Sigma}\left(y_k\right)B$$

$$S_k^2\left(y_k\right)=S_k^{uv}+\left(S_k^{vu}\right)^T+\frac{1}{M}S_k^{vv}+\frac{1}{M}\left(S_k^{vv}\right)^T$$

$$S_k^3\left(y_k\right)=\left(S_k^{xu}\right)^T+\frac{1}{M}\left(S_k^{xv}\right)^T+S_k^{ux}+\frac{1}{M}S_k^{vx}+2B^T\hat{\Sigma}\left(y_k\right)A$$

$$S_k^4\left(y_k\right)=\left(S_k^{zu}\right)^T+\frac{1}{M}\left(S_k^{zv}\right)^T+S_k^{uz}+\frac{1}{M}S_k^{vz}$$

Nash Equilibria

- The equations

$$S_k^1(y_k)u_k^i + S_k^2(y_k)v_k + S_k^3(y_k)x_k^i + S_k^4(y_k)z_k = 0 \quad \text{for } i \in I_k \qquad (*)$$

describe a Nash equilibrium of Feedback strategies (for the sub-game starting at time step k). The existence and uniqueness of a Nash equilibrium is equivalent to the inevitability of the following set of $My_k \times My_k$ matrices:

$$G(y_k) = \begin{bmatrix} S_k^1 + S_k^2/M & S_k^2/M & \mathrm{L} & S_k^2/M \\ S_k^2/M & S_k^1 + S_k^2/M & \mathrm{L} & S_k^2/M \\ \mathrm{M} & \mathrm{M} & \mathrm{O} & \mathrm{M} \\ S_k^2/M & S_k^2/M & \mathrm{L} & S_k^1 + S_k^2/M \end{bmatrix}$$

for any $y_k = 1/M, \ldots, 1$

Nash Equilibria

- Adding the equations we have:

$$\left(S_k^1(y_k)+y_k S_k^2(y_k)\right)v_k=-\left[S_k^3(y_k)+y_k S_k^4(y_k)\right]z_k$$

if $S_k^1(y_k)+y_k S_k^2(y_k)$ is <u>nonsingular</u> then:

$$v_k=-\left(S_k^1(y_k)+y_k S_k^2(y_k)\right)^{-1}\left[S_k^3(y_k)+y_k S_k^4(y_k)\right]z_k$$

If $S_k^1(y_k)$ is also <u>nonsingular</u> then:

$$u_k^i=-\left(S_k^1(y_k)\right)^{-1}S_k^3(y_k)x_k^i+\left(S_k^1(y_k)\right)^{-1}\cdot$$

$$\cdot\left[S_k^2(y_k)\left(MS_k^1(y_k)+My_k S_k^2(y_k)\right)^{-1}\left(MS_k^3(y_k)+My_k S_k^4(y_k)\right)-S_k^4(y_k)\right]z_k$$

Thus u_k^i is in the form:

$$u_k^i=L_k^1(y_k)z_k+L_k^2(y_k)x_k^i$$

Nash Equilibria

Lemma: The following are equivalent:

(1) The matrices $S_k^1(y_k) + y_k S_k^2(y_k)$ are nonsingular for any y_k and the matrices $S_k^1(y_k)$ are nonsingular for $My_k > 1$.

(2) The matrices $G(y_k)$ are nonsingular for any y_k.

Proof: $(1) \Rightarrow (2)$. If (1) holds then for any $x_k^i,\ i \in I_k$ there exist a unique solution to the equations $(*)$. Thus the matrices $G(y_k)$ are nonsingular.

$(2) \Rightarrow (1)$ Adding equations $(*)$ we conclude that $S_k^1(y_k) + y_k S_k^2(y_k)$ is nonsingular for any y_k. The inequalities $rank(S_k^1) < m$ and $My_k \geq 2$ contradicts the inequality:

$$My_k m = rank(G(y_k)) \leq My_k rank(S_k^1) + rank(S_2^k) \leq My_k rank(S_k^1) + m$$

Nash Equilibria

- Using the feedback control laws obtained we go back one step in the Dynamic Programming algorithm.

$$\Lambda_k^{xx}\left(y_k\right)=S_k^{xx}+S_k^{xu}L_k^2+\left(L_k^2\right)^T S_k^{ux}+\left(L_k^2\right)^T S_k^{uu}L_k^2$$

$$\Lambda_k^{xz}\left(y_k\right)=S_k^{xz}+S_k^{xu}L_k^1+S_k^{xv}\left(y_kL_k^1+L_k^2\right)+\left(L_k^2\right)^T S_k^{uz}+\left(L_k^2\right)^T S_k^{uu}L_k^1+\left(L_k^2\right)^T S_k^{uv}\left(y_kL_k^1+L_k^2\right)$$

$$\Lambda_k^{zx}\left(y_k\right)=S_k^{zx}+S_k^{zu}L_k^2+\left(L_k^1\right)^T S_k^{ux}+\left(L_k^1\right)^T S_k^{uu}L_k^2+\left(y_kL_k^1+L_k^2\right)^T S_k^{vx}+\left(y_kL_k^1+L_k^2\right)^T S_k^{vu}L_k^2$$

$$\begin{aligned}\Lambda_k^{zz}\left(y_k\right)=\;&S_k^{zz}+S_k^{zu}L_k^1+S_k^{zv}\left(y_kL_k^1+L_k^2\right)+\left(L_k^1\right)^T S_k^{uz}+\left(L_k^1\right)^T S_k^{uu}L_k^1+\left(L_k^1\right)^T S_k^{uv}\left(y_kL_k^1+L_k^2\right)+\\&+\left(y_kL_k^1+L_k^2\right)^T S_k^{vz}+\left(y_kL_k^1+L_k^2\right)^T S_k^{vu}L_k^1+\left(y_kL_k^1+L_k^2\right)^T S_k^{vv}\left(y_kL_k^1+L_k^2\right)+\\&+M\left[\left(A+BL_k^2\right)^T\hat{\Sigma}\left(y_k\right)BL_k^1+\left(L_k^1\right)^T B^T\hat{\Sigma}\left(y_k\right)\left(A+BL_k^2\right)+y_k\left(L_k^1\right)^T B^T\hat{\Sigma}\left(y_k\right)BL_k^1\right]\end{aligned}$$

$$\Sigma_k\left(y_k\right)=\left(A+BL_k^2\right)^T\hat{\Sigma}\left(y_k\right)\left(A+BL_k^2\right)$$

Nash Equilibria

Proposition: Consider the matrices Λ_k, S_k, Σ_k and $\hat{\Sigma}_k$ computed recursively and suppose that the Condition (1) of the Lemma holds. Then there exist a unique time consistent Nash equilibrium of feedback strategies in the form:

$$u_k^i = L_k^1\left(y_k\right)z_k + L_k^2\left(y_k\right)x_k^i \quad .$$

Conversely suppose that there exist a unique Nash equilibrium of Feedback Strategies and that it holds $\pi_{k,l,l'} > 0$. Then the Condition (1) of the Lemma holds.

Remark: The condition (1) of Lemma is Generic. Thus if the parameters of the game are chosen randomly we expect that there exist a unique Nash equilibrium of Feedback Strategies.

Structure

- Related Topics
- Description
- Nash Equilibria
- **Large Population Case**
- Special Cases
- Conclusion and Future Work

Large Population Case: Description

- Consider a sequence of games g^M. The M-th game has M players.
- Dynamics:

$$x_{k+1}^{i,M} = Ax_k^{i,M} + Bu_k^{i,M} + F\left(y_k^M\right)z_k^M$$

where:

$$z_k^M = \sum_{i=1}^{M} \xi_k^{i,M} x_k^{i,M} / M$$

- The cost functions $Q_k\left(y_k^M\right), R_k\left(y_k^M\right)$ are samples of continuous matrix functions.

- Consider the sequence of stochastic kernels

$$\bar{K}^M\left(y,B\right) = \Pr\left(y_{k+1}^M \in B \mid y_k^M = round\left(My\right)/M\right)$$

and suppose it converges to a Feller continuous limit $\bar{K}^{\lim}\left(\cdot,\cdot\right)$

Large Population Case: The Limit Problem

- Consider a solution to the limit of the finite number of players solutions.
- Define recursively the quantities:

$$\left(J_k^i\right)^{\lim}\left(X_k,\Xi_k\right)=\xi_k^i\left[\left(x_k^i\right)^T \quad z_k^T \quad \left(u_k^i\right)^T \quad v_k^T\right]S_k^{\lim}\left(y_k\right)\left[\left(x_k^i\right)^T \quad z_k^T \quad \left(u_k^i\right)^T \quad v_k^T\right]^T$$

where $S_k^{\lim}\left(y_k\right)$ is the limit of the matrices $S_k^M\left(y_k\right)$ and the matrices $\Sigma_k^M\left(y_k\right)\rightarrow 0$ as $M\rightarrow\infty$.

- The control is given by:

$$u_k^{i,\lim}=L_k^{1,\lim}\left(y_k\right)z_k+L_k^{2,\lim}\left(y_k\right)x_k^{i,\lim}$$

Large Population Case: Parameters

Where:

$$S_k^{xu,\lim}(y_k) = A^T \tilde{\Lambda}_{k+1}^{xx}(y_k) B$$

$$S_k^{xz,\lim} = A^T \tilde{\Lambda}_{k+1}^{xx}(y_k) F(y_k) + A^T \bar{\Lambda}_{k+1}^{xz}(y_k)\left[A + y_k F(y_k)\right]$$

$$S_k^{xx,\lim} = A^T \tilde{\Lambda}_{k+1}^{xx}(y_k) A$$

$$S_k^{xv,\lim} = A^T \bar{\Lambda}_{k+1}^{xz}(y_k) B$$

$$S_k^{zz,\lim} = F^T(y_k)\tilde{\Lambda}_{k+1}^{xx}(y_k)F(y_k) + F^T(y_k)\bar{\Lambda}_{k+1}^{xz}(y_k)\left[A + y_k F(y_k)\right] + \left[A^T + y_k F^T(y_k)\right]\bar{\Lambda}_{k+1}^{zx}(y_k)F(y_k) + $$
$$+ \left(A + y_k F(y_k)\right)^T \bar{\bar{\Lambda}}_{k+1}^{zz}(y_{k+1})\left(A + y_k F(y_k)\right)$$

$$S_k^{zu,\lim} = F^T(y_k)\tilde{\Lambda}_{k+1}^{xx}(y_k) B + \left(A^T + y_k F^T(y_k)\right)^T \bar{\Lambda}_{k+1}^{zx}(y_k) B$$

$$S_k^{zv,\lim} = F^T(y_k)\bar{\Lambda}_{k+1}^{xz}(y_k) B + \left(A + y_k F(y_k)\right)^T \bar{\bar{\Lambda}}_{k+1}^{zz}(y_{k+1}) B$$

$$S_k^{uu,\lim} = B^T \tilde{\Lambda}_{k+1}^{xx}(y_k) B$$

$$S_k^{uv,\lim} = 0$$

$$S_k^{vv,\lim} = B^T \bar{\bar{\Lambda}}_{k+1}^{zz}(y_{k+1}) B$$

$$\tilde{\Gamma}_{k+1}(y_k) = \int \frac{s}{y_k}\Gamma_{k+1}(s)\bar{K}^{\lim}(y_k, ds)$$

$$\bar{\Gamma}_{k+1}(y_k) = \int \frac{s^2}{(y_k)^2}\Gamma_{k+1}(s)\bar{K}^{\lim}(y_k, ds)$$

$$\bar{\bar{\Gamma}}_{k+1}(y_k) = \int \frac{s^3}{(y_k)^3}\Gamma_{k+1}(s)\bar{K}^{\lim}(y_k, ds)$$

Large Population Case: Parameters

$$L_k^{1,\lim}(y_k) = \left(S_k^{1,\lim}(y_k)\right)^{-1} S_k^{2,\lim}(y_k)\left(S_k^{1,\lim}(y_k) + y_k S_k^{2,\lim}(y_k)\right)^{-1}\left(S_k^{3,\lim}(y_k) + y_k S_k^{4,\lim}(y_k)\right) - \left(S_k^{1,\lim}(y_k)\right)^{-1} S_k^{4,\lim}(y_k)$$

$$L_k^{2,\lim}(y_k) = -\left(S_k^{1,\lim}(y_k)\right)^{-1} S_k^{3,\lim}(y_k)$$

Where:

$$S_k^{1,\lim}(y_k) = 2S_k^{uu,\lim} \qquad S_k^{2,\lim}(y_k) = S_k^{uv,\lim} + \left(S_k^{vu,\lim}\right)^T$$

$$S_k^{3,\lim}(y_k) = \left(S_k^{xu,\lim}\right)^T + S_k^{ux,\lim} \qquad S_k^{4,\lim}(y_k) = \left(S_k^{zu,\lim}\right)^T + S_k^{uz,\lim}$$

- In order to go back one step in the recursive computations

$$\Lambda_k^{xx,\lim}(y_k) = S_k^{xx} + S_k^{xu}L_k^2 + \left(L_k^2\right)^T S_k^{ux} + \left(L_k^2\right)^T S_k^{uu}L_k^2$$

$$\Lambda_k^{xz,\lim}(y_k) = S_k^{xz} + S_k^{xu}L_k^1 + S_k^{xv}\left(y_k L_k^1 + L_k^2\right) + \left(L_k^2\right)^T S_k^{uz} + \left(L_k^2\right)^T S_k^{uu}L_k^1 + \left(L_k^2\right)^T S_k^{uv}\left(y_k L_k^1 + L_k^2\right)$$

$$\Lambda_k^{zx,\lim}(y_k) = S_k^{zx} + S_k^{zu}L_k^2 + \left(L_k^1\right)^T S_k^{ux} + \left(L_k^1\right)^T S_k^{uu}L_k^2 + \left(y_k L_k^1 + L_k^2\right)^T S_k^{vx} + \left(y_k L_k^1 + L_k^2\right)^T S_k^{vu}L_k^2$$

$$\Lambda_k^{zz,\lim}(y_k) = S_k^{zz} + S_k^{zu}L_k^1 + S_k^{zv}\left(y_k L_k^1 + L_k^2\right) + \left(L_k^1\right)^T S_k^{uz} + \left(L_k^1\right)^T S_k^{uu}L_k^1 + \left(L_k^1\right)^T S_k^{uv}\left(y_k L_k^1 + L_k^2\right) + \left(y_k L_k^1 + L_k^2\right)^T S_k^{vz} + \left(y_k L_k^1 + L_k^2\right)^T S_k^{vu}L_k^1 + \left(y_k L_k^1 + L_k^2\right)^T S_k^{vv}\left(y_k L_k^1 + L_k^2\right) +$$

Large Population Case: ε – Nash Equilibrium

Proposition: Suppose that the matrices $S_k^{1,\lim}(y_k) + y_k S_k^{2,\lim}(y_k)$ and $S_k^{1,\lim}(y_k)$ computed recursively using the formulas above are nonsingular for any y_k .
Then the set of strategies given by:

$$u_k^{i,M} = L_k^{1,\lim}(y_k) z_k^M + L_k^{2,\lim}(y_k) x_k^{i,M}$$

for $i = 1,....,M$ constitute an ε – Nash equilibrium for sufficiently large value of M

Ideas of the proof : Introduce the quantities:

$$\bar{J}_k^{i,M}(X_k, \Xi_k, u_k^i) = \underset{\substack{u_t^j = u^{\lim} \\ t=k+1,\dots,N-1 \\ u_k^j = u^{\lim} \quad j \neq i}}{E} \Bigg\{ \left[(x_N^i)^T \quad z_N^T \right] Q(y_N) \left[(x_N^i)^T \quad z_N^T \right]^T \xi_N^i + \\ + \sum_{t=k}^{N-1} \xi_t^i \left(\left[(x_t^i)^T \quad z_t^T \right] Q(y_t) \left[(x_t^i)^T \quad z_t^T \right]^T + (u_t^i)^T R(y_t) u_t^i \right) \Bigg\}$$

$$\tilde{J}_k^{i,M}(X_k, \Xi_k, u_k^i) = \underset{\substack{u_t^j = u^{\lim}, j \neq i \\ t=k,\dots,N-1 \\ u_t^i := \min J^i \\ t=k+1,\dots,N\text{-}1}}{E} \Bigg\{ \left[(x_N^i)^T \quad z_N^T \right] Q(y_N) \left[(x_N^i)^T \quad z_N^T \right]^T \xi_N^i + \\ + \sum_{t=k}^{N-1} \xi_t^i \left(\left[(x_t^i)^T \quad z_t^T \right] Q(y_t) \left[(x_t^i)^T \quad z_t^T \right]^T + (u_t^i)^T R(y_t) u_t^i \right) \Bigg\}$$

Large Population Case: ε – Nash Equilibrium

Ideas of the proof : It holds:

$$\bar{J}_k^{i,M}\left(X_k,\Xi_k,u^{\lim}\right)=\xi_k^i\left[\left(x_k^{i,M}\right)^T \quad \left(z_k^{i,M}\right)^T\right]\Lambda_k^{M,\lim}\left[\left(x_k^{i,M}\right)^T \quad \left(z_k^{i,M}\right)^T\right]^T+\xi_k^i\sum_{j=1}^{M}\xi_k^j\left(x_k^{j,M}\right)^T\Sigma_k^{M,\lim}x_k^{j,M}$$

$$\min_{u_{k+1}^i}\tilde{J}_{k+1}^{i,M}\left(X_{k+1},\Xi_{k+1},u_{k+1}^i\right)=\xi_k^i\left[\left(x_{k+1}^{i,M}\right)^T \quad \left(z_{k+1}^{i,M}\right)^T\right]\tilde{\Lambda}_{k+1}^{M,\lim}\left[\left(x_{k+1}^{i,M}\right)^T \quad \left(z_{k+1}^{i,M}\right)^T\right]^T+\xi_{k+1}^i\sum_{j=1}^{M}\xi_{k+1}^j\left(x_{k+1}^{j,M}\right)^T\tilde{\Sigma}_{k+1}^{M,\lim}x_{k+1}^{j,M}$$

$$\bar{J}_k^{i,M}\left(X_k,\Xi_k,u_k^i\right)=\xi_k^i\left[\left(x_k^i\right)^T \quad z_k^T \quad \left(u_k^i\right)^T \quad v_k^T\right]S_k^{M,\lim}\left(y_k\right)\left[\left(x_k^i\right)^T \quad z_k^T \quad \left(u_k^i\right)^T \quad v_k^T\right]^T+$$
$$+\xi_k^i\sum_{j=1}^{M}\left[\xi_k^j\left(Ax_k^j+Bu_k^j\right)^T\hat{\Sigma}_k^{M,\lim}\left(y_k\right)\left(Ax_k^j+Bu_k^j\right)\right]$$

$$\tilde{J}_k^{i,M}\left(X_k,\Xi_k,u_k^i\right)=\xi_k^i\left[\left(x_k^i\right)^T \quad z_k^T \quad \left(u_k^i\right)^T \quad v_k^T\right]\tilde{S}_k^{M,\lim}\left(y_k\right)\left[\left(x_k^i\right)^T \quad z_k^T \quad \left(u_k^i\right)^T \quad v_k^T\right]^T+$$
$$+\xi_k^i\sum_{j=1}^{M}\left[\xi_k^j\left(Ax_k^j+Bu_k^j\right)^T\tilde{\Sigma}_k^{M,\lim}\left(y_k\right)\left(Ax_k^j+Bu_k^j\right)\right]$$

Large Population Case: ε – Nash Equilibrium

Ideas of the proof : It is shown that for any $\varepsilon > 0$ there exist a positive integer M_0 such that:

$$\left|\tilde{S}_k^{M,\lim}(y_k) - S_k^{M,\lim}(y_k)\right| < \varepsilon \quad \left\|\tilde{\Lambda}_k^{\mathrm{M},\lim} - \Lambda_k^{\mathrm{M},\lim}\right\| < \varepsilon \quad \left\|\Lambda_k^{\mathrm{M},\lim} - \Lambda_k^{\lim}\right\| < \varepsilon \quad \left\|\tilde{\Sigma}_k^{M,\lim}(y_k)\right\| < \varepsilon / \mathrm{M}$$

for any $M \geq M_0$.

Proof by induction. Principle of optimality implies:

$$\begin{aligned}\min_{u_k^i} \tilde{J}_k^{M}\left(X_k, \Xi_k, u_k^i\right) = \min_{u_k^i} \Bigg\{ & \xi_k^i \left(\left[\left(x_k^i\right)^T \quad z_k^T \right] Q(y_k) \left[\left(x_k^i\right)^T \quad z_k^T \right]^T + \left(u_k^i\right)^T R(y_k) u_k^i \right) \\ & + \xi_k^i \left[\left(x_{k+1}^{i,M}\right)^T \quad \left(z_{k+1}^{i,M}\right)^T \right] \Lambda_{k+1}^{\mathrm{M},\lim} \left[\left(x_{k+1}^{i,M}\right)^T \quad \left(z_{k+1}^{i,M}\right)^T \right]^T + \xi_{k+1}^i \sum_{j=1}^{M} \xi_{k+1}^j \left(x_{k+1}^{j,M}\right)^T \tilde{\Sigma}_{k+1}^{M,\lim} x_{k+1}^{j,M} + \\ & + \xi_k^i \left[\left(x_{k+1}^{i,M}\right)^T \quad \left(z_{k+1}^{i,M}\right)^T \right] \left[\tilde{\Lambda}_{k+1}^{\mathrm{M},\lim} - \Lambda_{k+1}^{\mathrm{M},\lim} \right] \left[\left(x_{k+1}^{i,M}\right)^T \quad \left(z_{k+1}^{i,M}\right)^T \right]^T \Bigg\}\end{aligned}$$

Large Population Case: ε – Nash Equilibrium

Ideas of the proof :

Thus:

$$\left|\min_{u_0^i} \tilde{J}_0^{i,M}\left(X_0,\Xi_0,u_0^i\right) - \bar{J}_0^{i,M}\left(X_0,\Xi_0,u_0^{i,\lim}\right)\right| < \varepsilon\left(1+\left\|x_0^i\right\|^2+\left\|z_0\right\|^2+\frac{1}{M}\sum_{j=1}^{M}\xi_k^j\left\|x_0^j\right\|^2\right)$$

for M sufficiently large.

The fact that x_0^i are i.i.d. r.v with finite second moments completes the proof.

Large Population Case: ε – Nash Equilibrium

Remark: The assumption that the matrices $S_k^{1,\lim}\left(y_k\right)+y_k S_k^{2,\lim}\left(y_k\right)$ and $S_k^{1,\lim}\left(y_k\right)$ are nonsingular for any $y_k \in \left[0,1\right]$ is not "generic".

Structure

- Related Topics
- Description
- Nash Equilibria
- Large Population Case
- **Special Cases**
- Conclusion and Future Work

Special Cases: Independent Exit

- Consider the case where exit is independent and the number of players is very large. Assume that any player that participate in the game at time step k continue at the step $k+1$ with probability a. Then the limit stochastic kernel becomes:

$$\bar{K}^{\lim}(y,B) = \begin{cases} 1 & \text{if } \; ay \in B \\ 0 & \text{if } \; ay \notin B \end{cases}$$

The operators $\tilde{\Gamma}, \bar{\Gamma}, \bar{\bar{\Gamma}}$ become: $\tilde{\Gamma}_{k+1}(y_k) = a\Gamma_{k+1}(ay_k)$, $\bar{\Gamma}_{k+1}(y_k) = a^2\Gamma_{k+1}(ay_k)$ and $\bar{\bar{\Gamma}}_{k+1}(y_k) = a^3\Gamma_{k+1}(ay_k)$.

The values of the parameters $S_k^{\lim}(y_k)$, $\Lambda_k^{xx,\lim}(y_k)$, $L_k^{1,\lim}(y_k)$ and $L_k^{2,\lim}(y_k)$ are computed only on:

$$y_k = a^k$$

Thus the computations in this case are simplified.

Special Cases: Independent Exit

- The evolution of z_k is asymptotically deterministic.

  Due to the independence of exit and independence of initial conditions we have:

$$z_{k+1} = \frac{1}{M}\sum_{j=1}^{M} \xi_{k+1}^{j}\left(A + BL_k^2\left(y_k\right)\right)x_k^j + \frac{1}{M}\sum_{j=1}^{M} \xi_{k+1}^{j}\left(BL_k^1\left(y_k\right) + F\left(y_k\right)\right)z_k$$

$$z_{k+1} \xrightarrow{d} S_k z_k$$

  Where:

$$S_k = a\left(A + BL_k^2\left(y_k\right)\right) + a^k\left(BL_k^1\left(y_k\right) + F\left(y_k\right)\right)$$

  Thus the strategies $u_k^{i,M} = L_k^{1,\lim}\left(y_k\right)z_k + L_k^{2,\lim}\left(y_k\right)x_k^{i,M}$ could be implemented approximately without measuring z_k and y_k i.e. using the approximation:

$$z_k = S_{k-1}\ldots S_1 S_0 \overline{x}$$

Special Cases: Random Initial Population

- Suppose that the initial population is unknown i.e. M is a random variable. Assume that it takes values $1,\dots,M_{\max}$ with probabilities $P_1,\dots.,P_{M_{\max}}$.
- Reduce this case to the initial. Introduce a step with $k=-1$. The initial number of players is $M_{\max}$.
- Solve the same equations when:

$$A_{-1}=1$$

$$B_{-1}=F_{-1}\left(y_{-1}\right)=Q_{-1}\left(y_{-1}\right)=0$$

The values for $\pi_{-1,M_{\max},l}$ are:

$$\pi_{-1,M_{\max},l}=P_l$$

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Conclusion and future work: Conclusions

- LQ games with random exit are studied. Necessary and sufficient conditions for the existence of a unique Nash Equilibrium were obtained.
- Formulas for computing the Nash Equilibrium were derived.
- Games with large number of players were studied using a limit approximation. ε – Nash Equilibrium results were obtained.
- The case of games with large number of players with independent exit was studied. The formulas are quite simpler.

Conclusion and future work: Future work

- Study the game as the time horizon tends to infinity as well as the continuous limit.
- Weaken the symmetry assumptions imposed.
- Study numerically the dependence of the cost of the players on the exit distribution and the speed of convergence in the limit approach.
- Study games with random entrance and random exit

Thanks !!