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Games on Large Random Interaction Structures: Information and Complexity Aspects

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Abstract— **Games on large structures of interacting agents are considered. The participants of the game do not have a full knowledge of the interaction structure or the characteristics of the other players. Instead of that, an ensemble of possible interaction structures as well as a probability measure on that ensemble are assumed to be a common knowledge among the players. Furthermore, we assume that the agents have also local information. Specifically, they know the characteristics of some players, important for them. A new notion of equilibrium, describing approximate Nash equilibrium with high probability, is introduced. A concept of complexity of a game is also defined, as the minimum amount of information needed, in order to play almost optimally. Some special cases are then analyzed. Particularly, games on random graphs are considered and are shown to be simple, under high connectivity assumptions. Games on rings, under quadratic and non quadratic cost functions, are finally studied. Bounds on the complexity of the ring games are derived.**

I. INTRODUCTION

In several game situations involving many agents, the strategic interactions depend on a large interaction structure such as a network. Several examples involve social networks [1], such as the selection of a telecommunication company, the opinion about an idea or a product, the selection of fashion group and the engagement in criminal behavior. In these examples, the choice of each agent depends on the choices of his/her friends. Examples of non-social interaction structures involve the interaction among the owners of stores for renting, the gas station prices and the producers connected in different places of a large electricity transmission grid, where there exists a local as well as a global competition.

Two kinds of approaches have been mainly used to predict the behavior of the participants in large games. The first approach is based on equilibrium concepts. The dominant notion in this approach is the Nash equilibrium and a complete knowledge of a large amount of information is needed. The second kind of approaches assumes limited rationality for the participants and it is based on dynamic formulations. In particular, some deterministic or stochastic rules describing the future actions of the agents as a function of their current actions are postulated and then evaluated experimentally or theoretically. This kind of approach does not require a complete knowledge of the game. However, the dynamic rules used are not universal, in the sense that there is no reason to believe that all the players will follow some specified rule to determine their future actions. This work aims to stand between these two approaches, assuming only a partial knowledge of the game and players of a full rationality.

A. Related Topics

The interest for the games with large number of players is not new. In [2], games with a continuum of players, called Oceanic Games, were introduced and a value for such games was defined. The Mean Field Games [3] have been introduced recently, to study games with large number of players. The closely related methodology of Nash Certainty Equivalence was also developed, in order to obtain asymptotic Nash equilibrium results, as the number of players tends to infinity [4]. These approaches study games, where each player interacts with the mass of the other players, which is approximated by a continuum.

Another related topic is Games with Local Interactions, in which each player interacts with some players important to him/her on some organized structure. In [5], equilibria for complete and incomplete information Local Interaction Games where found, based on contraction mapping ideas. The dynamic game counterpart is presented in [6]. Models with discrete choice were introduced in [7].

Games, where players move on a graph were analyzed in [8] and finite games on graphs, where each node corresponds to a participant of the game, were studied in [9]. Repeated games with random matching of the opponents were introduced in [10], in the context of sustainability of cooperation and social norms. The probability of existence of a Nash equilibrium for games on random graphs is studied in [11]. A quadratic game on networks is studied in [12] using centrality notions. Games on networks with incomplete information are studied in [13]. Dynamic games on evolving (state dependent) graphs were studied in [14] and stochastic games in [15]. A review of network games is given in [16].

Dynamic rules for updating the actions of the agents on lattices were studied in the context of Interacting Particle Systems [17] and in [18]. Several dynamic rules for games on graphs were introduced and studied analytically and computationally [19]. Several social applications of evolutionary games on graphs were studied in [20].

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B. Our Approach

We assume that the agents have a statistical knowledge about the game they are involved, instead of a full knowledge. Particularly, there is an ensemble (set, collection) of possible games that the players may participate in. A probability measure on the ensemble, common to all the players, is assumed. The players can also measure the interaction structure locally, i.e. they know the characteristics and the interactions of some players, that are important for them.

Based on such a model, a new approximate equilibrium concept is defined. Specifically, we study sets of strategies depending only on local information that constitute an ϵ - Nash equilibrium with very high probability. A new notion of complexity for an ensemble of games is defined as the minimum amount of information needed, in order to achieve an approximate equilibrium.

C. Organization

The rest of the paper is organized as follows: In Section II, the game is described. In section III, an approximate equilibrium concept is defined and compared with other equilibrium concepts. In Section IV, the complexity function for an ensemble of games is defined. Sections V and VI contain some examples of special cases. Section V studies games on random directed graphs and shows that are asymptotically simple under high connectivity assumptions. Section VI studies quadratic and non-quadratic games on rings and derives some bounds on the complexity functions. Section VII concludes.

D. Notation

A directed or undirected graph will be denoted by $G = (V, E)$, where V is the set of vertices and E the set of edges. For a $v \in V$, the neighborhood of v is defined as $N_v^1(G) = \{v\} \cup \{v' \in V : (v', v) \in E\}$. The neighborhood of order n of v is defined as $N_v^n(G) = \cup_{j \in N_v^1(G)} N_j^{n-1}(G)$. An ordered tuple $(u_1, \ldots, u_N)$ is denoted by $(u_i)_i$ and the ordered tuple $(u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_N)$ by u_{-i}.

The asymptotic notation will be also used. For real functions f and g, we write $f(x) \in O(g(x))$, if there exists a constant $c > 0$, such that $0 \leq f(x) \leq cg(x)$ for large x. We write $f(x) \in \omega(g(x))$ if for any $c > 0$ it holds $0 \leq cg(x) \leq f(x)$ for large x and $f(x) \in \Theta(g(x))$ if there exist constants c_1 and c_2 such that $0 \leq c_1 g(x) \leq f(x) \leq c_2 g(x)$ for large x. Finally, $f(x) \in \Omega(g(x))$ if for some constant c, it holds $f(x) \geq cg(x)$, for large x.

In what follows, we assume that all the functions involved are measurable within appropriate measurable spaces.

II. Game Description

The game depends on an interaction structure, which involves a set of N players $p_1, \ldots, p_N$. Each player has a state variable or type x_i. The action of player i is denoted by u_i and belongs to a set U. The interaction structure is defined as:

$$S = (\Pi, G), \tag{1}$$

where $\Pi = ((p_1, x_1), \ldots (p_N, x_N))$ and $G = (V, E)$ is a graph (directed or not), describing the interactions. Each vertex of the graph G represents an agent from $\{p_1, \ldots, p_N\}$ and each edge represents the influence of an agent to another.

The players do not know the interaction structure. Instead, they know that the interaction structure belongs to a set of interaction structures denoted by $\mathcal{E}$. Using the Statistical Physics terminology, we shall call this set $\mathcal{E}$, an ensemble. The players also have a probability measure P on a $\sigma-$algebra $\mathcal{F}$ of subsets of $\mathcal{E}$. We assume that the probability space $(\mathcal{E}, \mathcal{F}, P)$ is a common knowledge among the players.

Apart from the statistical model, each agent also knows its own type and has some local information about the interaction structure. Particularly, the information of player i includes the structure of the subgraph of its neighborhood of order n, denoted by $N_i^n(G)$, as well as the types of the players involved in that subgraph. Let us denote by I_i the information of player i.

The strategies are defined as functions of the information and have the form: $u_i = \gamma_i(I_i)$. We shall focus on symmetric sets of strategies, i.e. sets of strategies where players with the same information (and hence type) behave the same. The strategies under consideration have, thus, the form:

$$u_i = \gamma(I_i). \tag{2}$$

The cost function for player i, depends on his/her own action, u_i and the actions of the agents that the player i has a connection from:

$$J_i = \sum_{j \in N_i^1(G)} g_1(x_i, x_j, u_i, u_j) + \sum_{j \in V} g_2(x_i, x_j, u_i, u_j). \tag{3}$$

Remark 1: (i) The cost functions given by equation (3) describe two types of interactions. The first sum corresponds to local interactions and the second term to mean field interactions.

(ii) The members of the ensemble do not need to have the same number of players and the ensemble is not necessarily finite.

(iii) Models involving graphs with information on their edges could also be studied, as well as structures more general than the graphs relating more than two agents. However, for simplicity reasons, only the undirected and directed graph cases are considered.

(iv) The stochastic model presented describes the lack of knowledge of the players and not necessarily that the game is in fact stochastic. Thus, if we consider the repeated game case, the players would play the same game at each time step.

(v) The model defined could be generalized also in cases where the players have slightly different probabilities on the ensemble. However, the model here is different from the hypergame models [21].

In the following sections, asymptotic results for large interaction structures are derived. In order to do so, we define a sequence of ensembles describing games with increasing number of players.

III. Approximate Equilibrium

Consider a large game in which the actions of the players depend only on local and statistical information. Particularly, the strategies have the form (2). It is reasonable to expect that, due to the lack of complete information, such a set of strategies could not typically constitute a Nash equilibrium. Thus, an approximate equilibrium concept is studied. The following definition describes sets of strategies that are in ϵ - Nash equilibrium with high probability.

Definition 1: Consider an ensemble of interaction structures $\mathcal{E}$ and a set of cost functions given (3). Then a set of strategies $u_i = \gamma(I_i)$ is ϵ - fine if it holds:

$$P((u_i)_i \text{ is an } \epsilon\text{- Nash equilibrium}) > 1 - \epsilon, \qquad (4)$$

where the probability is computed over the several possible interaction structures.

The agents act without knowing in which game they play. Thus, the best they could expect is to play well in many of the possible games. For large information neighborhoods, i.e. for large n, we may expect that, under some conditions, there exists an ϵ - fine set of strategies with small ϵ and that $\epsilon \to 0$ as the information of the players approaches the full information.

Remark 2: The reason for studying ϵ - fine sets of strategies with small ϵ, is that in a large interaction structure, with a very high probability, no player has non-negligible benefit from changing his/her strategy. Furthermore, in case that the strategies are close to the optimum, under continuity assumptions, any player that moves to the optimum changes his/her strategy very little. Thus, each player expects, except himself, also the other players to change a little. Thus, he/she expects to stay for a long-time interval near the initial point.

Remark 3: Another notion describing equilibria in games with incomplete information is Bayesian Nash equilibrium. Let us point out some differences of the Bayesian Nash equilibrium concept and ϵ - fine concept.

(i) Consider a set of strategies, constituting a Bayesian Nash equilibrium. Each strategy minimizes the expected cost of the agent, given the strategies of the other players. The expectation is computed over all the possible games. However, the agents play in a particular game. Thus, each agent has probably a non-negligible motivation to change his/her strategy. In contrast to the Bayesian equilibrium, with an ϵ - fine set of strategies no player has a non-negligible motivation to change his/her strategy, in the vast majority of games.

(ii) In contrast to Bayesian Nash equilibrium, an ϵ - fine set of strategies is insensitive to new information. Particularly, assume that $(u_i)_i$ is an ϵ - fine set of strategies and some players receive more information $I_i' \supset I_i$. Then $(u_i)_i$ remains ϵ - fine.

(iii) The notion of ϵ - fine set of strategies is insensitive to the risk profile of the agents.

(iv) The notion of ϵ - fine set of strategies avoids the need to compute expectations, i.e. sums or integrals, on spaces of high dimensions.

IV. Complexity

With a small amount of information it is probably not possible to have an ϵ - fine set of strategies. Thus, we are interested in the following question:

Question 1: "Given a positive constant ϵ, what is the minimum amount of information that the agents need to have in order to achieve an ϵ - fine set of strategies?"

Based on the answer to this question, a complexity notion for an ensemble of games is defined. In several cases, it is easier to answer Question 1, when the game has a large number of players. The following definition refers to a sequence of ensembles $\mathcal{E}_\nu$ and a complexity function is defined.

Definition 2: (i) Consider an ensemble $\mathcal{E}$ and the cost functions given by equation (3). Let us define the following function:

$$\begin{aligned}\bar{n}(m) = \inf\{n \in \mathbf{N} : \exists (u_i)_i, u_i = \gamma(I(N_i^n(G)) \text{ which} \\ \text{is an } 2^{-m} \text{ - fine set of strategies}\}. \qquad (5)\end{aligned}$$

The Necessary Information Complexity (NIC) function is defined as:

$$C(m) = \max\{\# N_i^{\bar{n}(m)}\}, \qquad (6)$$

where the maximum is taken over the every player i, participating in each ensemble, that a 2^{-m} fine set of strategies exists.

(ii) Consider a sequence of ensembles $\mathcal{E}_\nu$ with the cost functions J_i^ν. Denote by $C^\nu(\cdot)$ the NIC function of the ν-th ensemble. The Asymptotic Necessary Information Complexity (ANIC) function is given by:

$$C_a(m) = \limsup_{\nu \to \infty} C^\nu(m). \qquad (7)$$

The sequence of ensembles will be called asymptotically simple if $C_a(m)$ is bounded and asymptotically complex if for some $m \in \mathbf{N}$ it holds $C_a(m) = \infty$.

Remark 4: The NIC and ANIC functions depend on the ensembles of games and not the particular realizations.

Remark 5: The complexity concepts defined could be used in order to give us some indication about the kind of strategies that the participants of a large game would use. Particularly, it is reasonable to assume that in a game with a large number of players, which is a member of an asymptotically complex sequence of ensembles, the players will behave using dynamic formulations. In contrast, in a game with a large number of players, which is a member of an asymptotically simple sequence of ensembles, it is sensible to assume that the players would behave based on equilibrium concepts.

Remark 6: Several static and dynamic games having only mean field interactions have been studied in the literature. In these cases, under some conditions, each player needs to know only his/her type in order to behave nearly optimally. Thus, the Mean Field Games is a first example of simple games.

In the following sections examples of asymptotically simple and complex ensembles will be given.

V. Random Graph Games

In this example, an ensemble of games where the players interact on a large random directed graph is studied. It is shown that the ensemble is asymptotically simple, if the connection probability is high and asymptotically complex if the connection probability is low.

Let us first describe the game. The agents know the number of their opponents and $N = \nu$. The type of each player i is denoted by x_i and belongs to the closed interval $[0, L]$. The random variables x_i are independent and have uniform distributions.

The random directed graph is described by its connection probability. Let a_{ij} be random variables describing the existence of a link from i to j, i.e. $a_{ij} = 1$ if the node i has a link to the node j and $a_{ij} = 0$ if it does not. The random variables a_{ij} are mutually independent and independent of x_i. The connection probability is $c_N = P(a_{ij} = 1)$.

The costs of the players are given by:

$$J_i = g(x_i - u_i) + g(u_i - \frac{1}{Nc_N}\sum_{j=1}^{N} a_{ij}u_j), \qquad (8)$$

where g is a strictly convex function with $g(0) = 0$.

Let us first consider strategies depending only on statistical information, $u_i = \gamma(x_i)$. A technique to derive such strategies is to approximate the terms in the cost function by their mean values. Specifically, we shall use the approximation $\bar{u} \simeq \sum_{j=1}^{N} a_{ij}u_j/(Nc_N)$. With this approximation, the cost functions depend only on statistical information. The strategies that minimize the approximate cost functions have the form:

$$u_i = h(x_i, \bar{u}) = \arg\min_{u_i}\{g(x_i - u_i) + g(u_i - \bar{u})\}. \qquad (9)$$

With the strategies given by equation (9), the mean value of the actions should satisfy the following compatibility condition:

$$\bar{u} = E[h(x, \bar{u})], \qquad (10)$$

where the expectation is taken over the distribution of x_i's.

The following proposition shows that, under high connectivity assumptions, the strategies given by (9) are ϵ - fine, for N sufficiently large.

Proposition 1: Under the specified assumptions, the equation (10) has a unique solution. Furthermore, if $c_N \in \omega(lnN/N)$, then for any $\epsilon > 0$, the strategies given by (9) are ϵ - fine for sufficiently large N. Thus, the game is asymptotically simple.

Proof: It is not difficult to show that the function $h(x, u)$ is strictly increasing, on its argument u, and that $\lim_{u\to\pm\infty} h(x, u) = \pm\infty$. Thus, the equation (10) has a unique solution.

The functions g and h are continuous. Using the strategies given by (9), the arguments of the functions belong to compact intervals. In those intervals, g and h are uniformly continuous. Thus, in order to show that the set of strategies given by (9) is ϵ - fine, for large N, it suffices to show that for any $\epsilon, \delta > 0$ it holds:

$$P(\exists i : |\bar{u} - \frac{1}{Nc_N}\sum_{j=1}^{N} a_{ij}u_j| > \delta) < \epsilon,$$

for large N.

It holds $\bar{u} = \int_0^L h(x, \bar{u})/L dx$. The empirical distribution, $\sum_{i=1}^{N} \delta_{x_i}/N$ converges a.s. to the uniform distribution as $N \to \infty$. Thus, there exists an integer N_{01}, such that:

$$\left|\bar{u} - \sum_{i=1}^{N} h(x_i, \bar{u})/N\right| < \delta/2,$$

with probability larger than $1 - \epsilon/2$ for any $N \geq N_{01}$.

Using (9), it remains to show that there exists an integer N_{02}, such that:

$$\left|\frac{1}{Nc_N}\sum_{j=1}^{N}(c_N - a_{ij})h(x_j, \bar{u})\right| < \delta/2,$$

with probability larger than $1 - \epsilon/2$, for any $N \geq N_{02}$.

The random variables $d_{i_j} = (c_N - a_{ij})h(x_j, \bar{u})$ are independent, given $(x_j)_{j=1}^N$. Thus, Bernstein inequality [22] and straightforward calculations imply that:

$$P\left[\left|\sum_{j=1}^{N} d_{i_j}\right| > tNc_N \mid (x_j)_{j=1}^N\right] < 2e^{-\frac{t^2 Nc_N}{2(M^2+M/3)}},$$

for $t < 1$, where $M = \max\{h(x, \bar{u}) : x \in [0, L]\}$. Hence:

$$P\left[\max_i \left[\frac{1}{Nc_N}\left|\sum_{j=1}^{N}(c_N - a_{ij})h(x_j, \bar{u})\right|\right] > t\right] <$$

$$< 1 - \left[1 - 2exp\left\{-\frac{t^2 Nc_N}{2(M^2 + M/3)}\right\}\right]^N,$$

The assumption $c_N \in \omega(lnN/N)$, completes the proof. ∎

Remark 7: If the connectivity is low, the game is not necessarily simple. For example, if $c_N \in o(lnN/N)$ and $c_N \in \Omega(1/N)$, then the game is asymptotically complex. We observe that there is an abrupt change from complex to simple around lnN/N.

VI. Games on Rings

A. Quadratic Games on Rings

This section studies an ensemble of games where the ANIC is asymptotically linear. Particularly, a quadratic game on a ring is considered.

Let us first describe the game. There is a set of $N = \nu$ players, each of which has a state variable x_i. The state variables, x_i, are independent uniformly distributed random variables with values in the closed interval $[-L, L]$. The cost functions are given by:

$$J_i = a\left(u_i - \frac{u_{i+1} + u_{i-1}}{2}\right)^2 + (u_i - x_i)^2, \qquad (11)$$

where $a > 0$ and $N + k \equiv k$ by convention.

In order to show that ANIC is asymptotically linear the following iterative procedure:

$$z_i(t+1) = \frac{a}{2(a+1)}(z_{i+1}(t) + z_{i-1}(t)) + \frac{1}{a+1}x_i, \quad (12)$$

is considered. The right hand side of equation (12) is the best response of the agent i given that the players $i-1$ and $i+1$ use the strategies z_{i-1} and z_{i+1} respectively. The proof of the following proposition, is based on the fact that the mapping $T : z(t) \mapsto z(t+1)$ is contractive, where $z(t) = [z_1(t) \dots z_N(t)]^T$.

Proposition 2: Denote by $C_a(\cdot)$ the ANIC of the game. For any constants, c_1, c_2, such that $c_1 < 1/log_2[2(a+1)/a]$ and $c_2 > 1/log_2[(a+1)/a]$ it holds $c_1 n \le C_a(n) \le c_2 n$, for large n. Thus, $C_a(n) \in \Theta(n)$.

Proof: We first show the second inequality, $C_a(n) \le c_2 n$. To do so, let us observe that the mapping T is contractive, for the infinity norm. Specifically, it has a Lipschitz constant $a/(a+1)$. Assuming that $z(0) = 0$, simple computations imply that:

$$\|z(k) - T(z(k))\|_\infty \le \frac{2La^k}{(a+1)^k}.$$

Thus, it holds:

$$J_i(z(k)) - \min_{u_i} J_i(z_{-i}(k), u_i) \le 4(a+1)L^2 \frac{a^{2k}}{(a+1)^{2k}}.$$

Consider an $n \in \mathbf{N}$. Then:

$$J_i(z(k)) - \min_{u_i} J_i(z_{-i}(k), u_i) \le 2^{-n}, \quad (13)$$

if it holds:

$$k \ge \frac{1}{2log_2(\frac{a+1}{a})}n + \frac{log_2(4(a+1)L^2)}{2log_2(\frac{a+1}{a})}. \quad (14)$$

Let us observe that, the value of $z_i(k)$, depends only on $(x_{i-k-1}, \dots, x_{i+k-1})$, i.e. the types of $2k-1$ players. Thus, due to the equations (13) and (14), for any $c_2 > 1/log_2(\frac{a+1}{a})$, it holds $C_a(n) \le c_2 n$, for large n.

It remains to show the lower bound for $C_a(\cdot)$. To do so, let us observe that there exists a single fixed point of T; denote by $z^\star$. This fixed point corresponds to the single Nash equilibrium and can be computed using (12) recursively in the form:

$$z_i^\star = a_0 x_i + \sum_{l \in \mathbf{N}} a_l [x_{i+l} + x_{i-l}].$$

It is not difficult to show that $a_l > b^{l-1}/(a+1)$, where $b = a/(2a+2)$.

For any strategy profile $\tilde{z}$, it holds:

$$\max_i \left\{ J_i(\tilde{z}) - \min_{u_i} \{J_i(\tilde{z}_{-i}, u_i)\} \right\} = (a+1)\|\tilde{z} - T(\tilde{z})\|_\infty^2.$$

The contractivity of T implies that $\|\tilde{z} - T(\tilde{z})\|_\infty \ge \|\tilde{z} - z^\star\|_\infty/(a+1)$. For any $0 < y < 1/2$, it holds:

$$P(x_{i+k+1} > (1-2y)L) = P(x_{i+k+1} < -(1-2y)L) = y.$$

Thus, for any strategy profile $\tilde{z}$, such that $\tilde{z}_i$ depends only on $x_{i-k}, \dots, x_{i+k}$ it holds:

$$\|\tilde{z} - z^\star\|_\infty \ge |\tilde{z}_i - z_i^\star| > (1-2y)Lb^k/(a+1),$$

with probability greater than or equal to y. Hence:

$$(a+1)\|\tilde{z} - T(\tilde{z})\|_\infty^2 > \frac{(1-2y)^2 L^2}{(a+1)^3} b^{2k} > 2^{-n}$$

with probability greater than or equal to y, if it holds:

$$2k < \frac{n}{log_2 \frac{2(a+1)}{a}} + \frac{log_2 \frac{(1-2y)^2 L^2}{(a+1)^3}}{log_2 \frac{2(a+1)}{a}}$$

Thus, for any $c_1 < 1/log_2 \frac{2(a+1)}{a}$, it holds $C_a(n) \ge c_1 n$ for large n. ∎

Remark 8: The proof for the upper bound part essentially depends only on the contractivity of the best response mapping (12). Thus, this result could be extended to a broader class of cost functions. Furthermore, the graph structure could be more general. For example, multidimensional lattices or graphs with known maximum number neighbors could be considered.

Remark 9: The lower bound result is shown using the computation of the single Nash equilibrium and the fact that a strategy profile far from the Nash equilibrium is not an ϵ - equilibrium, with small ϵ. In the next subsection, ϵ - fine sets of strategies, that are far from any Nash equilibrium, will be derived. It would be interesting to find lower bounds for the ANIC, without finding Nash equilibria.

B. A Non-Quadratic Game on a Ring

This subsection studies a non-quadratic game on a ring, where the best response is not contractive. The game has a linear upper bound for the ANIC. The set of strategies achieving ϵ - equilibrium use some form of cooperation.

Consider a set of $N = \nu$ players on a ring. All the agents are of the same type, type 1, except one which is of type 2. Each player has equal probability to be of type 2. The cost function for a player of type 1 is given by:

$$J_i = (u_i - f(u_{i-1}))^2, \quad (15)$$

where $f(y) = 4y(1-y)$, is the logistic map and the convention $N + k \equiv k$ is used. The cost function of the player of type 2 is given by:

$$J_{i_0} = (u_{i_0} - f(u_{i_0-1}))^2 + (u_{i_0} - x)^2 + \\ + \sum_{j \notin \{i_0, i_0+1\}} (u_j - f(u_{j-1}))^2/(N-2), \quad (16)$$

where i_0 is the player of type 2, x is a part of the state variable of the player i_0 and x is a random variable with uniform distribution in $[0, 1]$.

The best response of a player i of type 1, is $f(u_{i-1})$. Thus, the best response is not contractive. However, there exists a set of strategies which is ϵ - fine with small ϵ as shown in the following proposition.

Proposition 3: The ensemble of games described has an ANIC which is at most linear.

Proof: The proof is constructive. Consider a positive integer n. All the players except of the players $i_0, \dots, i_0+n$ use the strategy $u_i = 3/4$, which is the single nonzero fixed point of f. The agent i_0 uses a strategy which is very close to his/her optimal action, $y = x/2 + 3/8$. All the other players, react optimally to the player i_0, i.e. $\bar{u}_{i_0+1} = f(\bar{u}_{i_0})$, $\bar{u}_{i_0+2} = f(\bar{u}_{i_0+1})$ and so on.

Consider the set $A = \{z \in [0,1] : f^n(z) = 3/4\}$. Then, the strategy of the player i_0 is given by:

$$\bar{u}_{i_0} = \min\left\{\arg\min\{|z-y| : z \in A\}\right\}. \tag{17}$$

It holds: $J_{i_0}(\bar{u}_{i_0}) - \min_{u_{i_0}} J(u_{i_0}) < 2^{2n-1}$ and all the other players react optimally. Thus, the set of strategies described is 2^{2n-1} fine. The agents use the information about the n agents before them. Thus, for the ANIC it holds:

$$C_a(n) \leq \frac{n+1}{2}, \tag{18}$$

which completes the proof. ■

Remark 10: The game has a lot of Nash eqilibria. In a Nash equilibrium, the following equations hold:

$$u_{i_0} = \frac{x}{2} + \frac{f^N(u_{i_0})}{2}, \tag{19}$$

$$u_i = f(u_{i-1}), \quad \text{for} \quad i \neq i_0. \tag{20}$$

The equation (19) has approximately $N/2$ solutions. Denote by $\tilde{u}_{i_0}$ such a solution. The strategies of the players for this Nash equilibrium are given by $\tilde{u}_{i_0+k} = f^k(u_{i_0})$. Thus, a full knowledge of the information is needed. In contrast to the strategies given by equation (17), the strategies corresponding to Nash equilibria depend strongly on N. Let us observe that the ϵ - fine set of strategies is far from any Nash equilibrium.

Remark 11: The ϵ - fine set of strategies is in some sense cooperative. Particularly, the agent i_0 can improve his/her performance based on its own information. However, agent i_0 helps the other players to behave optimally with local information only. If the agent i_0 change his/her action to the optimal response, then he/she would expect that the other players would also use their best responses. Due to the chaoticity of f, we would expect that for a long amount of time the best responses would not converge. Thus, this is worse for player i_0. This example shares some ideas with ϵ - Nash cooperative outcomes of some repeated games [23].

Remark 12: If the players do not cooperate and do not have full information, for example in the case of an expected cost approach, the cost would be much higher.

VII. Conclusion

Information and complexity aspects for games on large random interaction structures were examined. Sets of strategies that constitute ϵ - Nash equilibria with probability higher than $1-\epsilon$, called ϵ - fine sets of strategies, were studied. The complexity of the game was defined as the minimum amount of information needed, in order to achieve an ϵ - fine set of strategies.

Games on random directed graphs were shown to be asymptotically simple under high connectivity and asymptotically complex under low connectivity assumptions. Upper and lower bounds for the complexity of the quadratic games on rings were derived, showing that the game has asymptotically linear complexity. An example of a non-quadratic game on ring was also considered. It was shown that the asymptotic complexity is at most linear, using strategies having some form of cooperation.

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