Proceedings of the American Control Conference
Albuquerque, New Mexico June 1997
0-7803-3832-4/97/$10.00 © 1997 AACC

Solving a Class of Rank Minimization Problems via Semi-Definite Programs, with Applications to the Fixed Order Output Feedback Synthesis

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Abstract

We apply certain recent results pertaining to the rank minimization problem under LMI constraints to determine non–trivial lower and upper bounds for the minimal order dynamic output feedback which stabilizes a given linear time invariant plant, via Semi-Definite Programs.

1. Introduction

1.1.

We consider the problem of minimizing the rank of a positive semi-definite matrix, subject the constraint that a particular affine transformation of it is also positive semi-definite. We shall refer to this problem as the Rank Minimization Problem (RMP). Since at the present time we do not know of a satisfactory characterization of the solution set to the general RMP (which leads to an efficient algorithm), we restrict our attention to particular instances of it, i.e., those with affine transformations of a particular form.

The RMP is defined as follows: Given a symmetry preserving linear map M on the space of symmetric matrices, and a particular symmetric matrix Q, solve

$$\min_X \mathbf{rank}\ X \tag{1}$$

$$Q + M(X) \geq 0, \tag{2}$$

$$X \geq 0, \tag{3}$$

where the inequality "$\geq$" is interpreted in the sense of Löwner, i.e., $A \geq B$ signifies that the matrix $A - B$ is positive semi-definite. For a given affine map $\mathcal{L}$, or a linear map M on the space of symmetric matrices and a symmetric matrix Q, the corresponding instances of the rank minimization problem are denoted by RMP($\mathcal{L}$) or RMP(Q, M), respectively.

1.2.

It is now recognized that what Lyapunov pioneered in his seminal 1849 paper [11] has flourished into one the most fruitful techniques for system analysis and synthesis, namely the Linear Matrix Inequality (LMI) approach [6]. The LMI which arises from a system or control problem could be originated from what Lyapunov technique is all about, i.e., finding a Lyapunov function; or it could have been obtained from the frequency domain or the input/output framework specifications (positivity, passitivity, positive realness, etc.), and turned into a LMI with the use of PR Lemma (when transition to time domain framework is least expected!). The most important issue however is that LMIs are *computationally tractable*, although there are still some very interesting issues that need to be resolved regarding their computational complexity.

When LMIs arise in the synthesis contexts, it turns out that imposing particular structure or order constraints on the controller becomes extremely important. For example one might be interested in the *minimal* order dynamic output feedback which stabilizes a given linear time invariant (LTI) plant (this problem is considered to be among the most important "open" problems in control [4], [5]). Decentralization is another "structure" that one might impose on the designed controller (in addition to other constraint, e.g., decentralized H_∞). Control researchers have now recognized that these "structural" constraints can in fact be formulated as rank constraints on the solution of a LMI [7], [8].

1.3.

Before presenting the main results we briefly go over some background material. Capital letters are used for matrices, as well as for the linear maps acting on them. For two symmetric matrices A and B, $A \geq B$ indicates that $A - B$ is positive semi-definite; similarly $A > B$ expresses the positive definiteness of $A - B$. The trace of AB is denoted by $A \bullet B$; A' denotes the transpose of the matrix A, $A^\dagger$ its pseudo-inverse. For two symmetric positive semi-definite matrices A and B, the parallel addition of A and B is denoted by $A : B$ and is defined by:

$$A : B := A(A+B)^\dagger B.$$

It is known that $0 \leq A : B = B : A \leq A, B$ [1], a result which will be used in the sequel. For a square symmetric matrix A, $\lambda_i(A)$ denotes the i-th eigenvalue of the matrix A, when they are arranged in an increasing order; it is known that,

$$A \leq B \Rightarrow \lambda_i(A) \leq \lambda_i(B) \quad (\forall\, i)$$

and

$$A \leq B \Rightarrow MAM' \leq MBM',$$

for all matrices M of appropriate dimensions [9]. We use the term Semi-Definite Programming (SDP) to refer to an optimization problem which has either (symmetric) matrices or scalar values as variables, and the objective is a linear functional on the product space of the spaces of the variables, and the constraint set is defined by linear equalities using components wise ordering, or matrix ordering ">" or "≥" defined above.

The outline of the paper is as follows. Section 2 starts with the introduction of certain classes of matrices. We then proceed to state the results that we employ to prove the main result of the paper which pertains to determining the minimum order dynamic output feedback which stabilizes a given LTI plant.

2.

As we pointed out previously, at the present time very little is known about the RMP, partly because the importance of this problem has only been recently recognized. The lack of an efficient algorithm for solving the RMP however, is due to more fundamental reasons, in particular the lack of convexity and the notion on NP-hardness.

We shall therefore consider only a particular class of RMPs, namely those with linear maps expressible as:

$$X \mapsto CXC' - \sum_i M_i X M_i',$$

where the matrix C is invertible. We shall refer to these linear maps as maps of the type generalized Z, and denote them by Z_G. These maps are a generalization of the type Z maps in [12] (and hence the prefix "generalized"). The motivation for calling the maps of the form

$$X \mapsto X - \sum_i M_i X M_i,$$

type Z in [12] is due to their analogy with the Z matrices, i.e., matrices which have non–positive off–diagonal elements.

An affine map $\mathcal{L}$ on the space of symmetric matrices is called p-concave (concave with respect to the parallel addition operation), if

$$\mathcal{L}(X : Y) \geq \mathcal{L}(X) : \mathcal{L}(Y).$$

We make use of the following results reported in [13].

Theorem 1 ([13]) *The RMP(Q, M) with $Q \leq 0$ and $M \in Z_G$ can be solved as a SDP.*

Our objective is to show how these results are applicable to the problem of output feedback synthesis.

Theorem 3 *Non-trivial upper and lower bounds for the order of the minimal order dynamic output feedback which stabilizes a given LTI plant can be obtained via a SDP.*

3.

Prior to presenting the proof of Theorem 5, we make the following observation.

Although the linear maps of type Z and Z_G appear naturally in the context of discrete time systems via the Lyapunov equation, they are also useful in the context of continuous time analysis.

Proposition 4 *The matrix inequalities of the form*

$$AX + XA' + Q \leq 0$$

can be written as

$$\overline{C}X\overline{C}' - \overline{M}X\overline{M}' + \overline{Q} \geq 0,$$

for appropriate choices of matrices $\overline{C}$ (invertible), $\overline{M}$, and $\overline{Q}$.

Proof: Let $\alpha > 0$ be large enough such that $\alpha I - A$ is invertible. Then,

$$\begin{aligned} AX + XA' + Q \leq 0 \;&\Leftrightarrow\; -A'X - XA - Q \geq 0 \\ &\Leftrightarrow\; 2\alpha(-A'X - XA - Q) \geq 0 \\ &\Leftrightarrow\; -2\alpha Q + (\alpha I - A)X(\alpha I - A)' \\ &\qquad -(\alpha I + A)X(\alpha I + A)' \geq 0 \end{aligned}$$

Let $\overline{Q} = -2\alpha Q$, $\overline{C} = \alpha I - A$, and $\overline{M} = \alpha I + A$. □

Remark 5 *It should be clear that we could choose the parameter $\alpha > 0$ such that $\overline{C}$ is singular.*

Consider the problem of stabilizing the LTI system,

$$\dot{x} = Ax + Bu \tag{4}$$
$$y = Cx \tag{5}$$

where $A \in R^{n\times n}$, $B \in R^{n\times m}$ and $C \in R^{p\times n}$.

For a given $k \leq n$, we would like to examine the existence of a stabilizing control law,

$$\dot{z} = A_K z + B_K y \tag{6}$$
$$u = C_K z + D_K y \tag{7}$$

where $A_K \in R^{k\times k}$.

In [7] it was shown that this problem can be solved by considering a rank minimization problem.

Theorem 6 ([7]) *The minimal order dynamic output feedback which stabilizes a given plant can be determined by solving,*

$$\rho_{\min} := \min_{R,S,\gamma>0} \textbf{rank} \begin{pmatrix} \gamma R & I \\ I & \gamma S \end{pmatrix} \tag{8}$$
$$\text{s.t.: } AR + RA' < BB', \tag{9}$$
$$A'S + SA < C'C, \tag{10}$$
$$\begin{pmatrix} \gamma R & I \\ I & \gamma S \end{pmatrix} \geq 0. \tag{11}$$

How would one go about solving this problem? Well, it seems that the least we can do is to consider a convex relaxation of it:

$$\min_{R,S,\gamma>0} \text{trace} \begin{pmatrix} \gamma R & I \\ I & \gamma S \end{pmatrix} \tag{12}$$
$$\text{s.t.: } AR + RA' < BB', \tag{13}$$
$$A'S + SA < C'C, \tag{14}$$
$$\begin{pmatrix} \gamma R & I \\ I & \gamma S \end{pmatrix} \geq 0, \tag{15}$$

hoping that a feasible point which has a minimum trace turns out to be of minimal rank as well. But this is of course not the case in general. One condition the guarantees that these two optimization problems are equivalent is the machinery which was introduced in [12], namely making sure that the feasible set is a hyper semi-lattice. But this condition might not hold in general either.

Let us denote by ρ_t the rank of the minimum trace solution of (12)–(15). Clearly one has

$$\rho_{\min} \leq \rho_t.$$

We are interested in obtaining a nontrivial lower bound for $\rho_{\min}$.

Let $W := \begin{pmatrix} \gamma R & I \\ I & \gamma S \end{pmatrix}$. By rewriting (8)–(11) in terms of W one obtains:

$$\min_{W,\gamma>0} \textbf{rank}\ W \tag{16}$$
$$\tilde{A}W + W\tilde{A} \leq Q_\gamma, \tag{17}$$
$$W \geq 0, \tag{18}$$
$$W \text{ is of the form: } \begin{pmatrix} W_1 & I \\ I & W_3 \end{pmatrix}, \tag{19}$$

where,

$$\tilde{A} := \begin{pmatrix} A & 0 \\ 0 & A' \end{pmatrix}$$

and

$$Q_\gamma := \begin{pmatrix} \gamma BB' - \epsilon_1 I & A + A' \\ A + A' & \gamma C'C - \epsilon_2 I \end{pmatrix}$$

for small $\epsilon_1 > 0$ and $\epsilon_2 > 0$.

Using Proposition 6 rewrite (16)–(19) as,

$$\min_{W,\gamma>0} \textbf{rank}\ W \tag{20}$$
$$\overline{Q}_\gamma + \overline{C}W\overline{C}' - \overline{M}W\overline{M}' \geq 0, \tag{21}$$
$$W \geq 0, \tag{22}$$
$$W \text{ is of the form: } \begin{pmatrix} W_1 & I \\ I & W_3 \end{pmatrix}, \tag{23}$$

where, $\overline{Q}_\gamma = 2\alpha Q_\gamma$, $\overline{C} := \alpha I - \tilde{A}$, and $\overline{M} := \alpha I + \tilde{A}$; the parameter $\alpha > 0$ is chosen such that $\overline{C}$ is singular. Now suppose that $\gamma > 0$ is fixed, in which case we use the notation $\overline{Q}$ instead of the more accurate $\overline{Q}_\gamma$. Consider the following problem,

$$\min_{W} \textbf{rank}\ W \tag{24}$$
$$\overline{C}_\perp(\overline{Q} + \overline{C}W\overline{C}' - \overline{M}W\overline{M}')\overline{C}_\perp' \geq 0, \tag{25}$$
$$W \geq 0, \tag{26}$$

where $\overline{C}_\perp$ is such that $\overline{C}_\perp\overline{C} = 0$, and $\overline{C}_\perp\overline{C}_\perp' = I$. According to Theorem 4, the latter RMP can be solved as a SDP. Let the rank of the optimal solution of the corresponding SDP be denoted ϑ. Observe that

$$\vartheta \leq \rho_{\min} \leq \rho_t,$$

and that both lower and upper bounds are computable via SDPs; we have thus proved the statement of Theorem 5.

4. Conclusion

We considered the fixed order dynamic output feedback synthesis in the framework of finding a minimal rank matrix in the feasible set of an LMI. We showed that non-trivial upper and lower bounds for this synthesis problem can be found by solving two semi-definite programs.

Acknowledgements
This research of M. Mesbahi was carried out at the Jet Propulsion Laboratory, California Institute of Technology, under a contract with the National Aeronautic and Space Administration. The research of G. P. Papavassilouplos was supported in part by the National Science Foundation grant CCR-9222734.

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