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Matrix Cones, Complementarity Problems, and the Bilinear Matrix Inequality

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Abstract

We discuss an approach for solving the Bilinear Matrix Inequality (BMI) based on its connections with certain problems defined over matrix cones. These problems are, among others, the cone generalization of the linear programming (LP) and the linear complementarity problem (LCP) (referred to as the Cone-LP and the Cone-LCP, respectively). Specifically, we show that solving a given BMI is equivalent to examining the solution set of a suitably constructed Cone-LP or Cone-LCP. This approach facilitates our understanding of the geometry of the BMI and opens up new avenues for the development of the computational procedures for its solution.

1. Introduction

The Bilinear Matrix Inequality (BMI) is considered to be an important problem in the field of robust control. The BMI feasibility problem is as follows: Given symmetric matrices $H_{ij} \in \mathbb{R}^{p\times p}$ ($i = 1,\ldots,n; j = 1,\ldots m$), does there exist $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, such that $\sum_{i=1}^{n}\sum_{j=1}^{m} x_i y_j H_{ij}$ is positive definite. As it was shown by Safonov *et al.* [13] it is possible to reduce a wide array of control synthesis problems such as the fixed-order H^∞ control, μ/k_m-synthesis, decentralized control, robust gain-scheduling, and simultaneous stabilization to a BMI. It is also known that the Linear Matrix Inequality (LMI) approach to control synthesis [2] is a special case of the BMI. Since the LMI is equivalent to the Semi-Definite Programming problem (SDP), the BMI can also be considered as a generalization of the SDP. It is therefore not surprising that the solution to the BMI is not only of central importance in the context of robust control [14], but also in its connections with the SDPs and the LMIs.

The BMI can be reformulated as a *nonconvex* programming problem. More specifically, Safonov and Papavassilopoulos [14] have shown that the BMI feasibility problem is equivalent to checking whether the diameter of a certain convex set is greater than two. Since this is equivalent to a *maximization* of a convex function subjected to a set of convex constraints (an NP-hard problem), no efficient algorithm is believed to exist for a general BMI. Moreover, Toker and Özbay have recently shown that the BMI feasibility is an NP-hard problem by reducing the Subset-Sum problem to it [16].

The computational procedures which have been suggested for solving the BMI rely on a global optimization approach [4]. There are at least three issues which have to be addressed in connection with the BMI and the global optimization methods: (1) What are the geometric interpretations of the BMI? (2) What are the specific properties of the global optimization problem which arises from the BMI, and whether these properties can be used to devise more efficient algorithms for the BMI? (3) Which instances of the BMI can be solved efficiently? Moreover, whether there are instances for which certain "structural" properties can be established, for example, the convexity of the solution set?

All of the above issues can be addressed by studying the BMI *on its own*. Nevertheless, we believe that many important structural and computational issues of the BMI can be studied by establishing a connection between the BMI and the problems which are more well-understood in the optimization theory. In this paper, we shall explore the connections between the BMI and two important problems in optimization: the linear programming problem (LP) and the linear complementarity problem (LCP) [3] defined over matrix cones (referred to as the Cone-LP and the Cone-LCP, respectively).

An intermediate step in adopting a Cone-LP and a Cone-LCP approach for solving the BMI, is the introduction of a problem which we shall refer to as the Extreme Form Problem (EFP). Formulated in the n-dimensional Euclidean space $\mathbb{R}^n$, and denoting the nonnegative orthant by $\mathbb{R}^n_+$, the EFP has the following formulation: Given $M : \mathbb{R}^n \to \mathbb{R}^n$, find $z \in \mathbb{R}^n$ (if it exists), such that, $z \geq 0$, $Mz > 0$, and

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z is an extreme form of $\mathbb{R}_+^n$ (an extreme form, or an extreme ray, of a cone is a face of the cone which is a half-line emanating from the origin [6], [12]). The above instance of the EFP is referred to as the $\mathrm{EFP}_{\mathbb{R}_+^n}(M)$. The EFP, as we just defined, is *not* an interesting problem. In fact, the $\mathrm{EFP}_{\mathbb{R}_+^n}(M)$ has a solution if and only if M has a positive column. On the other hand, the EFP becomes non-trivial when $\mathbb{R}_+^n$ is replaced by an arbitrary cone. We have used the cone generalization of the EFP as an intermediate step in the Cone-LP/LCP approach for solving the BMI. It can be argued that computational procedures could be developed for the EFP directly, without formulating it as a Cone-LP or a Cone-LCP. We have chosen this approach, since at the present time, the Cone-LP and the Cone-LCP seem to be more amenable for the application of the interior point methods than the EFP. It is still an open question whether an interior point method can be adapted for solving the EFPs directly.

The organization of the paper is as follows. In Preliminaries we present some basic definitions, certain matrix cones, as well as the precise formulation of the Cone-LP, the Cone-LCP, and the EFP. In the same section a glossary of notations that are used in this paper is provided. In Section 3, the "cone" formulations of the BMI are presented. We also discuss certain computational implications of these reductions. The proofs of the results are omitted for brevity and they can be found in the journal version of the paper [10].

2. Preliminaries

In this section we define the Cone-LP, the Cone-LCP, and introduce the Extreme Form Problem (EFP) over arbitrary cones. We also mention certain matrix classes, which when generalized appropriately, will make the transformation of the BMI to the EFP and the Cone-LP/LCP more explicit.

Prior to defining the Cone-LP, the Cone-LCP, and the EFP, few basic definitions are in order. The definitions include those of a cone, the dual cone of a set, and the notion of positivity and copositivity of a linear map with respect to a given cone. We shall restrict ourselves to the finite dimensional vector spaces in all subsequent sections.

Let $\mathcal{H}$ be a finite dimensional Hilbert space equipped with the inner product $<\cdot,\cdot>: \mathcal{H}\times\mathcal{H}\to\mathbb{R}$ (e.g., the n-dimensional Euclidean space or the space of $n\times n$ matrices, with the appropriate notion of an inner product defined on them). A set $\mathcal{K}\subseteq\mathcal{H}$ is a cone if for all $\alpha\geq 0$, $\alpha\mathcal{K}\subseteq\mathcal{K}$. $\mathcal{K}$ is a convex cone, if $\mathcal{K}$ is a cone and it is convex, i.e., for all $\alpha\in[0,1]$, $\alpha\mathcal{K}+(1-\alpha)\mathcal{K}\subseteq\mathcal{K}$, or equivalently, if $\mathcal{K}$ is a cone and $\mathcal{K}+\mathcal{K}\subseteq\mathcal{K}$. An extreme form (or an extreme ray) of a convex cone $\mathcal{K}$ is a subset $E=\{\alpha x:\alpha\geq 0\}$ of $\mathcal{K}$, such that if $x=\alpha y+(1-\alpha)z$, for $0<\alpha<1$, and $y,z\in\mathcal{K}$, one can conclude that $y,z\in E$ [6]. The dual cone of a set $S\subseteq\mathcal{H}$, denoted by S^*, is defined to be,

$$S^*=\{y\in\mathcal{H}:<x,y>\geq 0;\ \forall x\in S\}$$

It can easily be shown that S^* is always a convex set, and that if $S_1\subseteq S_2$, then $S_2^*\subseteq S_1^*$. In addition, $S=(S^*)^*$, if and only if S is a closed convex cone. For more on convexity, cones and their duals, the reader is referred to Berman [1], Rockafellar [12], and Stoer and Witzgall [15].

In Section 3, we shall be referring to two properties of a linear map that we now define. Given a convex cone $\mathcal{K}\subseteq\mathcal{H}$, a linear map $M:\mathcal{H}\to\mathcal{H}$ is call $\mathcal{K}$-positive if for all $0\neq X\in\mathcal{K}$, $M(X)\in\operatorname{int}\mathcal{K}^*$. Furthermore, a linear map $M:\mathcal{H}\to\mathcal{H}$ is called $\mathcal{K}$-copositive if $<X,M(X)>\geq 0$, for all $X\in\mathcal{K}$ [1], [5], [8].

We are now ready to formulate the cone problems that are dealt with in the paper. Given a cone $\mathcal{K}\subseteq\mathcal{H}$, a linear map $M:\mathcal{H}\to\mathcal{H}$, and the elements Q and C in $\mathcal{H}$, find $Z\in\mathcal{H}$ (if it exists) as a solution to:

$$\min <C,Z> \tag{2.1}$$
$$Z\in\mathcal{K} \tag{2.2}$$
$$Q+M(Z)\in\mathcal{K}^* \tag{2.3}$$

Similarly, the Cone-LCP is formulated as follows: Given a cone $\mathcal{K}\subseteq\mathcal{H}$, a linear map $M:\mathcal{H}\to\mathcal{H}$, and $Q\in\mathcal{H}$, find $Z\in\mathcal{H}$ (if it exists) such that:

$$Z\in\mathcal{K} \tag{2.4}$$
$$Q+M(Z)\in\mathcal{K}^* \tag{2.5}$$
$$<Z,Q+M(Z)>=0 \tag{2.6}$$

The above instances of the Cone-LP and the Cone-LCP shall be referred to as the $\text{Cone-LP}_{\mathcal{K}}(C,Q,M)$ and $\text{Cone-LCP}_{\mathcal{K}}(Q,M)$, respectively. When $\mathcal{K}$ is the nonnegative orthant in the n-dimensional Euclidean space, the $\text{Cone-LP}_{\mathcal{K}}(C,Q,M)$ (2.1)–(2.3) and the $\text{Cone-LCP}_{\mathcal{K}}(Q,M)$ (2.4)–(2.6), are equivalent to the familiar LP and the LCP. We shall also find it convenient to refer to the problem of finding a feasible element in the Cone-LP, i.e., an element that satisfies (2.2)–(2.3), as a $\text{Cone-LP}_{\mathcal{K}}(*,Q,M)$.

A problem which serves as a bridge between the BMI and the Cone-LP/LCPs is what we have referred to as the Extreme Form Problem (EFP): Given a cone $\mathcal{K} \subseteq \mathcal{H}$, a linear map $M : \mathcal{H} \to \mathcal{H}$, find $X \in \mathcal{H}$ (if it exists), such that,

$$X \in \mathcal{K} \quad (2.7)$$

$$M(X) \in \text{ int } \mathcal{K}^* \quad (2.8)$$

$$X \text{ is an extreme form of } \mathcal{K} \quad (2.9)$$

where the "int $\mathcal{K}$" denotes the interior of the cone $\mathcal{K}$. The above instance of the EFP is referred to as the $\text{EFP}_{\mathcal{K}}(M)$. As we mentioned in Introduction, when $\mathcal{K}$ is the nonnegative orthant in the n-dimensional Euclidean space, the EFP is a trivial problem. It should be noted that the solution set of $\text{EFP}_{\mathcal{K}}(M)$ is nonconvex in general; given the two extreme forms of $\mathcal{K}$ that solve the $\text{EFP}_{\mathcal{K}}(M)$, a strict convex combination of them is not even an extreme form of $\mathcal{K}$.

A few words on the notation before we present the cone formulations of the BMI. A' and diag(y) denote the transpose of the matrix A and the diagonal matrix with vector y on its diagonal, respectively. We denote by $\mathbb{SR}_+^{p\times p}$ and $\mathbb{SR}_{++}^{p\times p}$, the real, symmetric $p\times p$ positive semi-definite and positive definite matrices. To indicate that $A - B$ is symmetric positive definite, or positive semi-definite, we use the notations $A \succ B$, or $A \succeq B$ (we shall mainly reserve these notations for the $p \times p$ matrices, p being the dimension of the matrices H_{ij}'s in the BMI problem). The inner product for the space of matrices is denoted by "$\bullet$", i.e., $A \bullet B$ = Trace AB'; **vec** M is the vector obtained by stacking up the columns of the matrix M. Finally, int A stands for the interior of the set A.

3. Cone Formulations of the BMI

In this section we discuss the formulation of the BMI feasibility problem as a Cone-LP over a suitable generalization of the cone of completely positive matrices (to be defined below), and subsequently, as a Cone-LCP over the cone of positive semi-definite matrices. This is done by first reducing the BMI to an EFP, and subsequently reducing the EFP to a Cone-LP/LCP. As it becomes evident, various embeddings of matrices in different dimensions are needed to make these reductions as transparent as possible. For this purpose the **vec** notation, which is used in the studying of Kronecker products, has become specially handy. The **vec** operator, applied to a matrix in $\mathbb{R}^{p\times p}$, simply stacks up the columns of the matrix from left to right, and forms a vector in $\mathbb{R}^{p^2}$ [7].

Few Initial Steps
Consider again the BMI feasibility problem: Given $H_{ij} = H'_{ij} \in \mathbb{SR}^{p\times p}$ (the symmetric $p \times p$ matrices with real entries), does there exist x_i's ($1 \le i \le n$), and y_j's ($1 \le j \le m$), such that:

$$\sum_i \sum_j x_i y_j H_{ij} \succ 0 \quad (3.1)$$

Let us rewrite (3.1) as:

$$\sum_i x_i \sum_j y_j H_{ij} = \sum_i x_i H_i^y \succ 0 \quad (3.2)$$

where,

$$H_i^y = \sum_j y_j H_{ij} \in \mathbb{SR}^{p\times p}$$

As it becomes apparent by the subsequent developments, it is convenient to assume, without loss of generality, that $m = p$ and that y_j's ($1 \le j \le m$), are nonnegative. The first assumption is made to avoid defining inner products between matrix classes of different dimensions. The second assumption is made to facilitate the dual cone characterization in the EFP and Cone-LP approaches discussed in Section 3 (we shall later drop the nonnegativity assumption on the vector y when the Cone-LCP is presented).

We now employ the following generalization of the Gordan theorem of alternative [3], [15] for the cone of positive semi-definite matrices.

Proposition 3.1 *Given $x \in \mathbb{R}^n$ and the symmetric matrices A_i's $\in \mathbb{SR}^{p\times p}$ ($1 \le i \le n$), the system $\sum_{i=1}^n x_i H_i \succ 0$, has a solution if and only if the system $A_i \bullet Z = 0, Z \succeq 0, Z \ne 0$, has no solution.*

From Proposition 3.1, one concludes that the BMI (3.1) does not have a solution if and only if,

$$(\forall y \ge 0)\ (\exists Z \succeq 0, Z \ne 0): \quad H_i^y \bullet Z = 0 \quad (3.3)$$

Therefore, the BMI (3.1) has a solution if and only if,

$$(\exists y \ge 0)\ (\forall Z \succeq 0, Z \ne 0): \quad \sum_i (H_i^y \bullet Z)^2 > 0 \quad (3.4)$$

Now let,

$$H_i = [\overbrace{\textbf{vec } H_{i1}\, 0 \ldots 0}^{p}, \overbrace{0\, \textbf{vec } H_{i2} \ldots 0}^{p}, \ldots, \overbrace{0\, 0 \ldots \textbf{vec } H_{ip}}^{p}] \quad (3.5)$$

and Y=diag(y) $\in \mathbb{SR}^{p\times p}$. Since $(H_i^y)' = H_i^y = \sum_j y_j H_{ij}$,

$$\textbf{vec } H_i^{y\prime} = H_i \,\textbf{vec } Y$$

Thereby,

$$H_i^y \bullet Z = (\textbf{vec } Y)' H_i' (\textbf{vec } Z) \quad (3.6)$$

Combining (3.4) and (3.6) we conclude that (3.1) has a solution if and only if there exists $Y \succeq 0$, Y=diag(y), for some $y \geq 0$, such that for all $Z \succeq 0, Z \neq 0$,

$$(\mathbf{vec}\ Z)' \{\sum_i H_i(\mathbf{vec}\ Y)(\mathbf{vec}\ Y)'H_i'\} (\mathbf{vec}\ Z) > 0 \tag{3.7}$$

Let $X = (\mathbf{vec}\ Y)(\mathbf{vec}\ Y)'$ and,

$$M(X) = \sum_i H_i X H_i' \tag{3.8}$$

Remark 3.2 *Suppose that the vector y is not required to be nonnegative in the above analysis. It is clear that the above steps are still valid with the obvious modifications, and that the end result would read as follows: The BMI has a solution if and only if there exists a diagonal matrix Y, such that for all $Z \succeq 0, Z \neq 0$, the inequality (3.7) holds.*

The inequality (3.7) can be interpreted as requiring $M(X)$ to belong to a certain matrix class. The matrices in this class are symmetric (given that X is symmetric) and have quadratic forms which are positive over the **vec** form of the non-zero matrices in $\mathbb{SR}_+^{p\times p}$. This observation justifies the introduction of the matrix classes which we shall now discuss. In what follows it is assumed that all the matrix classes are subsets of $\mathbb{SR}^{p^2\times p^2}$.

Let $\mathcal{PSD}$ denote the class of symmetric $p^2 \times p^2$ matrices with quadratic forms nonnegative over the **vec** form of the symmetric $p \times p$ matrices, i.e.,

$$\mathcal{PSD} = \{A \in \mathbb{SR}^{p^2\times p^2} \ : \ (\mathbf{vec}\ Z)' A (\mathbf{vec}\ Z) \geq 0; \ Z \in \mathbb{SR}^{p\times p}\} \tag{3.9}$$

Let $\mathcal{C}$ denote the class of symmetric PSD-copositive matrices, i.e.,

$$\mathcal{C} = \{A \in \mathbb{SR}^{p^2\times p^2} : (\mathbf{vec}\ Z)' A (\mathbf{vec}\ Z) \geq 0; Z \succeq 0\} \tag{3.10}$$

Finally, let $\mathcal{B}$ denote the class of symmetric PSD-completely positive matrices,

$$\mathcal{B} = \{A \in \mathbb{SR}^{p^2\times p^2} \ : \ A = \sum_{i=1}^{t} (\mathbf{vec}\ Z_i)(\mathbf{vec}\ Z_i)'; \ Z_i \succeq 0, t \geq 1\} \tag{3.11}$$

We now summerize few facts regarding the above classes of matrices. The matrix classes $\mathcal{PSD}$, $\mathcal{C}$ and $\mathcal{B}$ are closed convex cones. For all rank t matrices $A \in \mathcal{PSD}$, there are non-zero symmetric matrices $W_i \in \mathbb{SR}^{p\times p} (1 \leq i \leq t)$, such that, $A = \sum_{i=1}^t (\mathbf{vec}\ W_i)(\mathbf{vec}\ W_i)'$, $W_i \bullet W_j = 0$, $(i \neq j)$. Moreover, any matrix which can be represented as such is a $\mathcal{PSD}$ matrix. Also, $\mathcal{PSD}^* = \mathcal{PSD}$, $\mathcal{B}^* = \mathcal{C}$, $\mathcal{C}^* = \mathcal{B}$, and the extreme forms of $\mathcal{PSD}$ and $\mathcal{B}$ are matrices $(\mathbf{vec}\ W)(\mathbf{vec}\ W)', W \in \mathbb{SR}^{p\times p}$, and $(\mathbf{vec}\ Z)(\mathbf{vec}\ Z)'$, $Z \succeq 0$, respectively [10].

The EFP formulation of the BMI

We now use the previously mentioned results to reformulate the BMI as the $\text{EFP}_{\mathcal{B}}(M)$, where M is defined by (3.8), and $\mathcal{B}$ is the cone of PSD-completely positive matrices. The next proposition states that in fact, the BMI is a special instance of the EFP.

Proposition 3.3 *Let $\mathcal{B}$ be the class of PSD-completely positive matrices (3.11), and the linear map M be defined by (3.8). Then the BMI has a solution if and only if the $EFP_{\mathcal{B}}(M)$ has a solution. Moreover, the solution of one yields the solution of the other.*

Quite analogous to the proof of Proposition 3.3, it can be shown that if the nonnegativity assumption on the vector y is dropped, the BMI is also equivalent to finding a extreme form X of the $\mathcal{PSD}$ cone (3.9), such that $M(X) \in \text{int}\,\mathcal{B}^* \equiv \mathcal{C}$. We shall use this version of Proposition 3.3 when we later present the Cone-LCP approach.

The implication of Proposition 3.3 is that the BMI is equivalent to checking whether the image of an extreme form of the matrix cone $\mathcal{B}$ under the linear map M (which is constructed from the original data of the BMI) is in the interior of the dual cone $\mathcal{B}^*$. This equivalence thus provides a rather simple geometric interpretation of the BMI feasibility problem.

An immediate consequence of the EFP formulation is the following characterization of the BMI instances for which a solution exists.

Proposition 3.4 *The BMI has a solution if the linear map M (3.8) is $\mathcal{B}$-positive (see Section 2 for the definition of the positivity of a linear map).*

The Cone-LP formulation of the BMI

In this subsection we shall explore an approach for solving the BMI based on its connection with the Cone-LP over the cone of PSD-completely positive matrices, $\mathcal{B}$. For this purpose we use the EFP formulation of the BMI, as discussed previously. In particular, in light of Proposition 3.3, we shall think of the BMI as the $\text{EFP}_{\mathcal{B}}(M)$, where M is defined by (3.8).

Since the Cone-LP approach for the $\text{EFP}_{\mathcal{B}}(M)$ is rather straight forward, we begin our discussion with the main result. Let $S(\alpha) = \{X \in \mathcal{B} : -\alpha I + M(X) \in \mathcal{B}^*\}$ and $\dot{\mathcal{B}}$ denote the extreme forms of $\mathcal{B}$.

Theorem 3.5 *The $EFP_{\mathcal{B}}(M)$ (BMI) has a solution if and only if for any $\alpha > 0$, $S(\alpha) \cap \dot{\mathcal{B}} \neq \emptyset$.*

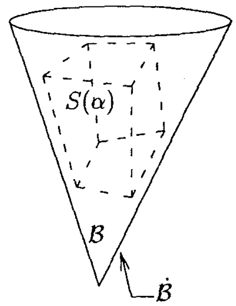

Figure 1: BMI has a solution if and only if $S(\alpha) \cap \dot{\mathcal{B}} \neq \emptyset$

The implication of Theorem 3.5 is the following: Given any $\alpha > 0$, check whether $S(\alpha)$ contains an extreme form of the cone $\mathcal{B}$, which in fact, has to be of rank one. This is the case if and only if the BMI has a solution. Geometrically, one can "attempt" to illustrate this implication as in Figure 1.

Initial observations indicate that checking whether a "matrix polyhedron" over the cone of *positive semi-definite* matrices, contains a rank one element can be done efficiently for certain classes of linear maps [9]. On the other hand, the above Cone-LP approach calls for checking for the rank one element in a "matrix polyhedron" over the matrix cone $\mathcal{B}$. However, it is known that checking whether a matrix belongs to the cone of copositive matrices (for which $\mathcal{B}^* \equiv \mathcal{C}$ is a generalization of) is a difficult computational task [11]. Consequently, although the preceeding Cone-LP approach provides us with a way of understanding the geometry of the BMI, its computational realization runs into difficulty.

The above considerations have led us to adopt yet another approach for solving the BMI. The approach relies on establishing a connection between the BMI and a linear complementarity problem over the $\mathcal{PSD}$ cone (3.9). The main advantage of the complementarity approach is that one can formulate the resulting problem over a matrix cone for which an interior point algorithm can be developed. The complementarity approach also provides us with a way of addressing certain structural issues. This approach is examined next.

The Cone-LCP formulation of the BMI

In this subsection we explore the idea of viewing the BMI as a certain linear complementarity problem over a matrix cone (Cone-LCP). The EFP formulation of the BMI discussed previously is again the main tool for making the Cone-LCP approach possible.

Let us denote by $\bar{p} = p(p+1)/2$ the dimension of the space of symmetric $p \times p$ matrices.

The starting point for the Cone-LCP approach is the observation made right after Proposition 3.3: the BMI has a solution if and only if the image of an extreme form of the matrix cone $\mathcal{PSD}$ under the linear map M, is in the interior of $\mathcal{C}$. The following lemma is employed to transform the above condition in terms of the solution set of a Cone-LCP.

Lemma 3.6 *Let Y be an extreme form of the cone $\mathcal{PSD}$. Then there exists a symmetric $W \in \mathcal{PSD}$, such that $Y \bullet W = 0$ and rank $(W) = \bar{p} - 1$.*

Consider the Cone-LCP$_{\mathcal{PSD}}(Q, M)$ and let M be defined by the equation (3.8): Find $X \in \mathbb{SR}^{p^2 \times p^2}$ (if it exists) such that:

$$X \in \mathcal{PSD} \quad (3.12)$$

$$Q + M(X) \in \mathcal{PSD}^* \equiv \mathcal{PSD} \quad (3.13)$$

$$X \bullet (Q + M(X)) = 0 \quad (3.14)$$

The main result of this section now follows.

Theorem 3.7 *The BMI has a solution if and only if there exists a matrix $Q \in$ int $(-\mathcal{C})$, such that the Cone-LCP$_{\mathcal{PSD}}(Q, M)$ has a solution X^*, and rank $(Q + M(X^*)) = \bar{p} - 1$. Moreover, if for all solutions X^* of the EFP$_{\mathcal{B}}(M)$, $X^* \bullet M(X^*) < Y \bullet M(X^*)$, for all extreme forms Y of $\mathcal{B}$ ($Y \neq X^*$), then the matrix Q can be taken to be any positive multiple of $-I$.*

We shall refer to the special case of the linear complementarity problem over the positive semi-definite cone as the Semi-Definite Complementarity Problem (SDCP).

It is noteworthy that the linear map M in the SDCP formulation, which arises in the context of the BMI, is itself copositive with respect to the matrix cone $\mathcal{PSD}$. Therefore, if one assumes that the *copositive* SDCPs can be solved efficiently, then the above proposition implies that the SDCPs which arise from the BMIs can be solved efficiently for each Q.

If the BMI has a solution, then knowledge of the direction of Q in the corresponding Cone-LCP$_{\mathcal{PSD}}(Q, M)$ is sufficient for finding the solution of the BMI. In other words, if this direction is known, the solution set of a *single* SDCP should be examined. This result is established by the following corollary.

Corollary 3.8 *The BMI has a solution if and only if there exists a symmetric $Q \in \text{int}\,(-\mathcal{C})$, such that for any $\alpha > 0$, the Cone-LCP$_{\mathcal{PSD}}(\alpha Q, M)$ has a solution X^*, such that rank $(Q + M(X^*)) = \bar{p} - 1$.*

Corollary 3.8 reduces the BMI to examining the solution set of a SDCP with a $\mathcal{PSD}$-copositive linear map.

4. Conclusion

In this paper, we have established various connections between the Bilinear Matrix Inequality (BMI) and three problems over matrix cones. The first two problems, which we have referred to as the Extreme Form Problem (EFP) and the linear programming problem over matrix cones (Cone-LP), were formulated over a generalization of the cone of completely positive matrices. The above two cone problems facilitate our understanding of the geometry of the BMI. Nevertheless, the computational implications of these formulations run into difficulty, since the completely positive matrices can not be efficiently characterized by means of an algorithm. The last cone problem which we have established its connection with the BMI is the linear complementarity problem over the cone of positive semi-definite matrices (SDCP). The SDCP is readily amenable to an interior point approach. In this later case, the existence of the solution to a BMI is checked by examining the solution set of an SDCP. The actual solution to the BMI can then be constructed by solving a linear matrix inequality (LMI) which can be done via an interior point algorithm.

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