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A Global Optimization Approach for the BMI Problem

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Abstract

The Biaffine Matrix Inequality (BMI) is a potentially very flexible new framework for approaching complex robust control system synthesis problems with multiple plants, multiple objectives and controller order constraints. The BMI problem may be viewed as the nondifferentiable biconvex programming problem of minimizing the maximum eigenvalue of a biaffine combination of symmetric matrices. The BMI problem is non-local-global in general, i.e. there may exist local minima which are not global minima.

While local optimization techniques sometimes yield good results, global optimization procedures need to be considered for the complete solution of the BMI problem. In this paper, we present a global optimization algorithm for the BMI based on the branch and bound approach. A simple numerical example is included.

1 The Bilinear Matrix Inequality Problem

This paper will be focus on the following problem introduced in [33]:

Definition 1.1 (The BMI Feasibilty Problem) *Given prescribed matrices $F_{i,j} = F_{i,j}^T \in \mathbf{R}^{m\times m}$, for $i \in \{0,\ldots,n_x\}$, $j \in \{0,\ldots,n_y\}$, define the biaffine function $F : \mathbf{R}^{n_x} \times \mathbf{R}^{n_y} \to \mathbf{R}^{m\times m}$:*

$$F(x,y) := F_{0,0} + \sum_{i=1}^{n_x} x_i F_{i,0} + \sum_{j=1}^{n_y} y_j F_{0,j} + \sum_{i=1}^{n_x}\sum_{j=1}^{n_y} x_i y_j F_{i,j} \quad (1)$$

Find, if it exists, $(x,y) \in \mathbf{R}^{n_x} \times \mathbf{R}^{n_y}$ such that

$$F(x,y) < 0 \quad (2)$$

For the rest of this paper, restrict (x,y) to some closed bounded hyper-rectangle $X \times Y \subset \mathbf{R}^{n_x} \times \mathbf{R}^{n_y}$ where:

$$X := \left[B_{x_1}^L, B_{x_1}^U\right] \times \ldots \times \left[B_{x_{n_x}}^L, B_{x_{n_x}}^U\right] \quad (3)$$

$$Y := \left[B_{y_1}^L, B_{y_1}^U\right] \times \ldots \times \left[B_{y_{n_y}}^L, B_{y_{n_y}}^U\right] \quad (4)$$

for some bounds $-\infty < B_{x_i}^L \le B_{x_i}^U < \infty$, $i = 1\ldots,n_x$, and $-\infty < B_{y_j}^L \le B_{y_j}^U < \infty$, $j = 1\ldots,n_y$.

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Notation will be standard. In particular, for symmetric matrices A and B, $\overline{\lambda}\{A\}$ and $\underline{\lambda}\{A\}$ refer to the greatest (most positive) and smallest (most negative) eigenvalues of A, and $A > 0$ means $\underline{\lambda}\{A\} > 0$, $A > B$ means $A - B > 0$. Further for any vector $z \in \mathbf{R}^n$, $\|z\|_\infty := \max_{i\in\{1,\ldots,n\}} |z_i|$. For $-\infty < B^L \le B^U < \infty$, $\left[B^L, B^U\right]$ denotes a closed interval $\subset \mathbf{R}$.

This paper will mainly be concerned with obtaining a global solution to the following problem:

Definition 1.2 (BMI Eigenvalue Problem) *Given the function $F : X \times Y \to \mathbf{R}^{m\times m}$ of (1), define*

$$\Lambda(x,y) := \overline{\lambda}\{F(x,y)\} \quad (5)$$

The BMI Eigenvalue Minimization Problem is

$$\min_{(x,y)\in X\times Y} \Lambda(x,y) \quad (6)$$

Clearly, there exists a solution $(x,y) \in X \times Y$ to the BMI feasibility problem (2) if and only if $0 > \min_{(x,y)\in X\times Y} \Lambda(x,y)$.

The Linear or Affine Matrix Inequality (LMI/AMI) framework of, e.g. [12, 29, 8], characterized by the problem

$$\min_{x\in X} \overline{\lambda}\left\{F_0 + \sum_{i=1}^{n} x_i F_i\right\} \quad (7)$$

where $F_i = F_i^T$, is a special case of the BMI problem.

The properties of the problem (6) were discussed in [18]. In particular, it is shown that the function $\Lambda(x,y)$ is *biconvex*, i.e. it is convex in x for y fixed and convex in y for x fixed. Further, it is *non*-local-global in general, i.e. the function $\Lambda(x,y)$ may have local minima which are not global minima.

While to is fairly straightforward to find at least one local minimum of the $\Lambda(x,y)$ in $X\times Y$, the complete solution of the minimization problem, i.e. for some $\epsilon > 0$, find any $(\bar{x},\bar{y})$ such that $\Lambda(\bar{x},\bar{y}) \le \Lambda(x,y) + \epsilon$ for all $(x,y) \in X \times Y$, is a *global* optimization problem. Global optimization is very hard in general in terms of computational effort required [21, 31, 25].

The next section will discuss the background to the BMI problem formulation and hence provide the motivation for investigating global optimization approaches for completely solving the BMI problem (6).

2 Background and Motivation

This section will discuss several important robust control synthesis problems that in general can not be recast as LMI or even convex optimization problems. For these problems the BMI framework offers a viable approach [33, 17].

Consider first the μ/K_m-Synthesis formulation of the robust control problem [32, 11]. It has been shown [33, 17] that

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the μ/K_m-Synthesis problem for fixed order multipliers and a fixed order controller is equivalent to the BMI problem of solving for the matrices T, S, P, W and Q such that the following matrix inequalities hold:

$$\begin{aligned} & \text{herm}\left\{ \begin{bmatrix} -T & 0 \\ 0 & W \end{bmatrix} (R_{MT} + U_{MT}QV_{MT}) \right\} && > 0 \\ P > 0, \quad & -\text{herm}\{P(R_{G_A} + U_{G_A}QV_{G_A})\} && > 0 \\ & \text{herm}\left\{ \begin{bmatrix} S & 0 \\ 0 & W \end{bmatrix} M \right\} && > 0 \end{aligned}$$

where M, R_{MT}, U_{MT}, V_{MT}, R_{G_A}, U_{G_A}, V_{G_A} are known/prescribed matrices.

It is an added bonus that, under mild assumptions, a wide array of robust control synthesis problems may also be expressed in the BMI formulation, e.g., any robust controller synthesis problem consisting of *any combination* of the following may be expressed as a BMI feasibility problem:

- Multiple objective synthesis, i.e. combinations of μ/K_m-Synthesis, $\mathcal{H}^\infty$ and positive real synthesis.
- Synthesis of one controller for multiple plants, including the Two-Disk $\mathcal{H}^\infty$ problem.
- Controller synthesis with controller order, structure, and/or stability constraints.

Such problems have been studied previously in, e.g., [5, 23, 24, 28, 4, 20].

Further, the BMI is obviously applicable as a *controller parameter tuning* framework, particularly within an integrated computer-aided control systems analysis and synthesis package, e.g. such as the one suggested in [6].

As would be expected, the immense flexibilty the the BMI fomulation offers has a price in that the resulting optimization problem is no longer convex, quasi-convex or even local-global. However, the lack of such properties does not mean that BMI problems are unsolvable. On the contrary, for the BMI:

- Since the problem is bilinear, one obvious (although suboptimal) way to obtain *local* solutions is to solve the BMIs as alternating LMIs.
- There is no reason why a local minimum of $\Lambda(x, y)$ with respect to (x, y) jointly cannot be found.
- Using existing robust controller synthesis and model reduction techniques, it is possible to generate initial conditions which have a high likelihood of being near a solution to (2).

Local minimization procedures coupled with the ability to make good initial guesses are often sufficent to give significantly improved results compared with other currently available approaches, e.g. see [17].

It may also be noted the other robust control problems or problem formulations which lack the local-global property include the calculation of the multivariable stability margin ($K_m = \frac{1}{\mu}$) of a plant, e.g. [10, 3, 27], LMI problems with rank constraints, e.g. [13, 9], and the optimization over Riccati equation constraints approach of, e.g., [5, 19], for the reduced order controller case.

3 Properties and Local Algorithms

Several important facts about the BMI problem are listed. See [18] for details. First note that $\Lambda(x, y)$ is nondifferentiable but continuous. However, expressions for its subdifferential at any given (x_o, y_o) may be obtained.

Proposition 3.1 *$\Lambda(x, y)$ of (6) is Lipschitzian on $X \times Y$.*

This follows trivially from the fact that the subdifferential of $\Lambda(x, y)$ is uniformly bounded over the bounded domain $X \times Y$.

Consider any closed hyper-rectangle $\mathcal{Q} \subset X \times Y$ of the form

$$\begin{aligned} \mathcal{Q} \ := \ & [L_{x_1}, U_{x_1}] \times \ldots \times \left[L_{x_{n_x}}, U_{x_{n_x}}\right] \\ & \times [L_{y_1}, U_{y_1}] \times \ldots \times \left[L_{y_{n_y}}, U_{y_{n_y}}\right] \end{aligned} \tag{8}$$

where L_{x_i} and U_{x_i} are the lower and upper bounds on the variable x_i for the rectangle $\mathcal{Q}$, and $-B^L_{x_i} \le L_{x_i} \le U_{x_i} \le B^U_{x_i}$. Similarly for y_j variables. Note that if $x_i \in [L_{x_i}, U_{x_i}]$ and $y_j \in \left[L_{y_j}, U_{y_j}\right]$, then $x_i y_j \in \left[L_{w_{i,j}}, U_{w_{i,j}}\right]$ where

$$L_{w_{i,j}} \ := \min\left\{L_{x_i}L_{y_j}, L_{x_i}U_{y_j}, U_{x_i}L_{y_j}, U_{x_i}U_{y_j}\right\} \tag{9}$$

$$U_{w_{i,j}} \ := \max\left\{L_{x_i}L_{y_j}, L_{x_i}U_{y_j}, U_{x_i}L_{y_j}, U_{x_i}U_{y_j}\right\} \tag{10}$$

Given any $\mathcal{Q}$, define the hyper-rectangle $\mathcal{W}(\mathcal{Q}) \subset \mathbf{R}^{n_x \times n_y}$ as follows:

$$\mathcal{W}(\mathcal{Q}) \ := \ \left\{W \in \mathbf{R}^{n_x \times n_y} \ : \ w_{i,j} \in \left[L_{w_{i,j}}, U_{w_{i,j}}\right]\right\} \tag{11}$$

where $w_{i,j}$ denotes the i,jth element of the matrix W. Define the *affine* matrix function of x, y and W,

$$F_L(x, y, W) := F_{0,0} + \sum_{i=1}^{n_x} x_i F_{i,0} + \sum_{j=1}^{n_y} y_j F_{0,j} + \sum_{i=1}^{n_x}\sum_{j=1}^{n_y} w_{i,j} F_{i,j} \tag{12}$$

where $W \in \mathcal{W}(\mathcal{Q})$. Define also

$$\Lambda_L(x, y, W) := \overline{\lambda}\{F_L(x, y, W)\} \tag{13}$$

The following result is then obvious:

Proposition 3.2

$$\min_{(x,y)\in\mathcal{Q}} \overline{\lambda}\{F(x, y)\} \ge \min_{(x,y)\in\mathcal{Q},\, W\in\mathcal{W}(\mathcal{Q})} \overline{\lambda}\{F_L(x, y, W)\} \tag{14}$$

The proof follows immediately from the fact that if $(x, y) \in \mathcal{Q}$, then the dyad $xy^T \in \mathcal{W}(\mathcal{Q})$.

The minimization (6) is very closely related to the LMI eigenvalue minimization problem which has been extensively studied. The approaches based on the interior point methods [26, 22, 7, 35, 15] have been particularly succesful.

A straightforward way to use currently available LMI algorithms to obtain "local solutions" to a BMI problem by alternatingly minimizing $\Lambda(x, y)$ with respect to x with y fixed and vice versa. This approach is not guaranteed to converge to a stationary point of $\Lambda(x, y)$ due to the non-smoothness of the function.

Another simple extension of LMI techniques to the BMI problem is based on the Method of Centers [22, 7]. Given the biaffine matrix function $F(x, y)$ of (1), introduce

$$\phi_\alpha(x, y) := \begin{cases} -\log\ \det[\alpha I - F(x, y)], & \underline{\lambda}\{\alpha I - F(x, y)\} > 0 \\ \infty, & \text{otherwise} \end{cases} \tag{15}$$

Note that the barrier function $\phi_\alpha(x,y)$ is convex in (x,α), and also convex in (y,α). Furthermore, for all (x,y) such that $\underline{\lambda}\{\alpha I - F(x,y)\} > 0$, $\phi_\alpha(x,y)$ is smooth and at least twice differentiable, with derivatives given in [18]. Algorithm 3.1 as defined in Table 1 is then guaranteed to converge to a *local* minimum of $\Lambda(x,y)$.

Algorithm 3.1 Method of Centers for BMI [18].

I. Fix $\epsilon > 0$, $\delta > 0$ *and* $\theta \in [0,1]$.
Set $(x^{(0)}, y^{(0)}) := (0,0)$, $\alpha^{(0)} := \delta + \Lambda(x^{(0)}, y^{(0)})$, $k := 0$.

II. Repeat, {

R1 $(x^{(k+1)}, y^{(k+1)}) := \arg\min \phi_{\alpha^{(k)}}(x,y)$.

R2 $\alpha^{(k+1)} := (1-\theta)\Lambda(x^{(k+1)}, y^{(k+1)}) + \theta\alpha^{(k)}$.

R3 $k := k+1$.

} *until* $\Lambda(x^{(k-1)}, y^{(k-1)}) - \Lambda(x^{(k)}, y^{(k)}) < \epsilon$.

Table 1: Algorithm 3.1.

In Algorithm 3.1, the minimization in Step R2 is a local minimization with initial point $(x^{(k)}, y^{(k)})$. Note that the local minimization in Step R2 guarantees that the solution will converge to a local minimum of $\Lambda(x,y)$.

For more complete solution of the BMI problem (6), global optimization techniques need to be examined. Any global optimization problem is, of course, inherently difficult. It should also be noted that in general, global optimization problems are NP-hard, e.g. [25]. This, of course, does not mean that the problems are unsolvable, only that they will require algorithms which will not have polynomial time bounds, unlike, say, LMI or linear programming problems.

4 BMI Branch and Bound Algorithm

A branch and bound algorithm for the BMI is presented in this section. Much of the following notation and terminology is from [3, 2] although the global optimization text [21] may also be consulted.

Before presenting a branch and bound algorithm for the BMI, it should be note that it is theoretically possible to establish the global optimum to any given tolerence by *exhaustively gridding* the entire domain. This follows from the fact that $\Lambda(x,y)$ is Lipschitz over any bounded domain. The branch and bound approach may be regarded as a clever way of gridding that uses upper and lower bounds to progressively refine the areas it needs to grid, therefore avoiding the need to grid the entire domain.

4.1 Upper and Lower Bounds

The objective is to minimize $\Lambda(x,y)$ over the domain $X \times Y$. The basic requirement for a branch and bound algorithm for minimizing $\Lambda(x,y)$ is for the existence of two functions, Φ_L and Φ_U, on the family of hyper-rectangles of the form (8) such that the following conditions hold:

C1. $\Phi_L(\mathcal{Q})$ gives a lower bound and $\Phi_U(\mathcal{Q})$ an upper bound on $min_{(x,y)\in\mathcal{Q}}\Lambda(x,y)$, i.e.,

$$\Phi_L(\mathcal{Q}) \le \min_{(x,y)\in\mathcal{Q}} \Lambda(x,y) \le \Phi_U(\mathcal{Q}) \tag{16}$$

for every hyper-rectangle $\mathcal{Q} \subset X \times Y$.

C2. Let Size($\mathcal{Q}$) denote the length of the longest side of the hyper-rectangle $\mathcal{Q}$, then as Size$(\mathcal{Q}) \searrow 0$, $\Phi_U(\mathcal{Q}) - \Phi_L(\mathcal{Q}) \searrow 0$ uniformly, i.e.,

$$\begin{aligned} &\forall\, \epsilon > 0\ \exists\, \delta > 0 \quad \text{such that: } \forall\, \mathcal{Q} \subset X \times Y, \\ &\text{Size}(\mathcal{Q}) < \delta \ \Rightarrow\ \Phi_U(\mathcal{Q}) - \Phi_L(\mathcal{Q}) < \epsilon \end{aligned} \tag{17}$$

Define the following functions on the family of hyper-rectangles of the form $\mathcal{Q} \subset X \times Y$:

$$\Phi_U(\mathcal{Q}) := \text{local} \min_{(x,y)\in\mathcal{Q}} \Lambda(x,y) \tag{18}$$

$$\Phi_L(\mathcal{Q}) := \min_{(x,y)\in\mathcal{Q}, W\in\mathcal{W}(\mathcal{Q})} \Lambda_L(x,y,W) \tag{19}$$

Clearly, for any $\mathcal{Q}$, $\Phi_U(\mathcal{Q})$ may be obtained from Algorithm 3.1. Also, $\Phi_L(\mathcal{Q})$ is merely the solution to an LMI problem over $\mathcal{Q} \times \mathcal{W}(\mathcal{Q})$, and can be calculated fairly efficiently [15, 35].

Theorem 4.1 *Φ_U and Φ_L as defined in (18) and (19) fulfil conditions* **C1** *and* **C2***.*

Proof: That Φ_U and Φ_L satisfy condition **C1** follows from Proposition 3.2.

From the inequality $\overline{\lambda}\{A+B\} \le \overline{\lambda}\{A\} + \overline{\lambda}\{B\}$ for hermitian matrices, and defining $v_{i,j} := v_{i,j}(x_i, y_j, w_{i,j}) = x_i y_j - w_{i,j}$

$$\begin{aligned} \Lambda(x,y) &\le \Lambda_L(x,y,W) + \overline{\lambda}\left\{\sum_{i=1}^{n_x}\sum_{j=1}^{n_y}(x_i y_j - w_{i,j})F_{i,j}\right\} \\ \Rightarrow\quad \Lambda(x,y) - \Lambda_L(x,y,W) &\le \overline{\lambda}\left\{\sum_{i=1}^{n_x}\sum_{j=1}^{n_y} v_{i,j}F_{i,j}\right\} \\ \Rightarrow\quad \Lambda(x^*,y^*) - \Lambda_L(x^*,y^*,W^*) &\le \overline{\lambda}\left\{\sum_{i=1}^{n_x}\sum_{j=1}^{n_y} v^*_{i,j}F_{i,j}\right\} \end{aligned}$$

where $(x^*, y^*, W^*) := \arg\min_{(x,y)\in\mathcal{Q}, W\in\mathcal{W}(\mathcal{Q})} \Lambda_L(x,y,W)$, and $v^*_{i,j} := v_{i,j}(x^*_i, y^*_j, w^*_{i,j})$. Note that

$$\begin{aligned} &\text{Size}(\mathcal{Q}) &&\le \delta \\ \Rightarrow\ &\text{Size}(\mathcal{W}(\mathcal{Q})) &&\le \delta\,(B_x + B_y + \delta) \\ \Rightarrow\ &v_{i,j} &&\le 2\delta\,(B_x + B_y + \delta), \\ & &&\forall\, i \in \{1,\ldots,n_x\},\ \forall\, j \in \{1,\ldots,n_y\} \end{aligned}$$

and define $\Delta(\mathcal{Q})$:

$$\begin{aligned} \Delta(\mathcal{Q}) &:= \Phi_U(\mathcal{Q}) - \Phi_L(\mathcal{Q}) \\ &= \min_{(x,y)\in\mathcal{Q}} \Lambda(x,y) - \min_{(x,y)\in\mathcal{Q}, W\in\mathcal{W}(\mathcal{Q})} \Lambda_L(x,y,W) \end{aligned}$$

Now, since $\min_{(x,y)\in\mathcal{Q}} \Lambda(x,y) \le \Lambda(x^*,y^*)$ by definition, and $\Delta(\mathcal{Q}) \ge 0$, it follows that

$$\begin{aligned} 0 \le \Delta(\mathcal{Q}) &\le \overline{\lambda}\left\{\sum_{i=1}^{n_x}\sum_{j=1}^{n_y} v^*_{i,j}F_{i,j}\right\} \\ &\le \max_{|v_{i,j}| \le 2\delta(B_x+B_y+\delta)} \overline{\lambda}\left\{\sum_{i=1}^{n_x}\sum_{j=1}^{n_y} v_{i,j}F_{i,j}\right\} \end{aligned} \tag{20}$$

so that condition **C2** follows. *Q.E.D.*

4.2 A BMI Branch and Bound Algorithm

Given the functions $\Phi_U(\mathcal{Q})$ and $\Phi_L(\mathcal{Q})$, it is straightforward to adapt the branch and bound algorithm given in [3, 2] to globally minimize $\Lambda(x,y)$, see Table 2.

Algorithm 4.1 Branch and Bound Algorithm for BMI

I. Fix $\epsilon > 0$. *Set* $k := 0$, $\mathcal{Q}_0 := X \times Y$, $\mathcal{S}_0 := \{\mathcal{Q}_0\}$.
$L_0 := \Phi_L(\mathcal{Q}_0)$, $U_0 := \Phi_U(\mathcal{Q}_0)$.

II. Repeat {

R1. Select $\bar{\mathcal{Q}}$ *from* $\mathcal{S}_k$ *such that* $L_k = \Phi_L(\bar{\mathcal{Q}})$.
$\mathcal{S}_{k+1} := \mathcal{S}_k - \{\bar{\mathcal{Q}}\}$.

R2. Split $\bar{\mathcal{Q}}$ *along its longest edge into* $\bar{\mathcal{Q}}_1$ *and* $\bar{\mathcal{Q}}_2$.

R3. For $i = 1, 2$, {
If $\Phi_L(\bar{\mathcal{Q}}_i) \leq U_k$, {
Compute $\Phi_U(\bar{\mathcal{Q}}_i)$. $\mathcal{S}_{k+1} := \mathcal{S}_{k+1} \bigcup \{\bar{\mathcal{Q}}_i\}$.
}
}

R4. $U_{k+1} := \min_{\mathcal{Q}\in\mathcal{S}_{k+1}} \Phi_U(\mathcal{Q})$.

R5. Pruning: $\mathcal{S}_{k+1} := \mathcal{S}_{k+1} - \{\mathcal{Q} : \Phi_L(\mathcal{Q}) > U_{k+1}\}$.

R6. $L_{k+1} := \min_{\mathcal{Q}\in\mathcal{S}_{k+1}} \Phi_L(\mathcal{Q})$.

R7. $k := k + 1$.

} *until* $U_k - L_k < \epsilon$.

Table 2: Algorithm 4.1

In Algorithm 4.1, $\mathcal{S}_k$ is the collection of hyper-rectangles $\{\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_{\bar{k}}\}$ after k iterations, where $\bar{k} \leq k$. At the $(k+1)$th iteration, $\bar{\mathcal{Q}}$, the rectangle in $\mathcal{S}_k$ with the smallest lower bound, $\Phi_L(\mathcal{Q})$ is identified, and split along its largest side into rectangles $\bar{\mathcal{Q}}_1$, and $\bar{\mathcal{Q}}_2$ (Step R2). $\mathcal{S}_{k+1}$ is formed by discarding $\bar{\mathcal{Q}}$ from $\mathcal{S}_k$.

The lower bounds corresponding to $\bar{\mathcal{Q}}_1$, and $\bar{\mathcal{Q}}_2$, $\Phi_L(\bar{\mathcal{Q}}_1)$ and $\Phi_L(\bar{\mathcal{Q}}_2)$, are calculated. If $\Phi_L(\bar{\mathcal{Q}}_i) > U_k$, $\bar{\mathcal{Q}}_i$ is discarded, otherwise $\Phi_U(\bar{\mathcal{Q}}_i)$ is calculated and $\bar{\mathcal{Q}}_i$ is added to $\mathcal{S}_{k+1}$.

The new estimate for the upper bound, U_{k+1} is the lowest upper bound of the $\mathcal{Q} \in \mathcal{S}_{k+1}$, see Step R4. Using U_{k+1}, $\mathcal{S}_{k+1}$ is *pruned* to remove hyper-rectangles for which the global minimum cannot occur, since their lower bounds exceed the current upper bound U_{k+1}. The pruning step R5 is not strictly necessary, but is required to minimize the number of hyper-rectangles stored. The new estimate for the lower bound L_{k+1} is the smallest lower bound of the $\mathcal{Q} \in \mathcal{S}_{k+1}$.

Note that necessarily,

- $\Phi_L(\bar{\mathcal{Q}}_1)$ and $\Phi_L(\bar{\mathcal{Q}}_2)$ cannot be less than L_k, and
- Size($\bar{\mathcal{Q}}_1$) and Size($\bar{\mathcal{Q}}_2$) cannot exceed Size($\bar{\mathcal{Q}}$), and the "volume" of the rectangles $\bar{\mathcal{Q}}_1$ and $\bar{\mathcal{Q}}_2$ will be half that of $\bar{\mathcal{Q}}_1$.

i.e. heuristically, it can be seen that the lower bound L_k and the hyper-rectangle containing the smallest lower bound will be successively refined. The details on the finite time convergence of the above standard branch and bound algorithm may be found in, e.g. [2].

Theorem 4.2 *Algorithm 4.1 terminates in finite time.*

This follows from the fact that Φ_U and Φ_L fulfil conditions **C1** and **C2**. An example of the performance of Algorithm 4.1 is given in the next section.

It should also be noted that if only the BMI feasibility problem is of interest, i.e. if it is desired only to find (x,y) such that $\Lambda(x,y) < 0$, then the upper bounds U_k may be set to 0 always, and the algorithm is terminated if either $\Phi_U(\mathcal{Q}) < 0$ for some $\mathcal{Q}$ or if $L_k \geq 0$.

4.3 Remarks

The key to Algorithm 4.1 is the development of the lower bound function Φ_L, which exploits the *bilinearity* of the problem. However, the use of the lower bound of (14) is possibly very conservative. If the geometry of the BMI problem is such that the lower bound of (14) is conservative, then the convergence of Algorithm 4.1 will be slow.

Even though convergence to a global minimum is guaranteed in Algorithm 4.1, it may not be *practical* to apply the algorithm to obtain complete solutions to even moderate sized robust control synthesis BMI problems due to the large computational load imposed by Algorithm 4.1. However, Algorithm 4.1 is probably more efficient compared to, say, gridding the entire domain of the BMI in question.

Note that Algorithm 4.1 differs from the branch and bound approach suggested in [3, 2] in the sense that the upper and lower bounds functions used in [3, 2] are derived from control theory considerations. In contrast, the upper and lower bound functions Φ_U and Φ_L used in Algorithm 4.1 are derived purely from the BMI formulation of (2) itself.

The geometrical aspects of the BMI feasibility problem were discussed in [18]. In particular, the BMI problem was shown to be equivalent to that of finding a hyperplane separating a given matrix numerical range (field of values) and the origin, subject to the constraint that the hyperplane being generated by a dyad. The connection between that viewpoint and the lower bound of Proposition 3.2 is obvious.

A related approach to the BMI problem is that of [34], which showed that the BMI feasibilty problem may be reduced to the problem of finding the maximum norm element in an intersection of ellipsoids centered at the origin: a convex *maximization* problem, which again requires global optimization techniques.

4.4 Other Global Approaches

There are several references in the mathematical programming literature to branch and bound methods for biconvex and bilinear problems, e.g., [1, 21]. Also available is the Benders decomposition (see [16] and references therein) which leads to the primal-relaxed dual algorithm of [14]. The algorithm of [14] is of particular interest. However, it requires closed form formulae for the gradients of the function to be minimized, which is unavailable in the BMI context.

Another obvious approach to the BMI global optimization problem is to use multistart methods, e.g. [30]. Various "intelligent" optimization methods may also be used. However, unless the underlying structure of the BMI problem is exploited, it is doubtful whether these methods will offer any improvement over Algorithm 4.1.

$(x,y)_{\text{opt}}$	$\underline{\lambda}\{F(x,y)_{\text{opt}}\}$
$(0.0049, -2.0253)$	3.3886
$(0.4436, 4.0174)$	−0.4434
$(1.0488, 1.4179)$	−0.9565

Table 3: Local Minima of $\Lambda(x,y)$ of (21).

5 A Simple BMI Example

Consider a simple BMI problem with its its corresponding LMI:

$$\begin{aligned} F(x,y) &:= \begin{bmatrix} -10 & -0.5 & -2 \\ -0.5 & 4.5 & 0 \\ -2 & 0 & 0 \end{bmatrix} + y \begin{bmatrix} -1.8 & -0.1 & -0.4 \\ -0.1 & 1.2 & -1 \\ -0.4 & -1 & 0 \end{bmatrix} \\ &+ x \begin{bmatrix} 9 & 0.5 & 0 \\ 0.5 & 0 & -3 \\ 0 & -3 & -1 \end{bmatrix} + xy \begin{bmatrix} 0 & 0 & 2 \\ 0 & -5.5 & 3 \\ 2 & 3 & 0 \end{bmatrix} \quad (21) \\ F_L(x,y,w) &:= \begin{bmatrix} -10 & -0.5 & -2 \\ -0.5 & 4.5 & 0 \\ -2 & 0 & 0 \end{bmatrix} + y \begin{bmatrix} -1.8 & -0.1 & -0.4 \\ -0.1 & 1.2 & -1 \\ -0.4 & -1 & 0 \end{bmatrix} \\ &+ x \begin{bmatrix} 9 & 0.5 & 0 \\ 0.5 & 0 & -3 \\ 0 & -3 & -1 \end{bmatrix} + w \begin{bmatrix} 0 & 0 & 2 \\ 0 & -5.5 & 3 \\ 2 & 3 & 0 \end{bmatrix} \end{aligned}$$

For this particular example, the objective is to minimize $\overline{\lambda}\{F(x,y)\}$ for $(x,y) \in [-0.5, 2] \times [-3, 7]$. Note that $\bar{F}(1,0,1) = -I$, so that $\min \overline{\lambda}\{F_L(x,y,w)\} = -1$

It should first be noted that that there are *three* local minima, as given in Table 3.

Figure 1 gives the contour plots of $\overline{\lambda}\{F(x,y)\}$, and the trajectories of the BMI local optimization algorithm, Algorithm 3.1.

The global optimization algorithm, Algorithm 4.1, required 24 iterations to reduce the difference between the minimum upper bound and the minimum lower bound, $U_k - L_k$ to within 0.5% of U_k. The ϵ-global minimum was found to be $\hat{x} = 1.0488$ and $\hat{y} = 1.4178$, and $U_{24} = \Lambda(\hat{x}, \hat{y}) = 0.9565$. The best lower bound to $\min_{(x,y)} \Lambda(x,y)$, $L_{24} = 0.9603$. Figure 2 shows the partitions generated by Algorithm 4.1. The endpoints of the local minimization algorithm, Algorithm 3.1, for each partition are also shown.

The progress of Algorithm 4.1 is shown in Figure 3. Note that the global minimum is found fairly early, and the remainder of the iterations are devoted to tightening the lower bounds.

6 Summary and Conclusion

The Biaffine Matrix Inequality (BMI) formulation is a very flexible framework for approaching complex control system synthesis problems. The robust control synthesis problems that fall within the BMI framework include, e.g., the μ/K_m-Synthesis problem with fixed order multipliers and fixed order controllers, multiobjective synthesis and the synthesis of one controller for multiple plants.

The BMI formulation leads to biconvex optimization problems which are often difficult to completely solve, i.e. a local minimum may not be a global minimum, and it is generally hard to verify that a local minimum is indeed a global minimum. While local optimization approaches often yield

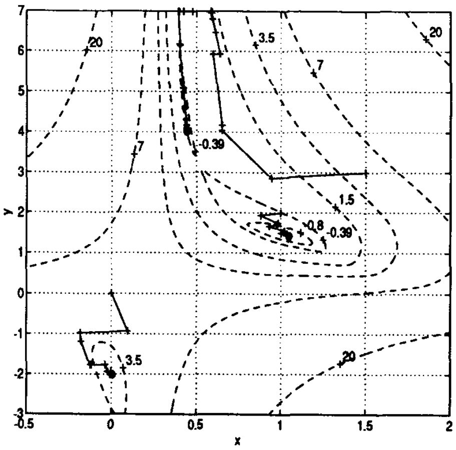

Figure 1: Contour Plot of $\overline{\lambda}\{F(x,y)\}$, with trajectories of Algorithm 3.1, the local optimization algorithm. Circles denote the local maxima.

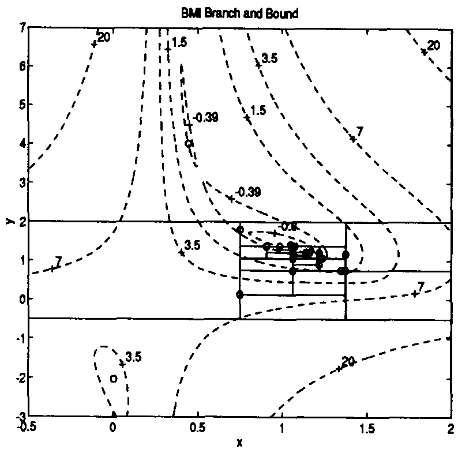

Figure 2: Contour Plot of $\overline{\lambda}\{F(x,y)\}$, with partitions generated by Algorithm 4.1 shown. Circles are local minimization endpoints.

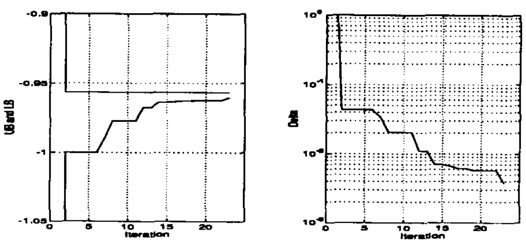

Figure 3: Progress of the upper and lower bounds, U_k and L_k, and the logarithmic plot of the difference, $U_k - L_k$.

sufficiently good solutions, it remains important to investigate methods for finding a solution that is guaranteed to be a global minimum within some given tolerence.

The contribution of this paper is that it provides a branch and bound global optimization algorithm for the BMI problem that is guaranteed to find a global minimum of the BMI problem to any arbitrary tolerence in finite time. The branch and bound algorithm provided exploits the bilinearity of the problem structure and builds on the recent advances in solving convex LMI type problems. A simple BMI example is included.

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