LEAST ELEMENTS AND THE MINIMAL RANK MATRICES*

MEHRAN MESBAHI[†] AND GEORGE P. PAPAVASSILOPOULOS [‡]

Abstract. The problem of minimizing the rank of a positive semi-definite matrix, subject to the constraint that an affine transformation of it is also positive semi-definite, is considered. In this direction, we demonstrate that certain instances of this problem can be solved by semi-definite programming. An illustrative example from control theory is also provided.

Key words. Rank Minimization Problem, Least Element Theory, Control Theory

AMS subject classifications. 15A45, 90C33, 93D09

1. Introduction. This note is concerned with the solution to the following, henceforth referred to as the MIN–RANK problem,

$$\min \mathbf{rank}\ X \tag{1.1}$$

$$\text{subject to: } Q + M(X) \succeq 0 \tag{1.2}$$

$$X \succeq 0 \tag{1.3}$$

In (1.1)–(1.3), M is a symmetry preserving linear map on the space of symmetric matrices, Q is a symmetric matrix (of appropriate dimensions), and the ordering "$\succeq$" is to be interpreted in the sense of Löwner, i.e., $A \succeq B$ if and only if $A - B$ is positive semi-definite; similarly $A \succ B$ indicates that $A - B$ is positive-definite.

The MIN–RANK problem has many applications in control and system theory. For example, the Bilinear Matrix Inequality problem (BMI) can be shown to be equivalent to a MIN–RANK problem (possibly with some additional constraints) [7], [11]. The BMI, on the other hand, has been shown by Safonov *et al.* [10] to be a unifying formulation for a wide array of control synthesis problems, including, the fixed-order H^∞ control, μ/k_m-synthesis, decentralized control, robust gain-scheduling, and simultaneous stabilization. Similarly in [4], El Ghaoui and Gahinet have shown that the important problems of static output feedback stabilization, dynamic reduced order output-feedback stabilization, reduced order H^∞ synthesis, and μ–synthesis with constant scaling, can be formulated as a rank minimization under an LMI constraint, clearly an instance of the MIN–RANK problem.

We shall restrict our attention to linear maps M in (1.2) of a particular structure; they are assumed to be of the type $\mathcal{Z}$:

DEFINITION 1.1. *A symmetry preserving linear map $M : SR^{n\times n} \rightarrow SR^{n\times n}$ is of the type $\mathcal{Z}$, if it can be represented as,*

$$M(X) = X - \sum_{i=1}^{k} M_i X M_i' \tag{1.4}$$

*Research supported in part by the National Science Foundation under Grant CCR-9222734.

[†]Department of Electrical Engineering–Systems, University of Southern California, Los Angeles, CA 90089-2563 (mesbahi@bode.usc.edu).

[‡]Department of Electrical Engineering–Systems, University of Southern California, Los Angeles, CA 90089-2563 (yorgos@bode.usc.edu)

for some matrices $M_i \in R^{n\times n}$ $(1 \le i \le k)$, and integer $k \ge 1$.

The approach that we adopt for solving MIN–RANK problems with the type $\mathcal{Z}$ linear maps, is strongly motivated by the results pertaining to the linear complementarity problems (LCPs) with a Z matrix (and hence the notation $\mathcal{Z}$ for maps of the form (1.4)) [2], [3], [6], [9]. Recall that a matrix is a Z matrix if all of its non–diagonal elements are non–positive. More specifically, we pose the following question:

> Can one solve a MIN–RANK problem with the type $\mathcal{Z}$ linear map via a semi-definite program (SDP) (a linear program over the cone of positive semi-definite matrices)?

The answer to the above question, as we shall show below, is affirmative, provided that Q in (1.2) is negative semi-definite.

The organization of this note is as follows. In Section 2, we show how a MIN–RANK problem with the type $\mathcal{Z}$ linear map can be solved by formulating it as a SDP. In this direction, we present an extension of the notion of a lattice (in fact, a meet semi-lattice), for the space of symmetric matrices. In Section 3, a control example demonstrating the applicability of our result is presented; few remarks then concludes the paper.

A few words on the notation. T' and $\lambda(T)$ denote the transpose and an eigenvalue of the matrix T, respectively. The space of $n \times n$ real matrices is denoted by $R^{n\times n}$, its symmetric subset by $SR^{n\times n}$, its symmetric positive semi-definite subset by $SR_+^{n\times n}$, and its identity matrix by I_n. Finally, the inner product of two square matrices A and B in $SR^{n\times n}$ is denoted by $A \bullet B$, which is equal to the trace of the product AB.

2. The MIN–RANK Problem. In this section, we first develop an extension of the notion of a lattice (for vectors, with component–wise ordering), for the space of symmetric matrices (with the Löwner ordering). We then demonstrate the usefulness of this notion by showing that a MIN–RANK problem with the $\mathcal{Z}$ linear map, reduces to a semi-definite program, provided that $Q \preceq 0$.

For a given pair of $n \times n$ symmetric matrices, consider the set

$$\Delta(A,B) := \{X \in SR^{n\times n} : 0 \preceq X \preceq A,\ 0 \preceq X \preceq B\}$$

In [1], Ando has shown that although the set $\Delta(A,B)$ does not possess a maximal point, it has in a sense, "many maximal elements," with respect to the Löwner ordering.

The set of the maximal points of $\Delta(A,B)$, which shall be denoted by $\Delta_{\sup}(A,B)$, has the following property:

$$\begin{aligned} &\forall D \in \Delta(A,B), \exists Z \in \Delta_{\sup}(A,B) : \\ &\qquad\qquad Z \in \Delta(A,B), D \preceq Z; \\ &\& \ \nexists W \in \Delta(A,B) : W \neq Z;\ W \succeq Z \end{aligned} \tag{2.1}$$

The matrix $Z \in \Delta_{\sup}(A,B)$ that satisfies the condition (2.1), not only depends on the matrices A and B, but also on the specific matrix D.

In [1], a complete characterization of the maximal points of the set $\Delta(A, B)$, along with an algorithm for their computation are provided. More explicitly, in [1] the set $\Delta_{\sup}(A, B)$ is parameterized by a subspace $\mathcal{N} \subset \text{range}(A) \cap \text{range}(B)$, and an n_2-by-n_1 matrix K, such that $K^*K \prec I_{n_1}$, where n_1 (respectively n_2) is the number of positive (respectively negative) eigenvalues of the matrix $[\mathcal{N}]A - [\mathcal{N}]B$, with multiplicity counted; the notation $[\mathcal{N}]A$ denotes the *short* of the matrix A to the subspace $\mathcal{N}$ [1]. Moreover, given a matrix $D \in \Delta(A, B)$, a matrix $Z \in \Delta_{\sup}(A, B)$ satisfying (2.1) is constructed as:

$$Z = \frac{1}{2}\{[\mathcal{N}]A + [\mathcal{N}]B - L|L^{-1}([\mathcal{N}]B - [\mathcal{N}]A)L^{-1}|L\} \tag{2.2}$$

where $L := ([\mathcal{N}]A + [\mathcal{N}](B) - 2D)^{1/2}$, L^{-1} is the inverse of L *restricted* to the range of $[\mathcal{N}]A - [\mathcal{N}]B$, and $|A|$ denotes the positive square root of the matrix A^2. For more details on this construction, and in particular, the reason for the existence of the restricted inverse of L, the reader is referred to [1] (page 5: lines 15–16; page 10: lines 5–7).

Analogous to the case of the component-wise ordering for vectors, we define the following generalization of a (meet semi–) lattice.

Definition 2.1. *A set $\Gamma \subseteq SR_+^{n\times n}$ is called a (meet semi-) hyper-lattice if for all pairs X and Y in Γ, there exists $Z \in \Delta(X, Y)$ such that $Z \in \Gamma$.*

Define,

$$\Gamma := \{X \succeq 0 : Q + M(X) \succeq 0\} \tag{2.3}$$

to be the feasible set of the MIN–RANK problem (1.1)–(1.3). We shall assume that the set Γ is non–empty.

We now demonstrate that Γ (2.3) is indeed a (meet–semi) hyper–lattice when Q is negative semi-definite.

Lemma 2.2. *Let the linear map M in the definition of Γ (2.3) be of the type $\mathcal{Z}$. Then Γ is a (meet semi-) hyper–lattice when Q is negative semi-definite.*

Proof. We would like to show that for two symmetric matrices A and B in Γ, there exists $Z \in \Delta(A, B)$ such that $Z \in \Gamma$.

We first note that the set $\Delta(A, B)$ is convex and compact. It suffices to show that for some $Z \in \Delta(A, B)$,

$$Z \succeq -Q + \sum_{i=1}^{k} M_i Z M_i'.$$

Since $Z \preceq A$ and $Z \preceq B$, one has

$$\sum_i M_i Z M_i' \preceq \sum_i M_i A M_i'$$

and

$$\sum_i M_i Z M_i' \preceq \sum_i M_i B M_i'$$

As a result of the assumption $A, B \in \Gamma$, one concludes that,

$$A \succeq -Q + \sum_i M_i A M_i' \succeq -Q + \sum_i M_i Z M_i' \succeq 0$$

and

$$B \succeq -Q + \sum_i M_i B M_i' \succeq -Q + \sum_i M_i Z M_i' \succeq 0$$

for all $Z \in \Delta(A, B)$ (recall that Q is assumed to be negative semi-definite). Hence for all $Z \in \Delta(A, B)$, $(-Q + \sum_i M_i Z M_i') \in \Delta(A, B)$.

In particular, for all $Z \in \Delta(A, B)$, there exists $Y \in \Delta_{\sup}(A, B)$ such that

$$Y \succeq -Q + \sum_i M_i Z M_i' \tag{2.4}$$

by the definition of the set $\Delta_{\sup}(A, B)$. Let $g : \Delta(A, B) \to \Delta(A, B)$ be the point-to-set map such that,

$$g(Z) := \{Y \in \Delta(A, B) : Y \succeq -Q + \sum_i M_i Z M_i'\} \tag{2.5}$$

The map g is upper semi-continuous. To see this, let $\{Z_k\}_{k\geq 1}$ and $\{Y_k\}_{k\geq 1}$ be a sequence of matrices such that

$$Y_k \succeq -Q + \sum_i M_i Z_k M_i'$$

and let $Z_k \to Z^*$, and $Y_k \to Y^*$. Since $\Delta(A, B)$ is compact, $Y^* \in \Delta(A, B)$. Define

$$M(Z_k, Y_k) := Q + Y_k - \sum_i M_i Z_k M_i'$$

The map M is linear on $SR^{n\times n} \times SR^{n\times n}$, and is therefore continuous. Since the cone of positive semi-definite matrices is closed,

$$0 \preceq \lim_{k\to\infty} M(Z_k, Y_k) = M(Z^*, Y^*)$$

and therefore,

$$Y^* \succeq -Q + \sum_i M_i Z^* M_i'$$

and hence $Y^* \in g(Z^*)$.

Since g is upper semi-continuous on the convex and compact set $\Delta(A, B)$, it has a fixed point via the Kakutani's Fixed Point Theorem [5]. That is, there exists a matrix $\widehat{Z} \in \Delta(A, B)$ such that $\widehat{Z} \succeq -Q + \sum_i M_i \widehat{Z} M_i'$. Hence, Γ is indeed a (meet semi–) hyper-lattice.

The following theorem answers the question posed in Introduction.

THEOREM 2.3. *A minimal rank element of Γ (2.3) can be found by a semi-definite program when Q is negative semi-definite matrix.*

Proof. Consider the following semi-definite program,

$$\min I \bullet X \tag{2.6}$$

$$\text{subject to: } Q + X - \textstyle\sum_i M_i X M_i' \succeq 0 \tag{2.7}$$

$$X \succeq 0 \tag{2.8}$$

Recall the Γ is the set defined by (2.7)–(2.8). Since Γ is assumed to be non-empty, let $A \in \Gamma$ (2.3) (such a matrix can be found be a semi-definite program itself). Now consider instead the problem,

$$\min I \bullet X \tag{2.9}$$

$$\text{subject to: } Q + X - \textstyle\sum_i M_i X M_i' \succeq 0 \tag{2.10}$$

$$X \succeq 0 \tag{2.11}$$

$$I \bullet X \leq I \bullet A \tag{2.12}$$

It should be clear that the optimum of both SDPs (2.6)–(2.8) and (2.9)–(2.12), are the same. The latter SDP has an optimum since, $\Gamma \cap \{X : I \bullet X \leq I \bullet A\}$ is a compact set, and $I \bullet X$ is a linear functional in X. Let $\widetilde{X}$ be the optimal solution of (2.6)–(2.8). We now claim that $\widetilde{X}$ is of minimal rank in Γ. To show this, let $Y \in \Gamma$ and $Z \in \Delta(\widetilde{X}, Y)$, such that $Z \in \Gamma$ (this is possible since Γ (2.3) is a (meet semi–) hyper-lattice). By the optimality of $\widetilde{X}$,

$$\sum_i \lambda_i(\widetilde{X}) \leq \sum_i \lambda_i(Z) \tag{2.13}$$

On the other hand since $Z \in \Delta(\widetilde{X}, Y)$, one has,

$$\lambda_i(Z) \leq \lambda_i(\widetilde{X}) \quad (i = 1, \ldots, n) \tag{2.14}$$

and

$$\lambda_i(Z) \leq \lambda_i(Y) \quad (i = 1, \ldots, n) \tag{2.15}$$

In view of (2.13), (2.14) implies that $\lambda_i(Z) = \lambda_i(\widetilde{X})$ $(i = 1, \ldots, n)$. Thus by (2.15), for an arbitrary matrix $Y \in \Gamma$,

$$\lambda_i(\widetilde{X}) \leq \lambda_i(Y) \quad (i = 1, \ldots, n) \tag{2.16}$$

Suppose now that $\widetilde{X}$ is not of minimal rank in Γ. Then there exists $\widetilde{Y}$ such that $\lambda_i(\widetilde{Y}) = 0$ and $\lambda_i(\widetilde{X}) \neq 0$, for some index i. Since $\widetilde{X} \succeq 0$, $\lambda_i(\widetilde{X}) > 0$, which violates (2.16). Hence $\widetilde{X}$ is of minimal rank in Γ.

3. A Control Example. Let Σ be the discrete time, linear time invariant dynamical system:

$$\Sigma: \quad x_{k+1} = Ax_k + Bu_k \tag{3.1}$$

$$y_k = Cx_k + Du_k \tag{3.2}$$

with matrix $A \in R^{n \times n}$ (and all other matrices of appropriate dimensions).

Suppose that it is desired to synthesis a controller for Σ such that the closed loop system is internally stable, as well as satisfying a H^∞ norm constraint from u to y, in face of a given structured uncertainty (the structured optimal performance control problem (SOPC)) [8].

In [8], Packard *et al.* show that this important problem in control theory can be reduced to a MIN-RANK problem.

THEOREM 3.1 ([8]). *The structured optimal performance control problem (SOPC) is solvable if for a given set of matrices M_1 and M_2 and an integer J, there exist matrices R and S (possibly structured), such that,*

$$M_1 R M_1^t - R \prec 0 \tag{3.3}$$

$$M_2^t S M_2 - S \prec 0 \tag{3.4}$$

and

$$\begin{pmatrix} R & I \\ I & S \end{pmatrix} \succeq 0 \tag{3.5}$$

$$rank \begin{pmatrix} R & I \\ I & S \end{pmatrix} \leq J \tag{3.6}$$

Let,

$$X = \begin{pmatrix} R & I \\ I & S \end{pmatrix}$$

Then it can be shown that the above problem reduces to solving the following instance of the MIN-RANK problem,

$$\min \operatorname{rank} X \tag{3.7}$$

$$\text{subject to: } \widetilde{A}X\widetilde{A}' - X \prec Q \tag{3.8}$$

$$X \in \mathcal{L} \tag{3.9}$$

$$X \succeq 0 \tag{3.10}$$

for an appropriate choice of the matrices $\widetilde{A}$ and (symmetric) Q; moreover the set $\mathcal{L}$ is defined as,

$$\mathcal{L} := \{X : X = \begin{pmatrix} U & I \\ I & V \end{pmatrix}; \; U, V \text{ symmetric}\} \tag{3.11}$$

The subset $\mathcal{L}$ can for example be defined by a set of linear equalities of the form $\frac{1}{2}E_{ij} \bullet X = 1$, where E_{ij} is a matrix whose all entries are zero, except the ij-th entry which is one (this fixes the ij-th entry of the matrix X to one).

Let us rewrite the above problem, for $\epsilon > 0$, as:

$$\min_X \operatorname{rank} X \tag{3.12}$$

$$\text{subject to: } (Q - \epsilon I) + X - \widetilde{A}X\widetilde{A}' \succeq 0 \tag{3.13}$$

$$X \in \mathcal{L} \tag{3.14}$$

$$X \succeq 0 \tag{3.15}$$

We now realize that the above problem is exactly a MIN-RANK problem with a linear map of type $\mathcal{Z}$, except that the solution has to be found in the affine set $\mathcal{L}$. Fortunately, this additional constraint does not introduce a difficulty for the applicability of the approach described earlier. This is due to the fact that if the matrices A and B are in $\mathcal{L}$, the set $\Delta(A, B)$ can be shown to belong to $\mathcal{L}$ [1].[1] Consequently, given that the set Γ (2.3), with the linear map

$$M(X) := X - \widetilde{A}X\widetilde{A}'$$

and $Q - \epsilon I \preceq 0$ is a (meet semi–) hyper–lattice, its restriction to $\mathcal{L}$, if non-empty, is a (meet semi–) hyper–lattice as well.

In order to solve this instance of the MIN-RANK problem arising from the SOPC problem, one thus consider the following semi-definite program, for $\epsilon > 0$,

$$\min_X I \bullet X \tag{3.16}$$

$$\text{subject to: } (Q - \epsilon I) + X - \widetilde{A}X\widetilde{A}' \succeq 0 \tag{3.17}$$

$$X \in \mathcal{L} \tag{3.18}$$

$$X \succeq 0 \tag{3.19}$$

where $Q - \epsilon I$ is required to be negative semi-definite. This approach consequently results in an efficient way of studying the structured optimal performance synthesis problem for the discrete time linear time invariant systems.

[1] Refer to the construction of Ando on pages 8–9 of [1].

4. Concluding Remarks. In this note, we have described an approach for solving the problem of minimizing the rank of a positive semi-definite matrix, subject to the constraint that an affine transformation of it is also positive semi-definite. In this direction, an approach analogous to finding the least element of a meet semi-lattice, which has been extensively studied in the context of the LCP, is developed. However our analysis uses some additional ideas and concepts since the positive semi-definite ordering can not be used to introduce a lattice structure on the space of symmetric matrices. The applicability of our results to certain synthesis problems in control theory is also discussed.

Acknowledgments. The authors gratefully acknowledge the comments and suggestions of the two anonymous referees, and thank Jong–Shi Pang for bringing to our attention references [2] and [9]. We would also like to thank T. Ando for pointing out an error in the original version the paper.

REFERENCES

[1] T. Ando. Parameterization of minimal points of some convex sets of matrices. *Acta Scientiarum Mathematicarum (Szeged)*, 57:3–10, 1993.

[2] R. W. Cottle and Jr. A. F. Veinott. Polyhedral sets having a least element. *Mathematical Programming*, 3:238–249, 1972.

[3] R. W. Cottle, J. S. Pang, and R. E. Stone. *The Linear Complementarity Problem*. Academic Press, 1992.

[4] L. El Ghaoui and P. Gahinet. Rank minimization under LMI constraints: a framework for output feedback problems. In *Proceedings of the European Control Conference*, July 1993.

[5] S. Kakutani. A generalization of Brouwer's fixed point theorem. *Duke Mathematical Journal*, 8:457–459, 1941.

[6] O. L. Mangasarian. Charactrization of linear complementarity problems as linear programs. *Mathematical Programming Study*, 7:74–87, 1978.

[7] M. Mesbahi and G. P. Papavassilopoulos. A cone programming approach to the bilinear matrix inequality and its geometry. *Mathematical Programming (Series B)*, to appear.

[8] A. Packard, K. Zhou, P. Pandey, and G. Becker. A collection of robust control problems leading to LMI's. In *IEEE Conference on Decision and Control*, pages 1245–1250, Brighton, England, 1991.

[9] J. S. Pang. *Least element complementarity theory*. PhD thesis, Department of Operations Research, Stanford University, Stanford, California, 1976.

[10] M. G. Safonov, K. C. Goh, and J. H. Ly. Control system synthesis via bilinear matrix inequalities. In *Proceedings of the 1994 American Control Conference*, Baltimore, M.D., July 1994.

[11] M. G. Safonov and G. P. Papavassilopoulos. The diameter of an intersection of ellipsoids and BMI robust synthesis. In *IFAC Symposium on Robust Control*, Rio de Janeiro, Brazil, September 1994.