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Adaptive rules for discrete-time Cournot games of high competition level markets

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Abstract For the Cournot-like oligopoly games of n firms, where the Cournot adjustment fails to converge, we propose adjustment processes originating from the family of the Moving Averages. In markets of linear demand, where firms have private and linear on quantities cost functions, these adaptive rules turn the games into discrete-time linear systems with delays. With an out of the box proof, we determine the least number of delays (m) that ensures the game of n players converges to its equilibrium. The Simple Moving Average rule (fixed number of delays) and the Cumulative Moving Average rule (constantly increasing number of delays), which is also known as "fictitious play", are the main rules considered. Along with a hybrid rule, result of their combination, they are all studied for their convergent properties and compared in a benchmarking framework to indicate the different trajectories and identify their suitability in applications.

Keywords Adaptation · Cournot game · Bounded rationality · Fictitious play · Convergence · Discrete system

Mathematics Subject Classification (2010) 91A06 · 91A50 · 37N40

1 Introduction

In markets of quantity-based competition with incomplete information, such as energy markets (Doukas et al., 2008), the simplest version of a Cournot

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oligopoly model fails to ensure convergence to the equilibrium in more competitive structures than that of the duopoly case (Theocharis, 1960). The adaptive rule found in the reasoning of the Cournot-Nash equilibrium's emergence in a partial information framework (Friedman, 1983; Daughety, 2005), the so called Cournot adjustment process, has been identified as the main reason of instability (McManus and Quandt, 1961). This simplistic rule allows players to estimate their rivals' output, using the most recent values of the state; even if it fails to serve its cause in games of more than two players, constitutes an early introduction of dynamics and adaptation in game theory. Nevertheless, this adequacy points out the link between the number of competitors and the convergence properties in this kind of games (von Mouche and Quartieri, 2016), indicating the crucial role the adjustment process plays. Although adaptation is considered to be a learning process, in the case of Cournot oligopoly it relies mostly on adaptive expectations (Okuguchi, 1970), with its rationale having experimental grounds (Rassenti et al., 2000; Huck et al., 2002) as well. Along with other quantity-based market models, where learning turns to be more sophisticated (Chrysanthopoulos and Papavassilopoulos, 2016), adaptation may pave the way for manipulation, which allows the occurrence of alternative equilibria (Kordonis et al., 2018; Chrysanthopoulos and Papavassilopoulos, 2016), or for cooperation and coalition formation (Stamtsis and Doukas, 2018). Beyond the sensitivity it adds, the adjustment process affects, indirectly, the social surplus's maximization and the market's sustainability (Gabszewicz and Vial, 1972) through the transient behavior and the convergence of the games.

Many researchers have contributed to the stability of the Cournot oligopoly solution, some of whom have focused on the adjustment process and proposed alternations towards an improved dynamic behavior of the game. In particular, the speed of adjustment has been proposed as a parameter able to stabilize games of more than two players (McManus and Quandt, 1961), a fact shown explicitly by Fisher (1961) for both continuous and discrete adjustment process cases, along with a sufficient condition for the speed of adjustment based on the number of players. Given this adjustment process with speed control, a first attempt for global stability conditions on (continuously differentiable) demand and cost functions has been made by Hahn (1962), while results for discrete and continuous adjustments have been summarized by al Nowaihi and Levine (1985). By introducing a feasible region of non-negative prices and outputs, Furth (1986), studied the stability of local and boundary equilibria. Many other works have dealt with the convergence characteristics of Cournot games as the number of competitors increases, either by extending those results (Puu, 2008; Puu and Sushko, 2002; Ahmed and Agiza, 1998) or by exploiting them towards other research directions (Askar, 2018). Models of varying number of players (Bernhard and Deschamps, 2017; Kordonis and Papavassilopoulos, 2015) offer far more accurate representation of real life market structures and form an ideal environment for introducing the dynamic adjustment of players. The idea of identifying suitable adaptive rules (Papavassilopoulos, 1986, 1988) that are able to ensure stability is broadly used in dynamic games. Its potential, especially, in a quantity-based competition context of increasing number

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of participants, along with its high applicability in decision making under incomplete information structures often found in energy market design (Doukas, 2013), has motivated this work .

This paper deals with a particular class of adaptive rules, originating from statistical and estimation theory, in the context of an incomplete information, n-firm, Cournot game. More precisely, the rules under consideration belong to the family of Moving Averages (MA), which is widely used in applications - ranging from economics and finance to filtering and signal processing - as it provides smoothing properties under simplicity. In our case, by letting players use the Simple Moving Average (SMA) and the Cumulative Moving Average (CMA) principles at the point where they need to estimate their rivals' actions, two different rules of adaptation are formed. Based on the aforementioned rules and also by taking into account a hybrid one, three different games are considered, in which the number of delays is considered to be the number of previous data used in the unweighted mean. Due to the oligopoly's assumptions made, the games examined result in linear discrete time systems of fixed or evolving structure. The main objective of this work is to study the sufficiency of the length of those backward-looking adaptation rules as the number of the market participants increases, with respect to the systems' stability. Complementary to this, we aim to provide a comparative analysis of the proposed rules and highlight the trade-off between stability and smooth transient behavior, even with the lack of information about the number of rivals. The main contributions of the paper are threefold:

(i) The proposition of adaptive rules characterized by the number of delays and based on simple principles, broadly used in business intelligence and technical analysis fields for supporting decision making,
(ii) The explicit identification of the least number of delays, required to ensure convergence of the n-player Cournot game along with the extensive study of its convergent behavior under fictitious play and
(iii) The development of a comparative analysis framework, capable to evaluate the proposed rules on their suitability in an ideal fitting case basis.

More precisely, for the discrete time model considered, the fixed number of delays rule, based on SMA, has been used initially to prove the condition that guaranties stability. Through an out of the box proof, inspired by Papavassilopoulos (1986), we determine that more than half of each firm's rivals delays, guaranty the game's convergence. Additionally, for the CMA rule, the transient behavior has been related to the initial choices of players and bounded, while the game's convergence has been characterized as sublinear. The CMA rule, also known in the literature as fictitious play, under its universal adoption, makes the game evolve and transit from instability to stability. Although it converges asymptotically, independently of the number of players, the evolution of the rule itself on every stage of the game compensates to some extent the critic that the bounded rationality and the myopic best response concepts have received. In a different way, this is also counteracted by models that impose strategic behavior in the selection of the rule (Hommes et al., 2018) and consequently allow alternations based on efficiency terms. The hybrid rule that switches from variable to fixed number of delays, shows that characteristics of both rules can be combined and forms a basis for more sophisticated mixing strategies.

The paper is structured as follows: A general formulation of the discrete time n-player Cournot game for linear demand and private constant marginal production costs is provided in Section 2, along with definitions of the observable game's history. In Section 3, we introduce the adaptive rules from the MA family and we describe the corresponding games. The condition between the number of delays and the number of players that ensures convergence of the SMA game is derived in Section 4, where the stability properties of the rules are, also, studied. For the CMA game, the evolution rate of the distance from the equilibrium is bounded and its sublinear convergence is shown. To allow comparisons of the proposed rules, in the first part of Section 5 simulations of an example market for different number of firms and delays are presented. In its second part, where the increase of the competition is internalised in the market, the main rules and a dynamic version of the hybrid one are compared based on their adaptability. Finally, the paper concludes in Section 6, where a discussion on the results, also, follows.

2 Game Formulation

We consider an industry of a homogeneous commodity, where n firms compete over quantities. In this "a la Cournot" oligopoly context, at every stage $t \in \{1, 2, \dots, T\}$, firm $i \in \{1, 2, \dots, n\} \equiv \mathcal{N}$ selects, simultaneously with its rivals, its output level $x_{i,t} \in \mathcal{X}_i \subseteq \mathbb{R}_+$, produced under a strictly increasing cost function $C_i(x_{i,t})$. Let $\mathbf{x}_t = [x_{1,t}, x_{2,t}, \dots, x_{n,t}]^T$ be the state vector of the market, consisting of all firms' outputs at stage t, and $P(\mathbf{x}_t) : \mathcal{X}^n \subset \mathcal{X}_1 \times \mathcal{X}_2 \times \cdots \times \mathcal{X}_n \to \mathcal{P} \subseteq \mathbb{R}_+$ be the inverse demand function, which as a clearing mechanism determines the price p_t. We assume (a) linear demand and (b) constant marginal production costs, thus the cost function of firm i and the inverse demand function are of the form $C_i(x_{i,t}) = c_i^0 + c_i x_{i,t}$ and $p_t \equiv P(\mathbf{x}_t) = a - b \sum_{i \in \mathcal{N}} x_{i,t}$, respectively. The linear form of the functions and the exogenous parameters $a, b > 0$ are common knowledge, while, in the contrary, cost parameters $c_i^0, c_i > 0$, $i \in \mathcal{N}$ are considered private (Daughety, 2005).

Let $h^t = \{\mathbf{x}_k\}_{k=1}^t$ be the full history of the game, available at stage t, and $h_m^t = \{\mathbf{x}_k\}_{k=\max(1,t-m)}^t$ be a short version of the history, i.e. a subsequence of h^t that consists at most of the last $m+1$ state vectors. Those two sequences coincide for $m = T-1$, where the short history becomes equivalent to the full history of the game. The sets of all possible, full and short, game histories at time t are $\mathcal{H}^t$ and $\mathcal{H}_m^t$, respectively. Firms select their actions by aiming to maximize their single-period profits. In the case where the full history of the game is available, firm i maximizes the strictly concave function $\pi_i(\mathbf{x}_t) = \mathrm{E}\left[P(\mathbf{x}_t) | h_m^{t-1}\right] x_{i,t} - C_i(x_{i,t})$ by choosing $x_{i,t} \in \mathcal{X}_i$. The first order condition leads to firm's i best reply, which provides the optimal output level given its expectations for the rivals outputs. The mappings $\xi_{i,t}^{m}(\cdot) : \mathcal{H}_m^{t-1} \to \mathcal{X}_i$, $\forall i \in \mathcal{N}$ associate an output value at time t with the observed history h_m^{t-1} and constitute the strategies of the players.

The discrete time game, resulting from a certain instance of the expectation operator, corresponds to the system of equations given in (1), where $\mathbf{A}_G$ is the adjacency matrix of the complete directed graph G with n vertices of Fig. 1(a).

$$\mathbf{x}_t = -\frac{1}{2}\mathbf{A}_G \mathrm{E}\left[\mathbf{x}_t | h_m^{t-1}\right] + \mathbf{B} \tag{1}$$

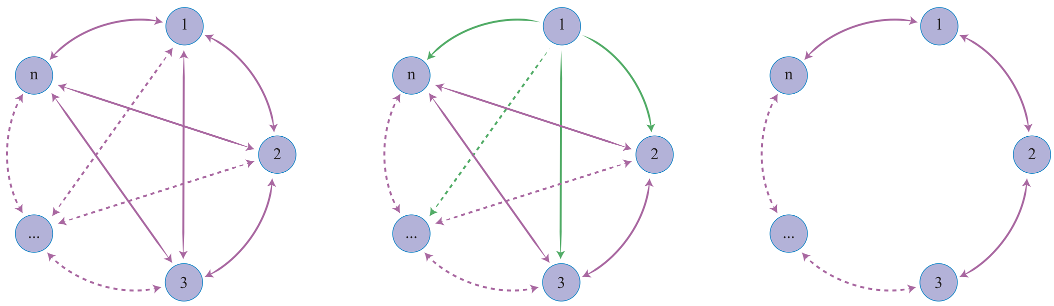
$$\mathbf{A}_G = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 0 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} \frac{a-c_1}{2b} \\ \frac{a-c_2}{2b} \\ \vdots \\ \frac{a-c_n}{2b} \end{bmatrix} \tag{2}$$

Considering the complete directed graph G, the spectral radius of its adjacency matrix $\mathbf{A}_G$ is equal with its dominant eigenvalue $\lambda_{max}^{\mathbf{A}_G}$ and the Perron-Frobenius Theorem implies that any other eigenvalue is strictly smaller, in absolute value (MacCluer, 2000). The dominant eigenvalue of any irreducible non-negative square matrix satisfies the following inequalities (3), where a_{ij} corresponds to the element of its i-th and j-th, row and column, respectively (Rumchev and James, 1990). As G is a complete directed graph, the dominant eigenvalue $\lambda_{max}^{\mathbf{A}_G}$ of its adjacency matrix $\mathbf{A}_G$ is equal to $n-1$, a number which also stands for the degree of the graph. Since a complete graph has the highest degree among all the other simple graphs of the same number of vertices, e.g. Fig. 1(b,c), its spectral radius will also be greater than those of all other simple graphs. Therefore, the corresponding to graph G game, can be considered as the most demanding one in terms of stability, with the results obtained being able to cover other cases as well.

$$\min_i \sum_j a_{ij} \leq \lambda_{max}^{\mathbf{A}_G} \leq \max_i \sum_j a_{ij} \tag{3}$$

3 Adaptive rules

In the case of the estimation being unbiased, e.g. the complete information case, the expectation of firm i for its rival's output can be replaced by the state vector at stage t it self, i.e. $\mathrm{E}\left[\mathbf{x}_t\right] = \mathbf{x}_t$. That leads directly the firms to the equilibrium vector $\mathbf{x}^* = \left(\mathbf{I} + \frac{1}{2}\mathbf{A}_G\right)^{-1}\mathbf{B}$. For any other case, this expectation should be replaced by the estimation of the rivals' outputs, taking under consideration the past states available. An example based on h_1^{t-1} is the so called Cournot adjustment process, an adaptive rule where $\mathrm{E}\left[\mathbf{x}_t | h_1^{t-1}\right]$ is replaced by an estimator $\hat{\mathbf{x}}_t = \mathbf{x}_{t-1}$. This rule works well in the duopoly case, as firms, by adopting this myopic best response strategy, manage to reach the



(a) Complete graph (b) Unbalanced graph (c) Cycle graph

Fig. 1: Directed simple graphs representing different n-player games

equilibrium point, but it fails for $n \geq 3$ in terms of system's convergence[1]. Its consistent adoption throughout the game, has been a controversial assumption mainly because players could have revised their adaptive rule after observing the bias of their estimation. By considering the adoption of the adaptive rule to be a non-strategic decision, it suffices the underlying assumption to be symmetric and to ensure convergence.

Every adaptive rule may arise either as the result of a learning process or by the need for estimating unknown parameters. We focus on rules that belong to the family of moving averages, emanating from players' need for practical estimation of rivals' output. In other words, we consider that players, before selecting their output, want to quantify their expectations using an estimator; by assuming Method of Moments (MM) estimators, we restrict our interest on a class of rules originating from the first raw moment estimators. Due to the nature of the information update mechanism, players perceive the history enrichment process through the arrival of data in an ordered datum stream and they end up observing rivals' output in a time series format. Under this context, the Simple Moving Average, the Cumulative Moving Average and the Cumulative-Simple Moving Average (CSMA) constitute useful tools for tackling such form of data for filtering and estimating purposes, by creating series of averages on different data sets. Thus, in the following subsections, we consider three variations of the discrete time Cournot oligopoly, where the adaptive rule differs in accordance with the aforementioned versions of moving averages.

3.1 Simple Moving Average (SMA) rule

Let us be at stage t, exactly before firms make their decisions and when the full history of game is h^{t-1}. Firms have access only to the truncated version of the history h_m^{t-1}. Assume that firm i estimates the action of firm j using the

[1] Alternations of the adjustment process (continuous or discrete form), which are based on the speed of adaptation concept (Fisher, 1961), are able to result convergence (Okuguchi, 1970; Szidarovszky and Okuguchi, 1988).

unweighted arithmetic mean of her previous m output levels, which are available in the short game history. The adoption of the Simple Moving Average (SMA) as an estimator for all her rivals allows the replacement of her expectation $\mathrm{E}_i\left[x_{j,t}|h_m^{t-1}\right]$ with (4). The way SMA process updates the estimation value is by incorporating, scaled, the difference between the new output and the one removed from memory. This can be seen by the inductive rule in (5), which can't be directly used by players because of the imposed game history structure.

$$\hat{x}_{j,t}^{SMA_m} = \frac{x_{j,t-1} + x_{j,t-2} + \ldots + x_{j,t-m}}{m}, \forall j \in \mathcal{N} \setminus \{i\} \tag{4}$$

$$\hat{\mathbf{x}}_t^{SMA_m} = \hat{\mathbf{x}}_{t-1}^{SMA_m} + \frac{1}{m}\left(\mathbf{x}_{t-1} - \mathbf{x}_{t-m-1}\right) \tag{5}$$

By assuming symmetric behavior for all n firms, the game is described by the backward looking system of m delays in (6).

$$\mathbf{x}_t = -\frac{1}{2}\mathbf{A}_G \frac{1}{m}\left(\mathbf{x}_{t-1} + \mathbf{x}_{t-2} + \ldots + \mathbf{x}_{t-m}\right) + \mathbf{B} \tag{6}$$

3.2 Cumulative Moving Average (CMA) rule

In the case of the Cumulative Moving Average (CMA), firms are assumed to have access to the full history of the game. Compared with the SMA case, the estimation of firm i is again the unweighted arithmetic mean of previous output levels, but this time the cardinality of those data sets increases by one at every stage. As the time passes, history h^{t-1} becomes richer and new information are exploited by players to update their estimation. The expectation $\mathrm{E}_i\left[x_{j,t}|h^{t-1}\right]$ is replaced by a constantly increasing number of terms estimator (7), while the inductive rule (8) utilizes the scaled bias to revise the estimation.

$$\hat{x}_{j,t}^{CMA} = \frac{x_{j,t-1} + x_{j,t-2} + \ldots + x_{j,0}}{t}, \forall j \in \mathcal{N} \setminus \{i\} \tag{7}$$

$$\hat{\mathbf{x}}_t^{CMA} = \frac{t-1}{t}\hat{\mathbf{x}}_{t-1}^{CMA} + \frac{1}{t}\mathbf{x}_{t-1} = \hat{\mathbf{x}}_{t-1}^{CMA} + \frac{1}{t}\mathbf{bias}_{t-1}^{CMA} \tag{8}$$

The best replies of all n firms, with the CMA estimators replacing the expectations, form the game described by the structural evolving backward looking system of (9), which is the Cournot game under fictitious play.

$$\mathbf{x}_t = -\frac{1}{2}\mathbf{A}_G \frac{1}{t}\left(\mathbf{x}_{t-1} + \mathbf{x}_{t-2} + \ldots + \mathbf{x}_0\right) + \mathbf{B} \tag{9}$$

3.3 Hybrid Cumulative-Simple Moving Average (CSMA) rule

The combination of the two main rules give rise to a hybrid rule, which shares properties of both. The CSMA rule, which we deploy here, refers to just one alternation between the two. Due to lack of initial data, players start by adopting the CMA rule and then turn to the SMA of respective number of delays. The number of delays d_t, used in forming the expectation of rivals' quantity, is provided for this static version by (10). By assuming that all firms adopt the CSMA rule in a symmetric way, the dynamics of the game are described by (11).

$$d_t = \begin{cases} t \; for \; t < m \\ m \; for \; t \geq m \end{cases} \tag{10}$$

$$\mathbf{x}_t = -\frac{1}{2}\mathbf{A}_G \frac{1}{d_t} \left(\mathbf{x}_{t-1} + \mathbf{x}_{t-2} + \ldots + \mathbf{x}_{t-d_t}\right) + \mathbf{B} \tag{11}$$

4 Convergence of adaptive rules

For the stability study of the games of fixed number of delays, a general framework is initially presented to introduce the topic and initialize the analysis. Consider that the m past outputs observed are weighted and used by firms to quantify their expectations and that all players adopt identical constant weighting functions, i.e. $w_\ell^i = w_\ell$, $\forall i \in \{1, 2, \ldots, n\}$. The weights are used to form an inner linear combination of the m past outputs to estimate the expectation about the current output. As $w_\ell \in (0, 1)$ and $\sum_{\ell=1}^{m} w_\ell = 1$, the weights can be also considered to be the probabilities that players assign to the recurrence of a past state. For arbitrary weights, the game acquires the form of the backward-looking system of m delays of (12), which for $w_\ell = \bar{w}$, $\forall \ell \in \{1, 2, \ldots, m\}$ turns to be the one presented in subsection 3.1 as $\bar{w} = {}^1/_m$.

$$\mathbf{x}_t = -\frac{1}{2}\mathbf{A}_G \left(w_1 \mathbf{x}_{t-1} + w_2 \mathbf{x}_{t-2} + \ldots + w_m \mathbf{x}_{t-m}\right) + \mathbf{B} \tag{12}$$

Given the homogenous system (Mickens, 1991) and its Z-transform (Jury, 1973), for the discrete time system of (12) to be stable[2], all the roots for Z of (13) should be strictly inside the unit circle.

$$\mathbf{X}(Z) = -\frac{1}{2}\mathbf{A}_G \left(w_1 \frac{\mathbf{X}(Z)}{Z} + w_2 \frac{\mathbf{X}(Z)}{Z^2} + \ldots + w_m \frac{\mathbf{X}(Z)}{Z^m}\right) \tag{13}$$

$$\left(\frac{1}{w_1 Z^{-1} + w_2 Z^{-2} + \ldots + w_m Z^{-m}} \mathbf{I} - \mathbf{A}\right) \mathbf{X}(Z) = \mathbf{0} \tag{14}$$

[2] System's stability refers to global, uniform, asymptotic stability of the equilibrium point $\mathbf{x}^*$, i.e. for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $\|\mathbf{x}_{t_0} - \mathbf{x}^*\| < \delta$ implies $\|\mathbf{x}_t - \mathbf{x}^*\| < \varepsilon$, $\forall t \geq t_0$ and $\lim\limits_{t\to\infty} \mathbf{x}_t = \mathbf{x}^*$, independently of t_0 and $\mathbf{x}_{t_0}$. (Khalil, 2002)

The eigenvalues λ_i, $i \in \mathcal{N}$ of the matrix of interest $\mathbf{A} = -{}^1\!/\!{}_2\mathbf{A}_G$, which can be identified directly by the characteristic polynomial form of (14), define n polynomials of degree m with respect to Z. The zeros of the polynomial of (15) should lie strictly inside the unite circle. By considering $\tilde{\lambda}_i = \lambda_i^{-1}$ to be the reciprocal of eigenvalue λ_i and $\tilde{Z} = Z^{-1} = \frac{1}{r}\mathrm{e}^{-j\varphi}$ (j stands for the imaginary unit) the reciprocal of Z variable, in (16) we restate those polynomials in reciprocal terms. For stability, in their latter expression, we require all the nm zeros to lie strictly outside the unite circle, i.e. $\|\tilde{Z}_R\| > 1$, $\forall R \in \{1, 2, \ldots, nm\}$ where $\tilde{Z}_R$ are the roots of (16). The Theorem that follows relates the polynomial's coefficients with the position of the roots and can be found useful in providing sufficient conditions for stability (Kakeya, 1912).

$$\frac{1}{w_1 Z^{-1} + w_2 Z^{-2} + \ldots + w_m Z^{-m}} = \lambda_i \tag{15}$$

$$w_1 \tilde{Z}^1 + w_2 \tilde{Z}^2 + \ldots + w_m \tilde{Z}^m = \tilde{\lambda}_i \tag{16}$$

Theorem 1 (Kakeya) *If $f(\tilde{Z}) = w_0 + w_1\tilde{Z} + w_2\tilde{Z}^2 + \ldots + w_m\tilde{Z}^m$ is a polynomial of degree m with real and positive coefficients then all the zeros of f lie in the annulus $\underline{R} \le \|\tilde{Z}_R\| \le \overline{R}$ where $\underline{R} = \min\limits_{1\le\ell\le m} \frac{w_{\ell-1}}{w_\ell}$ and $\overline{R} = \max\limits_{1\le\ell\le m} \frac{w_{\ell-1}}{w_\ell}$.*

An alternative approach may begin without considering the topology as given. The question that follows, is where the eigenvalues λ_i, $i \in \mathcal{N}$ of matrix $\mathbf{A}$ should be located so that the system is stable. To answer that, let us define as $\mathbb{S}_m$ and $\mathbb{S}_m^{\mathsf{B}}$ the subsets of $\mathbb{C}$, which contain the eigenvalues λ_i of $\mathbf{A}$ in the asymptotic and marginal stability cases[3]. Description of $\mathbb{S}_m$ is given in (17) and of $\mathbb{S}_m^{\mathsf{B}}$ in (18).

$$\mathbb{S}_m = \left\{ \frac{1}{w_1 Z^{-1} + w_2 Z^{-2} + \ldots + w_m Z^{-m}} : Z = r\mathrm{e}^{j\varphi}, r < 1, \varphi \in [0, 2\pi) \right\} \tag{17}$$

$$\mathbb{S}_m^{\mathsf{B}} = \left\{ \frac{1}{w_1 Z^{-1} + w_2 Z^{-2} + \ldots + w_m Z^{-m}} : Z = r\mathrm{e}^{j\varphi}, r = 1, \varphi \in [0, 2\pi) \right\} \tag{18}$$

Proposition 1 *The homogenous discrete time linear system of (13) is stable if all the eigenvalues λ_i of $\mathbf{A}$ belong to $\mathbb{S}_m$ and not unstable if they all belong in $\mathbb{S}_m \cup \mathbb{S}_m^{\mathsf{B}}$.*

[3] Since matrix $\mathbf{A}$ is a scaled instance of $\mathbf{A_G}$, one of its eigenvalues is dominant and all the other are strictly smaller than that, in absolute value (Section 2). With marginal stability, we mean the case where the system is neither asymptotically stable nor unstable, i.e. the case where the greatest magnitude of any of the eigenvalues is one and the multiplicity of this critical eigenvalue is one.

By considering $\tilde{\lambda}_i = \lambda_i^{-1}$ to be the reciprocal of eigenvalue λ_i and $\tilde{Z} = Z^{-1} = \frac{1}{r}\mathrm{e}^{-j\varphi}$ the reciprocal of Z variable, we define the sets $\tilde{\mathbb{S}}_m$ and $\tilde{\mathbb{S}}_m^{\mathsf{B}}$ with respect to those reciprocal quantities.

$$\tilde{\mathbb{S}}_m = \left\{ w_1\tilde{Z}^1 + w_2\tilde{Z}^2 + \dots w_m\tilde{Z}^m : \tilde{Z} = \frac{1}{r}\mathrm{e}^{-j\varphi}, r > 1, \varphi \in [0, 2\pi) \right\} \tag{19}$$

$$\tilde{\mathbb{S}}_m^{\mathsf{B}} = \left\{ w_1\tilde{Z}^1 + w_2\tilde{Z}^2 + \dots w_m\tilde{Z}^m : \tilde{Z} = \frac{1}{r}\mathrm{e}^{-j\varphi}, r = 1, \varphi \in [0, 2\pi) \right\} \tag{20}$$

Proposition 2 *The homogenous discrete time linear system of (13) is stable if none of the $\tilde{\lambda}_i$ belong to $\tilde{\mathbb{S}}_m \cup \tilde{\mathbb{S}}_m^{\mathsf{B}}$ and not unstable if none of them belong in $\tilde{\mathbb{S}}_m$.*

Lemma 1 *The set $\tilde{\mathbb{S}}_m^{\mathsf{B}}$ bounds the set $\tilde{\mathbb{S}}_m$ since for any $C_1 \in \tilde{\mathbb{S}}_m$ there exists a $C_2 \in \tilde{\mathbb{S}}_m^{\mathsf{B}}$ such that $\angle C_2 = \angle C_1$ and $\|C_2\| > \|C_1\|$.*

Proof Indeed, let $C_1 = \sum_{\ell=1}^{m} w_\ell (\frac{1}{r_1})^\ell \mathrm{e}^{-j\ell\varphi_1}$, where $r_1 > 1$ and $\varphi_1 \in [0, 2\pi)$. Consider the following limiting case:

$$\lim_{r\to1^+} \sum_{\ell=1}^{m} w_\ell (\frac{1}{r})^\ell \mathrm{e}^{-j\ell\varphi_1} = \sum_{\ell=1}^{m} w_\ell \mathrm{e}^{-j\ell\varphi_1} = C_2$$

We observe that $C_2 \in \tilde{\mathbb{S}}_m^{\mathsf{B}}, C_2 \notin \tilde{\mathbb{S}}_m$ and we compare one by one the corresponding m terms of the summation. Since the ratio $C_{1,\ell}/C_{2,\ell}$ is greater than one for all ℓ, we conclude that $\|C_2\| > \|C_1\|$ for any common $\varphi \in [0, 2\pi)$, i.e. $\angle C_2 = \angle C_1 = \varphi$, and thus $\partial\tilde{\mathbb{S}}_m = \tilde{\mathbb{S}}_m^{\mathsf{B}}$.

$$\frac{C_{1,\ell}}{C_{2,\ell}} = \frac{w_\ell \mathrm{e}^{-j\ell\varphi_1}}{w_\ell (\frac{1}{r})^\ell \mathrm{e}^{-j\ell\varphi_1}} = r > 1, \ \forall \ell \in \{1, 2, \dots, m\}$$

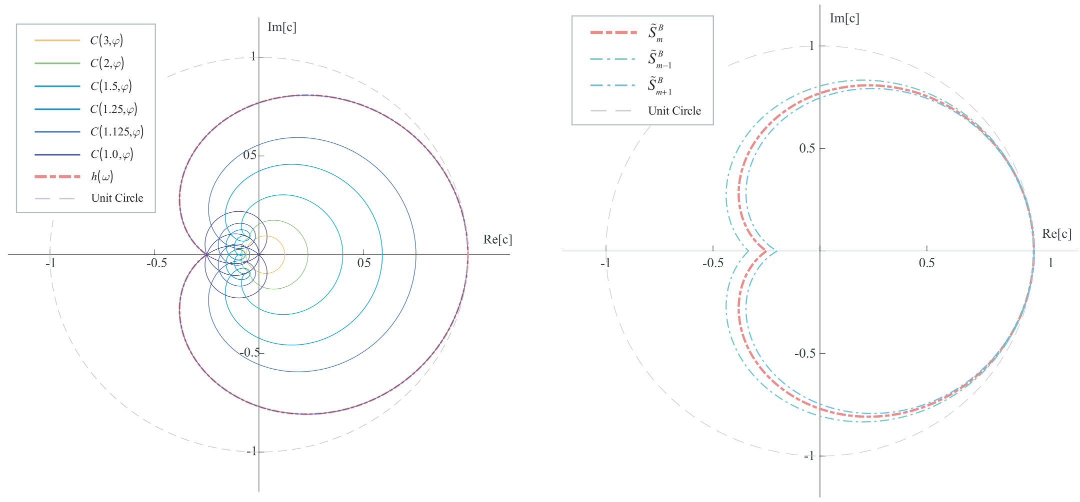
□

Based on Lemma 1, the set $\tilde{\mathbb{S}}_m^{\mathsf{B}}$ bounds the set $\tilde{\mathbb{S}}_m$ and from this point onward we will refer to it as the boundary set.

4.1 Simple Moving Average (SMA)

The weights for the SMA case, are $w_\ell = \bar{w} = 1/m$, $\forall \ell \in \{1, 2, \dots, m\}$. The analysis that follows extends the general framework and concludes by the end of the subsection, with Theorem 2, where the stability threshold of the memory window as a function of the number of players is derived.

We work on the description of the boundary set to obtain an equivalent description that will facilitate our study. Given that the complex number $w_1\tilde{Z}^1 + w_2\tilde{Z}^2 + \dots w_m\tilde{Z}^m$ of (20), for $\bar{w} = 1/m$, can be expressed as $1/m \sum_{\ell=1}^{m} \tilde{Z}^\ell$, we use Lemma 2 to obtain expression (21), which constitutes an intermediate result .



(a) The curve $h(\omega)$, along with instances of $C(r,\varphi)$ for $\|\tilde{Z}\| = {}^1\!/r \leq 1$.

(b) The boundary set $\tilde{\mathbb{S}}^{\mathsf{B}}_m$, which shrinks as the number of delays increases ($m = 4$).

Fig. 2: The curve $C(r,\varphi)$ for decreasing values of $\|\tilde{Z}\|$ and the boundary set $\tilde{\mathbb{S}}^{\mathsf{B}}_m$, which shrinks as the number of delays increases.

Lemma 2 *If* $\tilde{Z} = \mathrm{e}^{-j\varphi}$, $\varphi \in [0, 2\pi)$ *then the complex number* $\sum_{\ell=1}^{m} \tilde{Z}^{\ell} = \rho(\omega)\mathrm{e}^{-j\omega}$ *where* $\rho(\omega) = \frac{\sin\left(\frac{m}{m+1}\omega\right)}{\sin\left(\frac{1}{m+1}\omega\right)}$ *and* $\omega \in [0, (m+1)\pi)$.

Proof Let $\tilde{Z} \in \mathbb{C}$ and $C(r,\varphi) : \mathbb{C} \to \mathbb{C}$ where $C(r,\varphi) = \sum_{\ell=1}^{m} \tilde{Z}^{\ell}$, $\tilde{Z} = \frac{1}{r}\mathrm{e}^{-j\varphi}$ and m is a positive integer. For any $\tilde{Z} \neq 1$, the complex number $C(r,\varphi)$, instances of which are given in Fig. 2(a), can be expressed as a geometric sum and this is defined as an alternative expression $C_a(r,\varphi)$.

$$C(r,\varphi) = \tilde{Z}^1 + \tilde{Z}^2 + \ldots + \tilde{Z}^m = \frac{\left(1 - \tilde{Z}^m\right)\tilde{Z}}{1 - \tilde{Z}} \triangleq C_a(r,\varphi)$$

For $\tilde{Z} = \mathrm{e}^{-j\varphi}$, $\varphi \neq 2k\pi$, $k \in \mathbb{Z}$ we obtain for $C_a(1,\varphi)$ a simplified expression.

$$C_a(1,\varphi) = \frac{\left(1 - \mathrm{e}^{-jm\varphi}\right)\mathrm{e}^{-j\varphi}}{1 - \mathrm{e}^{-j\varphi}} = \frac{\mathrm{e}^{+j\frac{m\varphi}{2}} - \mathrm{e}^{-j\frac{m\varphi}{2}}}{\mathrm{e}^{+j\frac{\varphi}{2}} - \mathrm{e}^{-j\frac{\varphi}{2}}} \frac{\mathrm{e}^{-j\frac{m\varphi}{2}}}{\mathrm{e}^{-j\frac{\varphi}{2}}}\mathrm{e}^{-j\varphi}$$

$$C_a(1,\varphi) = \frac{\sin\left(\frac{m\varphi}{2}\right)}{\sin\left(\frac{\varphi}{2}\right)}\mathrm{e}^{-j\frac{m+1}{2}\varphi}$$

For $\tilde{Z} = \mathrm{e}^{-j\varphi}$, $\varphi = 2k\pi$, $k \in \mathbb{Z}$, the complex number $C(1, 2k\pi) = \sum_{\ell=1}^{m} 1 = m$. For any $k \in \mathbb{Z}$, the following limit of $C_a(1,\varphi)$ exists, as the side limits are equal, and its value coincides with that of $C(1, 2k\pi)$.

$$\lim_{\varphi \to 2k\pi} C_a(1,\varphi) = \lim_{\varphi \to 2k\pi} \frac{\sin\left(\frac{m\varphi}{2}\right)}{\sin\left(\frac{\varphi}{2}\right)}\mathrm{e}^{-j\frac{m+1}{2}\varphi} = m$$

Thus, $C(1,\varphi) \equiv C_a(1,\varphi)\ \forall\varphi \in \mathbb{R}$. By setting $\varphi = \frac{2\omega}{m+1}$ in the equivalent alternative expression we obtain:

$$C(1,\frac{2\omega}{m+1}) = \frac{\sin\left(\frac{m}{m+1}\omega\right)}{\sin\left(\frac{1}{m+1}\omega\right)}\mathrm{e}^{-j\omega},\ \forall\omega\in\mathbb{R}$$

□

The set $\tilde{\mathbb{S}}_m^{\mathsf{B}}$ contains the complex numbers $\rho(\omega)\mathrm{e}^{-j\omega}$ for $\omega \in \left[0, \frac{(m+1)\pi}{2}\right)$ along with their conjugates for $\omega \in \left[\frac{(m+1)\pi}{2}, (m+1)\pi\right)$.

$$\tilde{\mathbb{S}}_m^{\mathsf{B}} = \left\{ \frac{1}{m}\frac{\sin\left(\frac{m}{m+1}\omega\right)}{\sin\left(\frac{1}{m+1}\omega\right)}\mathrm{e}^{-j\omega} : \omega \in [0,(m+1)\pi) \right\} \tag{21}$$

Let us consider the curve $h_m(\omega) = \frac{1}{m}\frac{\sin\left(\frac{m}{m+1}\omega\right)}{\sin\left(\frac{1}{m+1}\omega\right)}\mathrm{e}^{-j\omega}$, $\omega \in \left[0, \frac{(m+1)\pi}{2}\right)$ and its symmetric about x-axis, i.e. for $\omega \in \left[\frac{(m+1)\pi}{2}, (m+1)\pi\right)$.

Lemma 3 *Let* $h_m(\omega) = \frac{1}{m}\frac{\sin\left(\frac{m}{m+1}\omega\right)}{\sin\left(\frac{1}{m+1}\omega\right)}\mathrm{e}^{-j\omega}$ *where* $m \in \mathbb{N}^+$. *Then for* $\omega \in [0,\pi]$ *it holds that* $\|h_m(\omega)\| \geq \|h_m(k\pi+\omega)\|$ *for* $k \in \mathbb{N}^+$, $k < \frac{m+1}{2}$.

Proof Suppose $\omega \in (0,\pi]$ and consider the following ratio for $k \in \mathbb{N}^+, k < \frac{m+1}{2}$.

$$\frac{\|h_m(k\pi+\omega)\|}{\|h_m(\omega)\|} = \left\| \frac{\sin\left(\frac{m}{m+1}k\pi\right)\cot\left(\frac{m}{m+1}\omega\right)+\cos\left(\frac{m}{m+1}k\pi\right)}{\sin\left(\frac{1}{m+1}k\pi\right)\cot\left(\frac{1}{m+1}\omega\right)+\cos\left(\frac{1}{m+1}k\pi\right)} \right\|$$

$$\frac{\|h_m(k\pi+\omega)\|}{\|h_m(\omega)\|} = \left\| \frac{\sin\left(k\pi-\frac{1}{m+1}k\pi\right)\cot\left(\frac{m}{m+1}\omega\right)+\cos\left(k\pi-\frac{1}{m+1}k\pi\right)}{\sin\left(\frac{1}{m+1}k\pi\right)\cot\left(\frac{1}{m+1}\omega\right)+\cos\left(\frac{1}{m+1}k\pi\right)} \right\|$$

$$\frac{\|h_m(k\pi+\omega)\|}{\|h_m(\omega)\|} = \frac{\left\|\sin\left(\frac{1}{m+1}k\pi\right)\cot\left(\omega-\frac{1}{m+1}\omega\right)-\cos\left(\frac{1}{m+1}k\pi\right)\right\|}{\left\|\sin\left(\frac{1}{m+1}k\pi\right)\cot\left(\frac{1}{m+1}\omega\right)+\cos\left(\frac{1}{m+1}k\pi\right)\right\|}$$

- For $\omega \in (0,\pi)$, since $\|\cot(\theta)\|$ is symmetric around $\theta = \pi/2$, the independent of ω terms are positive and $\left\|\cot\left(\omega - \frac{1}{m+1}\omega\right)\right\| < \left\|\cot\left(\frac{1}{m+1}\omega\right)\right\|$. Thus, the ratio is less than the unit and therefore $\|h_m(\omega)\| > \|h_m(k\pi+\omega)\|$.
- For $\omega = \pi$, it holds that $\cot\left(\frac{m}{m+1}\pi\right) = \cot\left(\frac{1}{m+1}\pi\right)$ and therefore $\|h_m(\pi)\| = \|h_m((k+1)\pi)\| = \frac{1}{m}$ since $\sin\left(\frac{m}{m+1}\pi\right) = \sin\left(\frac{1}{m+1}\pi\right)$.
- For $\omega = 0$, $\|h_m(0)\| = 1$ and $\|h_m(\pi)\| = \frac{1}{m}$ therefore $\|h_m(0)\| > \|h_m(k\pi)\|$.

□

Combining the previous results, we reach the final description of the set $\tilde{\mathbb{S}}_m \cup \tilde{\mathbb{S}}_m^{\mathsf{B}}$ in (22). For the game to converge, the reciprocal eigenvalues of $\mathbf{A}$ should not belong to this set.

$$\tilde{\mathbb{S}}_m \cup \tilde{\mathbb{S}}_m^{\mathsf{B}} = \left\{ \|c\|\, \mathrm{e}^{j\omega} \in \mathbb{C} : \ \|c\| \leq \frac{1}{m} \frac{\sin\left(\frac{m}{m+1}\omega\right)}{\sin\left(\frac{1}{m+1}\omega\right)},\ \omega \in (-\pi, \pi] \right\} \tag{22}$$

Lemma 4 *The sequence of sets* $\tilde{\mathbb{S}}_m \cup \tilde{\mathbb{S}}_m^{\mathsf{B}}$, $m \in \mathbb{N}^+$ *is strictly decreasing, i.e.* $\tilde{\mathbb{S}}_m \cup \tilde{\mathbb{S}}_m^{\mathsf{B}} \supset \tilde{\mathbb{S}}_{m+1} \cup \tilde{\mathbb{S}}_{m+1}^{\mathsf{B}}$.

Proof The curve $h_m(\omega), \omega \in (-\pi, \pi]$ is the boundary of $\tilde{\mathbb{S}}_m \cup \tilde{\mathbb{S}}_m^{\mathsf{B}}$. The derivative of $\|h_m(\omega)\|$ with respect to m for $\omega \in (-\pi, \pi)$ is negative as a summation of negative terms.

$$\begin{aligned}
&\frac{d}{dm}\left(\frac{1}{m}\frac{\sin\left(\frac{m}{m+1}\omega\right)}{\sin\left(\frac{1}{m+1}\omega\right)}\right) = \\
&= \frac{\cos\left(\frac{m}{m+1}\omega\right) m \sin\left(\frac{1}{m+1}\omega\right)\frac{1}{(m+1)^2}\omega - \sin\left(\frac{m}{m+1}\omega\right)\left(\sin\left(\frac{1}{m+1}\omega\right) + m\cos\left(\frac{1}{m+1}\omega\right)\frac{1}{(m+1)^2}\omega\right)}{\left(m\sin\left(\frac{1}{m+1}\omega\right)\right)^2} = \\
&= -\frac{\sin\left(\frac{m}{m+1}\omega\right)}{m^2\sin\left(\frac{1}{m+1}\omega\right)} + \frac{m}{(m+1)^2}\omega\frac{\cos\left(\frac{m}{m+1}\omega\right)\sin\left(\frac{1}{m+1}\omega\right) - \cos\left(\frac{1}{m+1}\omega\right)\sin\left(\frac{m}{m+1}\omega\right)}{\left(m\sin\left(\frac{1}{m+1}\omega\right)\right)^2} = \\
&= -\frac{\sin\left(\frac{m}{m+1}\omega\right)}{m^2\sin\left(\frac{1}{m+1}\omega\right)} + \frac{m}{(m+1)^2}\omega\frac{\sin\left(\frac{1}{m+1}\omega - \frac{m}{m+1}\omega\right)}{\left(m\sin\left(\frac{1}{m+1}\omega\right)\right)^2} = \\
&= -\frac{\sin\left(\frac{m}{m+1}\omega\right)}{m^2\sin\left(\frac{1}{m+1}\omega\right)} - \frac{m}{(m+1)^2}\omega\frac{\sin\left(\frac{m-1}{m+1}\omega\right)}{\left(m\sin\left(\frac{1}{m+1}\omega\right)\right)^2}
\end{aligned}$$

Therefore, the curve $h_m(\omega)$, $\omega \in (-\pi, \pi]$ shrinks as m increases. This can also be seen in Fig. 2(b) and consequently $\tilde{\mathbb{S}}_{m+1} \cup \tilde{\mathbb{S}}_{m+1}^{\mathsf{B}}$ is a proper subset of $\tilde{\mathbb{S}}_m \cup \tilde{\mathbb{S}}_m^{\mathsf{B}}$ for any $m \in \mathbb{N}^+$. □

The set $\tilde{\mathbb{S}}_m \cup \tilde{\mathbb{S}}_m^{\mathsf{B}}$ where the reciprocal eigenvalues $\tilde{\lambda}_i$ of matrix $\mathbf{A}$ should not belong is depicted in Fig. 3. The set that contains all the inverted elements of $\tilde{\mathbb{S}}_m \cup \tilde{\mathbb{S}}_m^{\mathsf{B}}$ is the set where the eigenvalues λ_i of matrix $\mathbf{A}$ should not belong. This is $\mathbb{C} \setminus \mathbb{S}_m$ and it is described by (23).

$$\mathbb{C} \setminus \mathbb{S}_m = \left\{ \|c\|\, e^{j\omega} \in \mathbb{C} : \ \|c\| \geq m \frac{\sin\left(\frac{1}{m+1}\omega\right)}{\sin\left(\frac{m}{m+1}\omega\right)},\ \omega \in (-\pi, \pi] \right\} \tag{23}$$

Of particular interest is the set $\mathbb{S}_m$, which contains all possible eigenvalues that ensure the stability of the system and the convergence of the SMA game to the equilibrium.

$$\mathbb{S}_m = \left\{ \|c\|\, \mathrm{e}^{j\omega} \in \mathbb{C} : \ \|c\| < m \frac{\sin\left(\frac{1}{m+1}\omega\right)}{\sin\left(\frac{m}{m+1}\omega\right)},\ \omega \in (-\pi, \pi] \right\} \tag{24}$$

The final authenticated version is available online at: http://dx.doi.org/10.1007/s12351-019-00522-z

Lemma 5 *For $\omega \in (-\pi, \pi]$, $Re\left(1/h_m(\omega)\right)$ has its maximum at $\omega = 0$ and its minimum at $\omega = \pi$, so $-m \le Re\left(1/h_m(\omega)\right) \le 1$.*

Proof Let $f(\omega) = Re\left(1/h_m(\omega)\right)$, $\omega \in (-\pi, \pi]$. For $\omega \in (-\pi, \pi)$, f is differentiable and has an interior critical point at $\omega = 0$.

$$\frac{d}{d\omega}\left(f(\omega)\right) = \frac{d}{d\omega}\left(Re\left(\frac{1}{h_m(\omega)}\right)\right) = \frac{d}{d\omega}\left(m\frac{\sin\left(\frac{1}{m+1}\omega\right)}{\sin\left(\frac{m}{m+1}\omega\right)}\cos(\omega)\right) = \ldots =$$
$$= m\left(\frac{\frac{1}{m+1}\sin\left(\frac{1}{m+1}\omega+\frac{m}{m+1}\omega\right)\cos(\omega)-\cos\left(\omega-\frac{m}{m+1}\omega\right)\sin\left(\frac{1}{m+1}\omega\right)}{\sin^2\left(\frac{m}{m+1}\omega\right)}\right) =$$
$$= \frac{m}{2}\left(\frac{\frac{1}{m+1}\sin(2\omega)-\sin\left(\frac{1}{m+1}2\omega\right)}{\sin^2\left(\frac{m}{m+1}\omega\right)}\right) = \frac{m}{2}\left(\frac{E(\omega)}{\sin^2\left(\frac{m}{m+1}\omega\right)}\right)$$

For $\omega \in (-\pi, 0)$, f is increasing since $E(\omega) > 0$ and for $\omega \in (-\pi, 0)$, f is decreasing since $E(\omega) < 0$.Thus, at $\omega = 0$ f obtains its global maximum, which is $f(0) = 1$.At the end point of the domain ($\omega = \pi$), the global minimum of f is obtained. This is $f(\pi) = -m$. Therefore, $-m \le Re\left(1/h_m(\omega)\right) \le 1$. □

Theorem 2 *The homogenous discrete time linear system of (13) for the SMA case where $w_\ell = \bar{w} = 1/m$, $\forall \ell \in \{1, 2, \ldots, m\}$ is stable if $m > \frac{n-1}{2}$, i.e. the number of delays is greater than half of each firm's rivals..*

Proof Based on Proposition 1, the eigenvalues of matrix $\mathbf{A}$ ($-\frac{n-1}{2}$ and $\frac{1}{2}$) should belong to $\mathbb{S}_m$ for the corresponding system to be stable and for the SMA game to converge to its equilibrium. Consider the description for $\mathbb{S}_m$ obtained

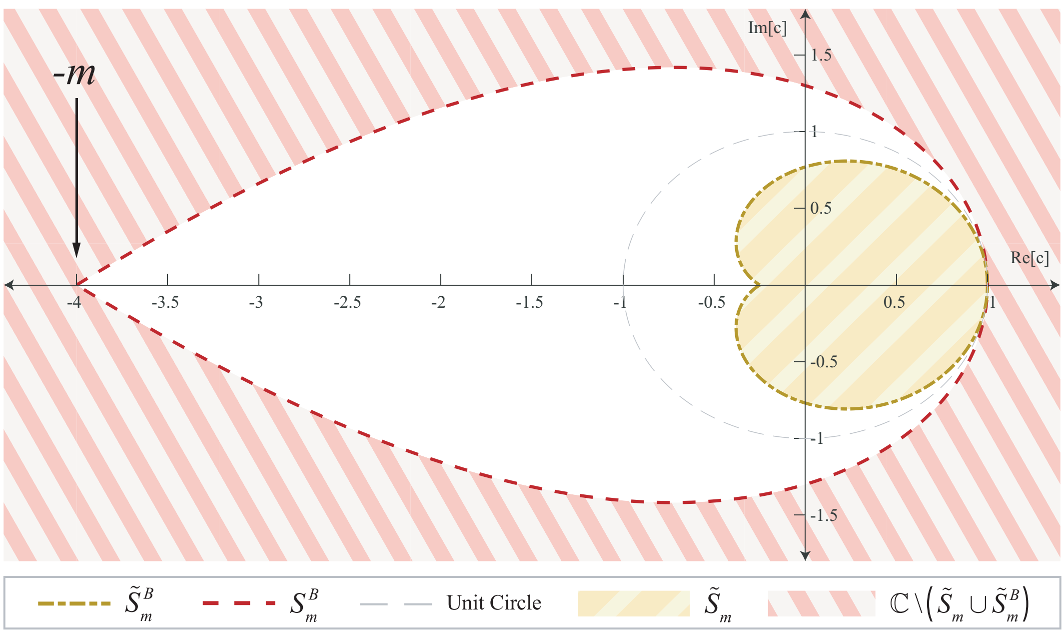


Fig. 3: Sets $\tilde{\mathbb{S}}_m \cup \tilde{\mathbb{S}}^{\mathsf{B}}_m$ and $\mathbb{C}\backslash\mathbb{S}_m$ where $\tilde{\lambda}_i$ and λ_i should not belong respectively for stability. Example presented for $m = 4$.

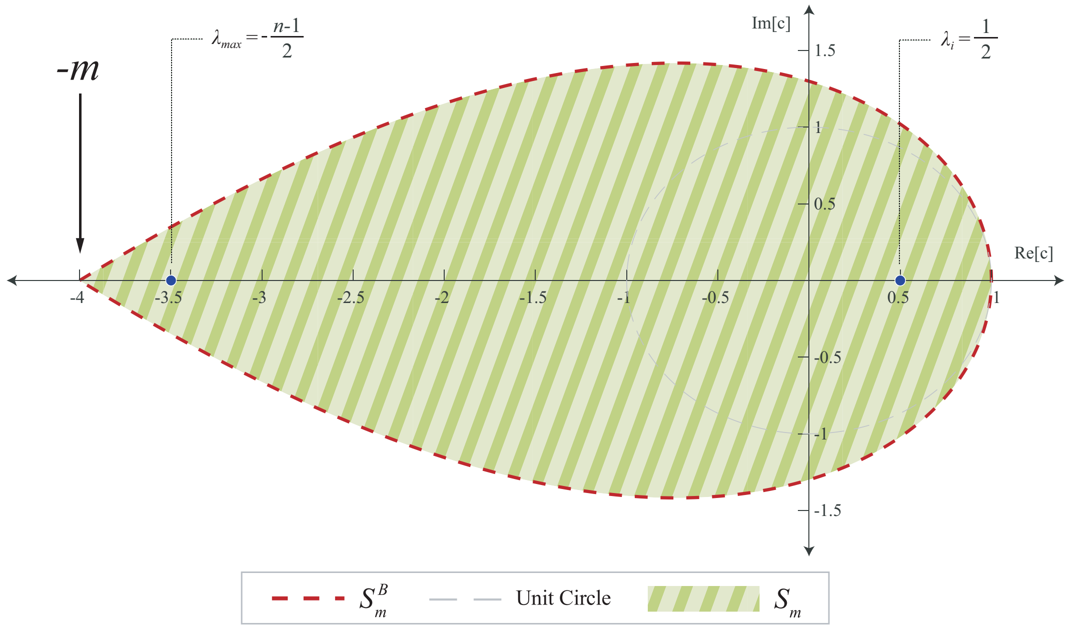


Fig. 4: The set $\mathbb{S}_m$ where all λ_i should belong for the system to be stable. The example presented here is for $m = 4$ delays and $n = 8$ players.

in (24). From the fact that for any $c \in \mathbb{S}_m$ it holds that $-m < Re(c) < 1$ (Lemma 5) and given that the sequence of sets $\mathbb{S}_m$, $m \in \mathbb{N}^+$ is strictly increasing (Lemma 4), all the eigenvalues of $\mathbf{A}$ belong to $\mathbb{S}_m$ if $m > \frac{n-1}{2}$. $\square$

Fig. 4 shows the set $\mathbb{S}_m$, along with the eigenvalues of matrix $\mathbf{A}$ for a case where convergence is ensured. Theorem 2 relates the number of players of the game with the number of delays required for convergence. Therefore, every game of the form (1), where n players adopt SMA adaptive rules of m delays (more than half the rivals), converges to the equilibrium. This result is consistent with the sufficient condition that can be obtained using Theorem 1. The eigenvalue of interest in terms of system's stability is the dominant eigenvalue $\lambda^{\mathbf{A}}_{max} = -\frac{n-1}{2}$. By considering the ratios of successive coefficients of (16) it can be observed that $\frac{w_{\ell-1}}{w_\ell} = 1$ for $2 \leq \ell \leq m$ and $\frac{w_0}{w_1} = \frac{2m}{n-1}$. By a straight forward application of Kakeya's Theorem, a sufficient condition for the roots of the reciprocal polynomial to lay outside of the unit circle, i.e. $\|\tilde{Z}_r\| > 1$, $\forall i \in \{1, 2, \ldots, nm\}$, is $m > \frac{n-1}{2}$.

4.2 Cumulative Moving Average (CMA)

In this case, the weights of the delays change uniformly at every stage as new history observations are introduced in the adaptation process. The weights depend on the stage of the game and consecuently on the number of instances of the outcomes available in the full history of the game, i.e. $w_{t,\ell} = \bar{w}_t = {}^1\!/_t$, $\forall \ell \in \{1, 2, \ldots, t\}$. The CMA game of (9) is a Discrete Linear Time-Varying

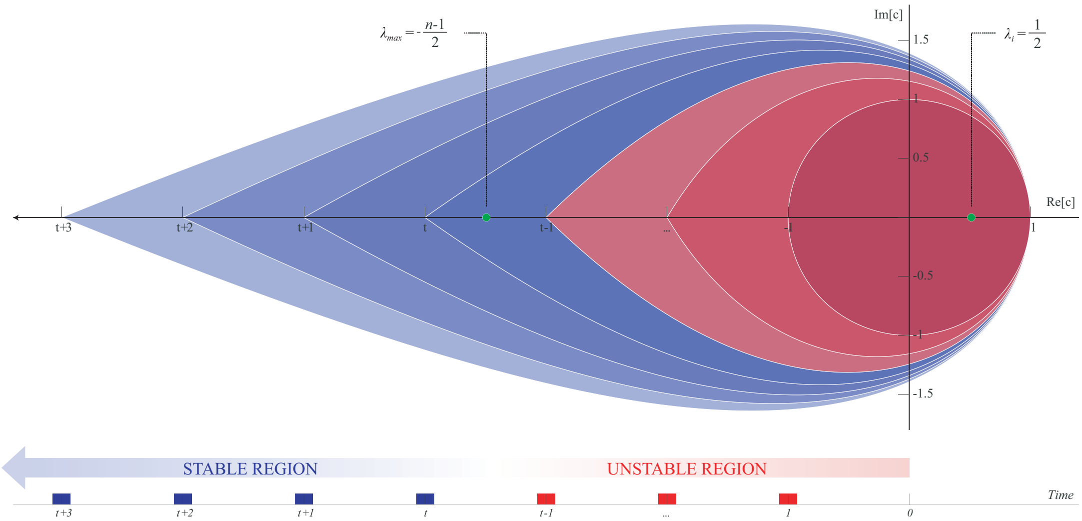


Fig. 5: Relative position of matrix $\mathbf{A}$ eigenvalues λ_i and of the strictly increasing sequence of sets $\mathbb{S}_t$, $t \in \mathbb{N}^+$. After a sufficient large step the game transits to the stable region and after that point convergence to the equilibrium is ensured.

(DLTV) system and can be expressed also in the form of (25). Its Time-Varying property can be seen clearly by observing matrix $\mathbf{A}_n(t-1)$, which has eigenvalues $\lambda_n(t-1) = \frac{2t-1-n}{2t}$ and $\lambda_c(t-1) = \frac{2t-1}{2t}$ of multiplicity 1 and $n-1$, respectively. Its only stationary point is the Cournot-Nash equilibrium $\mathbf{x}^* = \left(\mathbf{I} + \frac{1}{2}\mathbf{A}_G\right)^{-1}\mathbf{B}$.

$$\mathbf{x}_t = \left(\mathbf{I} - \left(\frac{1}{2}\mathbf{A}_G + \mathbf{I}\right)\frac{1}{t}\right)\mathbf{x}_{t-1} + \frac{1}{t}\mathbf{B} \tag{25}$$

$$\mathbf{A}_n(t-1) = \mathbf{I} - \left(\frac{1}{2}\mathbf{A}_G + \mathbf{I}\right)\frac{1}{t} = \begin{bmatrix} \frac{t-1}{t} & -\frac{1}{2t} & \cdots & -\frac{1}{2t} \\ -\frac{1}{2t} & \frac{t-1}{t} & \ddots & \vdots \\ \vdots & \ddots & \ddots & -\frac{1}{2t} \\ -\frac{1}{2t} & \cdots & -\frac{1}{2t} & \frac{t-1}{t} \end{bmatrix}_{n\times n} \tag{26}$$

The results obtained already for the SMA case can be found also useful in studying the convergence of the CMA game, even if it is a DLTV system. In this case, the set $\mathbb{S}_t$ is of interest. The sequence of sets $\mathbb{S}_t$, $t \in \mathbb{N}^+$ is strictly increasing since $\mathbb{S}_t \subset \mathbb{S}_{t+1}$ for any $t \in \mathbb{N}^+$. Following Theorem 2 there exists a stage $t_0 = \min\left\{t \in \mathbb{N}^+ : t > \frac{n-1}{2}\right\}$ after which all the eigenvalues of matrix $\mathbf{A}$ are contained in $\mathbb{S}_t$. Exactly at t_0 the CMA game is identical to a SMA game of t_0 delays and, due to the latter's stability, (27) holds for $\tau \geq t_0$. Consequently, for any following stage $t > t_0$ of the CMA game there exists a SMA game of $m = t$ delays and initial history $h_t^{\tau-1}$ the history h^{t-1} of the CMA game that is stable for $\tau \geq t$. Considering only the first stage of all those SMA games of the same unique equilibrium $\mathbf{x}^*$ and a per stage sequential transition we conclude that (27) holds for $t > t_0$. Since $\mathbf{x}^*$ is its fixed point, for any $t > \frac{n-1}{2}$ the CMA game presents an asymptotically stable behavior by following, converging to the equilibrium $\mathbf{x}^*$, trajectories.

$$\|\mathbf{x}_t - \mathbf{x}^*\| \leq \|\mathbf{x}_{t-1} - \mathbf{x}^*\| \tag{27}$$

Alternatively, a more appropriate way to study its stability is by considering the evolution of the difference of the state vector from the stationary point $\mathbf{x}^*$. By subtracting $\mathbf{x}^*$ from both sides of (25) an autonomous homogenous system description, i.e. $\mathbf{x}_t - \mathbf{x}^* = \mathbf{A}_n(t-1)\left(\mathbf{x}_{t-1} - \mathbf{x}^*\right)$, is obtained. The application of the Euclidean norm results in expression (28) and leads to Theorem 3 that bounds the distance from the equilibrium, with the upper bound presented in Fig. 6 in a log-linear scale. The asymptotic stability of the CMA game is not only consistent with our previous findings but confirms the continuous-time literature (Thorlund-Petersen, 1988; Deschamps, 1975).

$$\|\mathbf{x}_t - \mathbf{x}^*\| \leq \|\mathbf{A}_n(t-1)\| \, \|\mathbf{x}_{t-1} - \mathbf{x}^*\| \, , \, \forall t \in \mathbb{N}^+ \tag{28}$$

Theorem 3 *For $t \in \left\{t \in \mathbb{N}^+ : t > \frac{1}{4}(n+1)\right\}$, the CMA game, which is equivalent to the Cournot game under fictitious play, presents a convergent behavior with $\|\mathbf{x}_t - \mathbf{x}^*\| \leq \|\mathbf{x}_{t-1} - \mathbf{x}^*\|$ since the number of players is finite and the CMA game is asymptotically stable.*

Proof For (27) to hold, the distance from the equilibrium has to be descending over time, equivalently $\|\mathbf{A}_n(t-1)\|$ has to be less than the unit. The spectral norm $\|\mathbf{A}_n(t-1)\|$ is equal to the greatest singular value of $\mathbf{A}_n(t-1)$, which is the square root of the greatest eigenvalue of $\mathbf{A}_n(t-1)\mathbf{A}_n(t-1)$, i.e. $\|\mathbf{A}_n(t-1)\| = \sigma_{max}\left(\mathbf{A}_n(t-1)\right) = \sqrt{\lambda_{max}\left(\mathbf{A}_n(t-1)\mathbf{A}_n(t-1)\right)}$. Since both matrices have inherited the structure of $\mathbf{A}_G$, the singular values of $A_n(t-1)$ can be found explicitly as equal to the absolute value of its eigenvalues, with the maximum one given in (29). Considering (27) along with the asymptotic limit of (25), the convergence follows.

$$\|\mathbf{A}_n(t-1)\| = \sigma_{max}\left(\mathbf{A}_n(t-1)\right) = \max\left\{\left\|\frac{2t-1-n}{2t}\right\|, \left\|\frac{2t-1}{2t}\right\|\right\} \tag{29}$$

□

To gain a better insight on the dynamics of the CMA game, let us consider the ratio of the vector norms of the difference from the fixed point $\mathbf{x}^*$ of two sequential state vectors. The Theorem that follows along with Fig. 6 serve this cause, while the comments that conclude this section point out some important aspects of the CMA rule.

Theorem 4 *For every $t \in \mathbb{N}^+$, the ratio of the vector norms of two sequential states of the DLTV homogenous system $\mathbf{x}_t - \mathbf{x}^* = \mathbf{A}_n(t-1)\left(\mathbf{x}_{t-1} - \mathbf{x}^*\right)$, which describes the behavior of the CMA game, is bounded by the minimum and the maximum singular value of $\mathbf{A}_n(t-1)$ and the game converges sublinearly to its fixed point $\mathbf{x}^*$.*

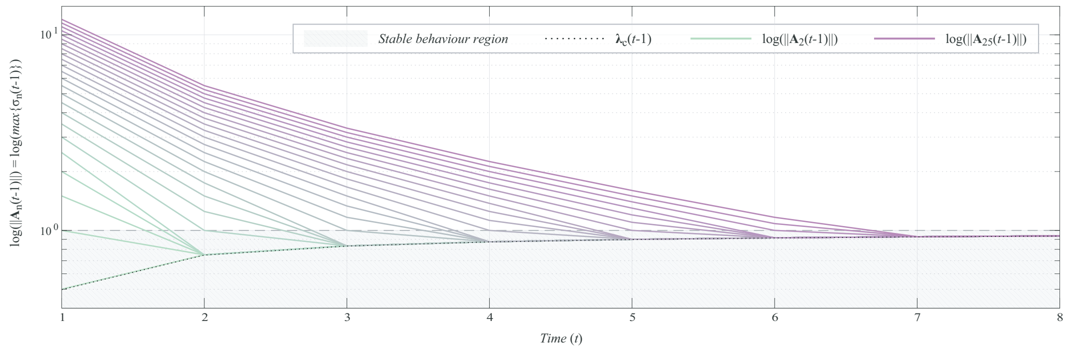


Fig. 6: Evolution of the spectral norm of $\mathbf{A}_n(t-1)$, which upper bounds the speed of convergence of the CMA game (fictitious play). Cases for 2 to 25 players.

Proof By setting $\mathbf{y}_t = \mathbf{A}_n(t-1)\mathbf{y}_{t-1}$ and $\mathbf{A}_n(t-1) = \frac{2t-1}{2t}\mathbf{I}_n - \frac{1}{2t}\mathbf{1}_n\mathbf{1}_n^{\mathrm{T}}$, consider the CMA game in the form of (30), where $\mathbf{I}_n$ is the identity $n \times n$ matrix and $\mathbf{1}_n$ the $n \times 1$ vector of ones. As it can be inferred by (30), the sequence $\mathbf{y}_t \in \operatorname{span}\{\mathbf{y}_0, \mathbf{1}_n\}$, $\forall t \in \mathbb{N}^+$. The unit vectors $\boldsymbol{i}_{\mathbf{1}_n} = {}^{\mathbf{1}_n}/\sqrt{n}$, $\boldsymbol{i}_\perp$ such that $\langle \boldsymbol{i}_\perp, \boldsymbol{i}_{\mathbf{1}_n} \rangle = 0$ and $\operatorname{span}\{\boldsymbol{i}_\perp, \boldsymbol{i}_{\mathbf{1}_n}\} = \operatorname{span}\{\mathbf{y}_0, \mathbf{1}_n\}$, form a natural basis of the corresponding subspace. Given that the angle between $\mathbf{y}_{t-1}$ and $\mathbf{1}_n$ is φ_{t-1}, expression (31) can be obtained.

$$\mathbf{y}_t = \left(\frac{2t-1}{2t}\mathbf{I}_n - \frac{1}{2t}\mathbf{1}_n\mathbf{1}_n^{\mathrm{T}}\right)\mathbf{y}_{t-1} = \frac{2t-1}{2t}\mathbf{y}_{t-1} - \frac{n}{2t}\frac{\mathbf{1}_n}{\sqrt{n}}\left\langle \frac{\mathbf{1}_n^{\mathrm{T}}}{\sqrt{n}}, \mathbf{y}_{t-1} \right\rangle \tag{30}$$

$$\mathbf{y}_t = \|\mathbf{y}_{t-1}\| \left(\frac{2t-1-n}{2t}\cos(\varphi_{t-1})\boldsymbol{i}_{\mathbf{1}_n} + \frac{2t-1}{2t}\sin(\varphi_{t-1})\boldsymbol{i}_\perp\right) \tag{31}$$

The ratio of the vector norms of two sequential states of the DLTV system is given in (32). Its critical points, obtained for either $\varphi = \kappa\pi$ or $\varphi = \kappa\pi + \frac{\pi}{2}$, $k \in \mathbb{Z}$, are a minimum and a maximum, which are equal to the minimum and the maximum singular value of $\mathbf{A}_n(t-1)$, respectively. The bounding expression follows, while the sublinear convergent behavior is obtained by finding the asymptotic limit of (32) equal to one.

$$\frac{\|\mathbf{y}_t\|}{\|\mathbf{y}_{t-1}\|} = \sqrt{\lambda_n^2(t-1)\cos^2(\varphi_{t-1}) + \lambda_c^2(t-1)\sin^2(\varphi_{t-1})} \tag{32}$$

$$\min\left\{\left\|\tfrac{2t-1-n}{2t}\right\|, \left\|\tfrac{2t-1}{2t}\right\|\right\} \leq \tfrac{\|\mathbf{y}_t\|}{\|\mathbf{y}_{t-1}\|} \leq \max\left\{\left\|\tfrac{2t-1-n}{2t}\right\|, \left\|\tfrac{2t-1}{2t}\right\|\right\}, \ \forall t \in \mathbb{N}^+$$

□

In Fig. 7, the evolution of singular values over time is depicted for an increasing number of players along with the evolution of the ratio for three different initial conditions for a particular case ($n = 8$). Among them is one of symmetric difference, one where only one player deviates and a third one where the initial vector lays very close to the eigenspace of $\lambda_c(t-1)$.

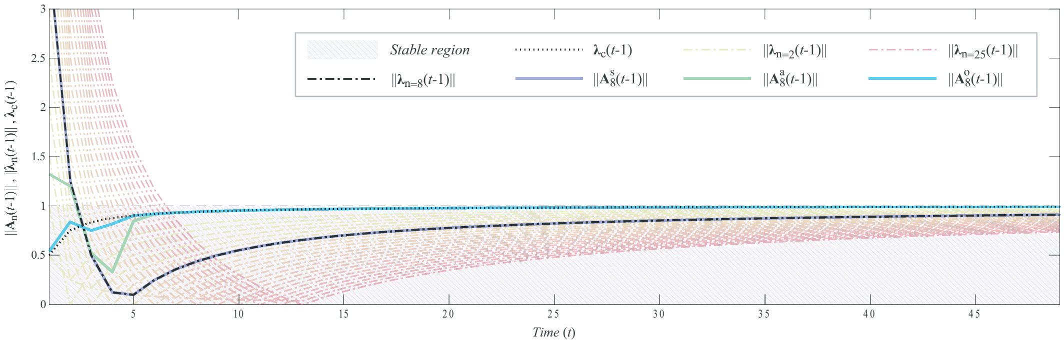


Fig. 7: Singular values of $\mathbf{A}_n(t-1)$ that bound the speed of convergence of the CMA game (fictitious play) for 2 to 25 players along with three particular cases for n=8.

Remark 1 In the cases where the initial condition $\mathbf{y}_0$ and one of the eigenvectors are linearly dependent, the ratio is equal to the absolute value of the corresponding eigenvalue. Typical example is the symmetric case, where all players begin having the same difference from the equilibrium, $\mathbf{y}_t$ and $\mathbf{1}_n$ are all the time linearly dependent and the ratio is equal to $\|\lambda_n(t-1)\|$. For $\mathbf{y}_0$ belonging to the subspace spanned by the other $n-1$ eigenvectors, the ratio is equal to $\|\lambda_c(t-1)\|$ and the game does not experience an exploding behavior at any time.

Remark 2 The ratio as time progresses lays on a strip of decreasing width (Fig. 7), close to one. This makes any adaptation to a new, shifted, equilibrium on a later time to be more difficult and the speed of convergence much slower. This is also identified, from a slightly different perspective, in Section 5.2 to be a serious drawback of the CMA adaptive rule and, generally, of fictitious play as a learning strategy.

5 Comparative Analysis

In this section, through simulations, we validate the theoretical results obtained on the convergence of the game and we examine the qualitative characteristics of the adaptive rules proposed to identify their suitability. By considering an example market under two main scenarios - of fixed and increasing competition level - we develop a benchmarking environment for studying the role different parameters play on players' interaction.

Let the markets consist of n identical firms, each one of them having marginal cost $c_i = 40$. The linear demand parameters considered in the cases that follow are $a = 100$ and $b = 0.1$. Before we proceed further with the two scenarios and the respective results, let us first introduce the Euclidean distance from the equilibrium in the n dimensional space which can be used as a metric of convergence. This metric will facilitate the comparison, in terms of convergence, of results obtained in markets of different number of players.

It will also allow the initialization of the games in a uniform and comparable way, by taking initial conditions from the n dimensional hyperspheres of radius $\rho_{n,t}$, centered at the corresponding equilibria points.

$$\rho_{n,t} = \sqrt{\sum_{i=1}^{n} \left(x_i^* - x_{i,t}\right)^2} \tag{33}$$

5.1 Markets of fixed number of players

For the fixed competition level scenario, we assume that each of the markets considered has a different but fixed number of players. The results presented below correspond to sets of 100 games of different initial conditions. For the games where the SMA rules have been adopted, m states are required for the proper initialization of the game. On the contrary, for the CMA and the CSMA rules one initial condition suffices for the interaction to begin. By considering that every player brings about a bias on the equilibrium quantities we take the radius $\rho_{n,t}$ of the n dimensional hypersphere increasing on the number of players by a constant bias. Although games of more players are initialized further away from the equilibrium, the different adaptive rules can be compared on a common basis since an identical distance is imposed before the beginning of interaction.

The games of the sets have been used for the formation of the region of potential outcome, serving the need of indicating the characteristics of the trajectories from both the time and the output perspectives. Those regions are presented in the figures along with one of the trajectories where the output of each player is distinguished as well. Markets of different competition levels are presented for three different cases of delays. Games of 5, 8 and 11 players (n) are simulated in the cases of 3, 6 and 9 delays (m) for the three adaptive rules presented. To facilitate the discussion about those games, let us define as $G^{R}_{n,m}$ the game of n players who adopt the adaptive rule R for the case of m delays. For comparability reasons, the SMA and the CMA rules are presented together in Fig. 8, while the CMA and the CSMA are presented in Fig. 9.

In Fig. 8, for the games $G^{SMA}_{5,m}$, $m \in \{3, 6, 9\}$, which converge for all the delays cases presented, the belated convergence due to more initial conditions required, can be seen clearly. Comparing the interaction from the step at which the SMA rule is activated, the increase in delays doesn't imply faster convergence to the equilibrium but only a less volatile one. Respectively for the CMA rule games ($G^{CMA}_{5,m}$), the direct entrance of the dynamics in their stable region can be observed with the duration of the interaction in the unstable region to be negligible. As the number of players increases, the unsuitability of the SMA rule in the $G^{SMA}_{11,3}$ and the $G^{SMA}_{8,3}$ games due to its divergent behavior, become obvious[4]. Longer stay in the unstable region leads to temporal deviat-

[4] Firms' output can take non-negative values and this has been imposed in simulations; The nonlinearities that may occur (Cánovas et al., 2008) in the case of bounded action sets, given the existence of a Nash Equilibrium are out of the scope of this paper.

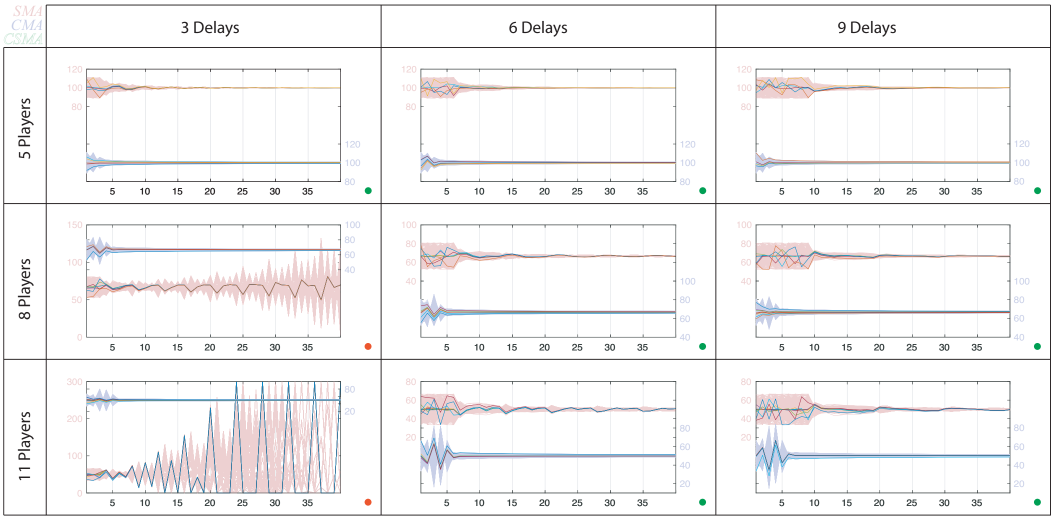


Fig. 8: Trajectories of games of 5, 8 and 11 players under the adoption of SMA and CMA rules. SMA is presented for the 3, 6 and 9 delays cases.

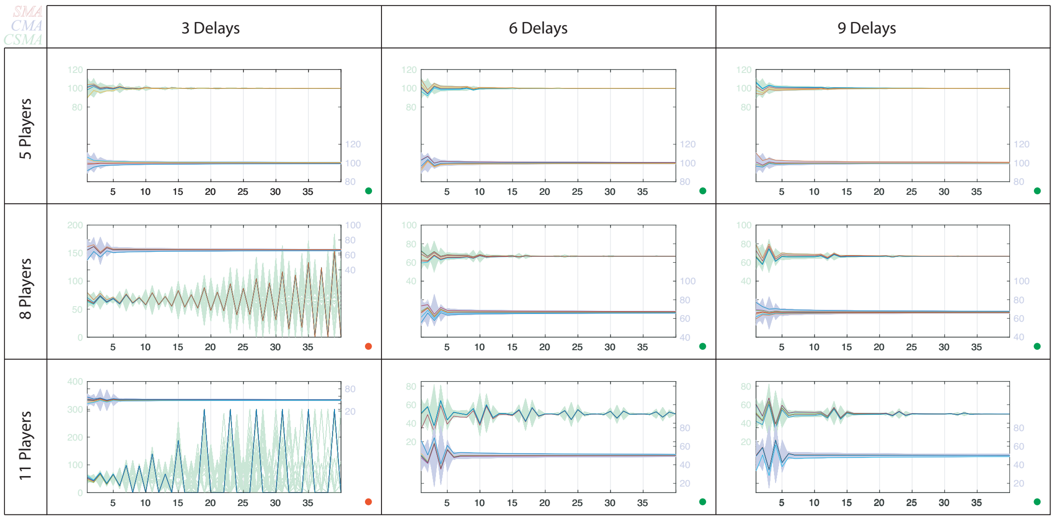


Fig. 9: Trajectories of games of 5, 8 and 11 players under the adoption of CSMA and CMA rules. CSMA is presented for the 3, 6 and 9 delays cases.

ing trends which affect the market price and the profits, resulting in negative effects. After the incorporation of richer information sets, the dynamics become stable and the outcome trajectories get into convergent paths towards the equilibrium.

The hybrid rule CSMA, originating from the combination of the two main rules, inherits characteristics present in both the SMA and the CMA rule. As can be seen in Fig. 9, the instability that occurred in the $G_{11,3}^{SMA}$ and the $G_{8,3}^{SMA}$ games is also maintained by the CSMA rule. In this figure, where the CSMA rule is in direct comparison with the CMA rule, the divergent course of the CMA games' outcome, during the first stages of interaction, can be identified as

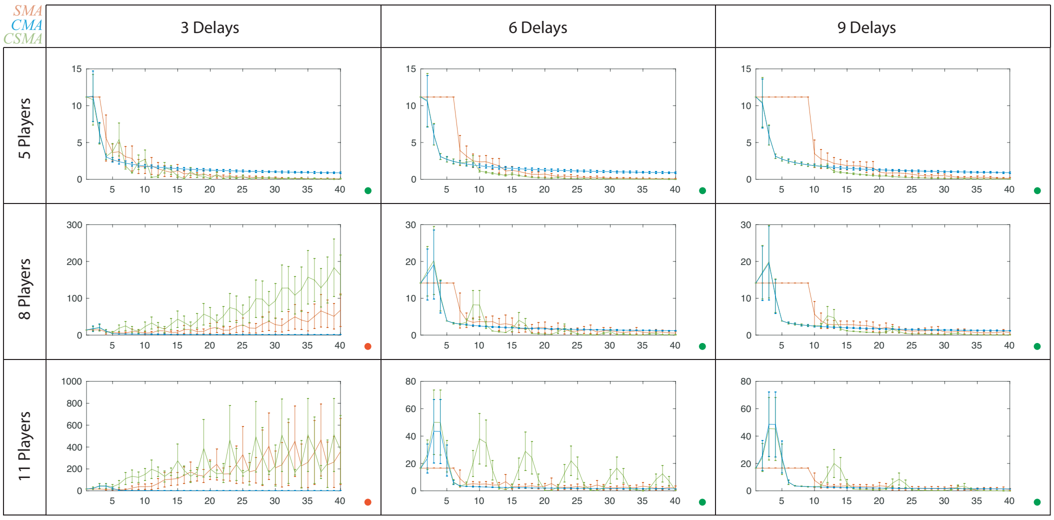


Fig. 10: Convergence behavior comparison between SMA, CMA and CSMA rules in set ofs games of 5, 8 and 11 players. Mean of the radius $\rho_{n,t}$ and its standard deviation.

another problematic element inherited by the CSMA rule. On the other hand, the CSMA adaptation exploits the zero need for excessive initial conditions and at the same time is exempted by the drawback of the CMA rule, referring to the very late exact convergence to the equilibrium. Specifically, this measure is captured by the radius $\rho_{n,t}$, of which the mean and the standard deviation of the sets of 100 games is presented in Fig. 10. In this figure, the inability of the CMA rule adaptation to reach the equilibrium in the short run due to its "no-forget" property becomes clear. It is worth mentioning that better performance in terms of minimization of the distance from the equilibrium is exhibited by the CSMA adaptation. It seems that this periodic re-triggering of oscillatory response with delaying amplitude recalibrates the outcome closer to the equilibrium. Due to the replacement of one of the unstable moments, with a moment closer to the equilibrium, on each recalibration round the responses are centered at an improved level in terms of distance from it.

The statistics of the radius $\rho_{n,t}$, as presented in Fig. 10, indicate that aside from the convergence property that a rule should ensure, the transient behavior that results also matters. It is not possible to conclude to an overall superior rule, since the suitability of the rule depends strongly to market's characteristics. The uncertainty about the number of decision makers and the intolerance of strongly competitive outcomes where losses incur are examples of characteristics that can make some rules to be more adequate and suitable than others. Adoption of different rules by different players can counteract the main disadvantages of each rule and enhance the overall efficiency of the outcome.

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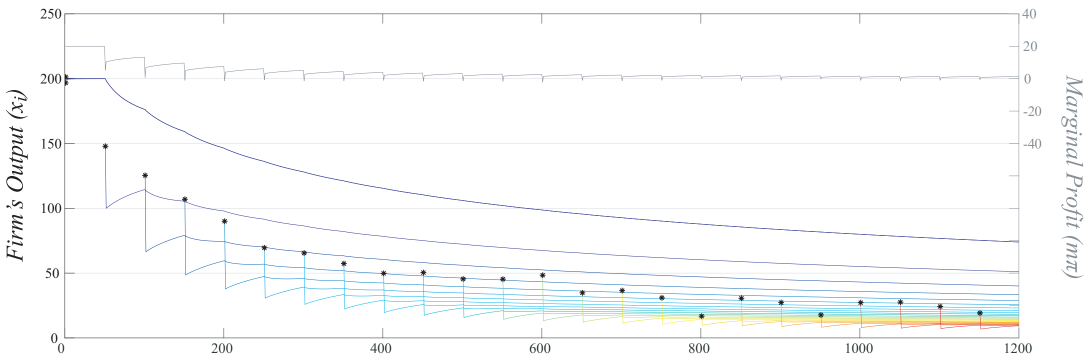
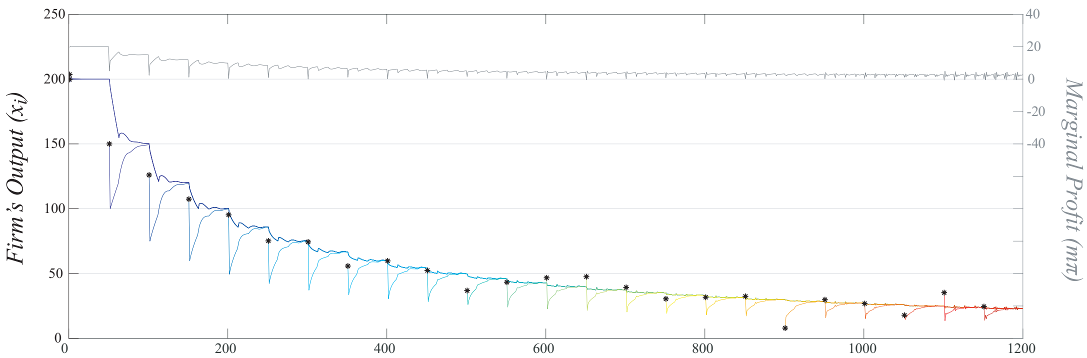
5.2 Markets of increasing competition level

Under this scenario, where the number of firms increases, we examine how the adoption of a CMA, a CSMA and a newly introduced Dynamic-CSMA rule affects the games outcome. In all cases the game starts as a duopoly and ends with $n_f = 25$ players. Firms enter the market in intervals of $\Delta t = 50$ steps, having only one biased observation of the state, which they use as an initial condition. The asymmetries occurred are mainly due to the different history h^{t-1} in which firms of different entry time have access to. Even if all the previous analysis has been made for symmetric cases, where firms start concurrently, the concept of new entrants in the market implies asynchronous adaptation. Therefore, the example market deployed here make use of the results obtained for games where players on every step exploit the same history and adopt identical adaptive rules, aiming to provide a further comparability base.

Fig. 11 presents the output of all firms as their number increases for both the CMA and the CSMA cases. The evolution of the marginal profit, i.e. the difference between the market price and the marginal cost, is also depicted.



(a) Output of firms and marginal profit in an example market where players adopt CMA rule



(b) Output of firms and marginal profit in an example market where players adopt CSMA rule of maximum $m = 12$ delays

Fig. 11: Outcome of games under CMA and CSMA rules in markets of increasing competition

It can be said that the CMA rule is found unsuitable, since the "no-forget" principle that characterizes it causes a very slow adaptation. Its asymptotic convergence causes an extremely belated approach of the Nash equilibrium, a fact that in the context of a game with varying number of players can not be neglected. This is mainly because the adoption of the adaptive rules is based on an assumption initially made; for preventing its questioning and consequently the emergence of divergent behaviors, the assumption should not lead to exorbitantly inefficient outcomes. The lack of symmetry in the length of history obtained by players of different arrival times exists all along the game, with the first firms abstaining more from the equilibrium due to their inability to exempt from past equilibria. Nevertheless, their profits are higher than those of the other firms for a long time, as the output is higher, a fact that contradicts with the identical cost the firms have.

On the other hand, the CSMA rule of $m = 12$ delays is found to be ideal for this game. In the beginning, the two firms concurrently approach the duopoly equilibrium, while in every new entrance they adjust rapidly to the new output level. The new entering players start with one biased observation of their rivals' quantity and they start to increase their history along with the interaction till they switch to the SMA part of that hybrid rule. Therefore, there are some intervals in which the symmetry is restored, with quantities and profits to be equally shared. With just one new entrant each time and with every other participating firm in the SMA phase of the rule, the transient behavior is even smoother and no "harmonics" effect can be identified.

Similar to the CSMA rule with a fixed number of delays are the results occurred with the adoption of a Dynamic-CSMA rule. The difference between those two cases is that in the latter the maximum number of delays each firm keeps is dynamically controlled, by being based on a strategy. The strategy should be based on the trend of the output each firm identifies and, for efficiency reasons, it should enrich the history in time, spaningly. An expected consequence is the firms to end up using slightly different adaptive rules, an asymmetry we have not considered so far. A dynamic adjustment process is expected to allow the number of delays to increase for maintaining conver-

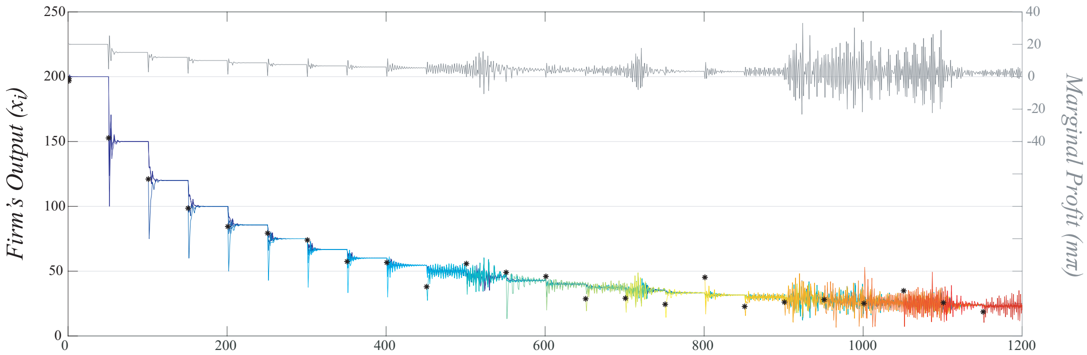


Fig. 12: Output of firms and marginal profit in an example market where players adopt the proposed Dynamic-CMA rule

gence as the number of players increases. At the same time it should aim to keep the number of delays in a low level so that the speed of adjustment is kept high enough, such that the convergence is fast. The suggested control strategy considers the rate of change along with the volatility of the output when available. This strategy is described by (34) where d_t^i is the length of the history available to player i at the time of decision making, σ_{01}^i and σ_{12}^i are the standard deviations of the last two pairs of its output, Δx_i is the last rate of change and ρ_0 is the initial divination from the equilibrium.

$$d_{t+1}^i = \begin{cases} d_t^i + 1 & if\ (\sigma_{01}^i > \sigma_{12}^i) \wedge (\Delta x_i > \rho_0) \\ d_t^i & otherwise \end{cases} \tag{34}$$

The firms' output and the marginal profit, under the adoption of the Dynamic-CSMA rule, are presented in Fig. 12. The volatility that is observed, in both the output and the marginal profit, can be explained by the low number of delays players dynamically select. Beyond that, the outcome level is comparable to the level of the most efficient CSMA rule, where all players behaved as they knew apriori the maximum number of players. Of course, there may exist more effective and efficient strategies for players choosing the length of history, which can be determined.

6 Conclusion

A discrete time "a la Cournot" game has been considered and its convergence under adaptive rules inspired by business intelligence tools has been the subject matter of this work. The game has been deployed in a dynamic framework and the private information of firms' costs lead players to update their best responses using particular rules. The game formulation has been enriched with the description of a feasible region, able to apply boundaries if the avoidance of irrational trajectories is a prerequisite. Since the proposed backward-looking rules exploit previous instances of the output, the history of the game is determined and two versions are identified, the full and the truncated one, namely.

The ordered datum stream of fixed window length corresponds to the truncated version of the history and is used by the adaptive rule that is based on a SMA. The least number of delays, or in other words, the smallest length of the window that guaranties convergence, depends on the number of players and is determined explicitly. This result is in accordance with the known property of smoothing that the MA posses and the lower speed of adjustment required for stability as the number of players increases. For the convergence of the CMA rule, which as a rule -after some point- presents smoother adaptation than the least required one, both the transient and the asymptotic behavior have been studied extensively.

The latter forms the basis for rules inspired by the MA family, suitable for games where the number of rivals varies or is uncertain. Even if CMA rule ensures convergence, some of its structural characteristics, such as its "no-forget" principle, makes it inefficient for many cases. Somewhere in the middle ground lays the hybrid rule, the CSMA, which inherits the step-wise increase of delays from the CMA while the game is in its unstable region and the fixed length of the history window from the SMA when the game turns to a stable region. Such a static rule has been taken into consideration and compared with the other rules for games of constant number of players. Its dynamic version should come with the monitoring of an extra measure to identify the trend behind the volatility and a threshold for triggering the switch between the basic rules. This goes beyond the scope of this paper but turning the number of delays to a strategic parameter will result in more sophisticated and more efficient strategies, which can ensure convergence and counteract the disadvantages of static rules.

To conclude, the adaptive rules proposed have been found suitable for the game at stake, since in their sufficient they ensure convergence. A great advantage of the rules, beyond their simplicity, is that they are based in techniques already used in entrepreneurial environments. Therefore, the assumption about their universal adoption sounds more rational. Cournot-like games have been studied to great extents from different perspectives and all the results have enriched oligopoly theory's literature. Our results are found to be in line with those already obtained for the continuous adjustment process case and their equivalent for the discrete one. The more delays the rule implies, the less the impact of each one of them to the adjustment and, therefore, the game tends towards a convergent behaviour. The condition determined for the adequate number of delays for a given level of competition along with the interesting way it has been obtained, are novel and create a new potential for further research as well. The application of those rules in the modelling of quantity-based competition markets, where several firms participate, can be a typical example. Additionally, the further study of the arbitrary weighting case (Weighted Moving Average) and of other rules from the MA family (Linearly Descending Weighted Moving Average, Exponentially Descending Weighted Moving Average, etc.) could lead to more efficient rules and constitute potential extensions.

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