On the Linear–Quadratic–Gaussian Nash Game with One-Step Delay Observation Sharing Pattern

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Abstract—This paper studies a two-player Nash dynamic, discrete-time, linear–quadratic game under the so-called "one-step delay observation sharing pattern." It is shown that under very weak assumptions the solution exists and is unique and linear in the information.

I. INTRODUCTION

THE PURPOSE of this paper is to analyze the two-player linear–quadratic–Gaussian Nash game under the so-called "one-step delay observation sharing pattern." This is an important problem in game theory which has been studied previously only in [6]. Our main result is that the solution almost always exists and is linear in the information. By "almost always" we mean that the conditions under which our results hold are satisfied for all except a set of measure zero of the parameters of the problem (i.e., the matrices involved). Also, if these conditions are not satisfied, then either there exists no solution or there exist infinitely many and they might be nonlinear in the information.

The importance of Nash games is by now widely accepted, but unfortunately, the theoretical difficulties associated with them are often discouraging. This is partly due to the causal appearance of nonclassical information patterns or nonuniqueness and nonlinearity of the solutions. Also, the behavior of the solution might change drastically for small variations of parameters or from the deterministic to the stochastic counterpart.

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The problem considered here is perhaps the only dynamic stochastic Nash game for which a relatively mature ensemble of results seems to be possible in the near future. A key point of the formulation is the employment of the "one-step observation sharing pattern" for the information of the players. Under this information setup, at each stage each player acquires a new measurement, recalls all his previous measurements and the previous measurements of his opponent, but he has no access to the last measurement of his opponent. This is a meaningful setup since the acquisition of the information of the other player always takes some time, either due to communication delays or due to the time needed to perform any spying activities. This type of information structure is different from the "one-step delay sharing pattern" of [3], where at each stage each player also has access to the previous control values of his opponent (in reference to this, see Remark 9 in Section IV). The only results available for this problem were established by Basar in [6] and were based on the study of the static case in [4] also by Basar. In [6] and [4], appropriate conditions were imposed on the magnitudes of the norms of the matrices involved so as to guarantee the use of the contraction mapping theorem and thus obtain existence and uniqueness of the solution, which under these conditions turns out to be linear in the information. Our analysis is based on [7] where the static case was studied and completely solved. The use of the results of [7] enables us to avoid the use of the contraction mapping theorem and consequently avoid imposing the very restrictive conditions of [6]. (It will become evident to the reader that our assumptions induce those of [6].) Under the conditions that we use we show that the game under consideration admits a unique solution which has to be linear in the information.

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The meaning of these conditions is the demand that at each stage the coupling of the costs to go of the two players is not inversely proportional to some power of the coupling of their information at this stage. The coupling of the costs to go at each stage is quantified by the eigenvalues of the product of the matrices which involve the cross terms of the two controls at that stage, whereas the coupling of the information at each stage is quantified by the canonical correlation coefficients of the information of the two players at that stage. Our conditions are expressed as invertibility conditions on several matrices and they do not involve any bounds on the norms of the matrices. The importance of our main result lies in our opinion, in that our invertibility conditions hold for almost all the values of the parameters and thus it is guaranteed that almost all such Nash games admit a unique solution which has to be linear in the information. Having this linearity is appealing not only due to its simplicity, but also because if the Nash game is employed as a decentralized approximation of an originally centralized problem which is linear–quadratic–Gaussian and thus has a linear solution, the solution of the approximate problem (i.e., of the Nash game) is also linear.

The structure of the paper is the following. In Section II we pose the problem and in Section III we present the analysis. In Section IV we discuss possible extensions and further research problems emanating from our analysis.

II. Problem Statement

Consider the state evolution equation

$$x_{n+1}=A_nx_n+B_{1n}u_{1n}+B_{2n}u_{2n}+w_n,\qquad n=0,1,2,\cdots,N, \quad (1)$$

the measurement equations

$$y_{in}=C_{in}x_n+v_{in},\qquad i=1,2,\ n=0,1,2,\cdots,N, \quad (2)$$

and the costs

$$J_i=E\left[x'_{N+1}Q_{i,N+1}x_{N+1}+\sum_{k=0}^{N}\left(x'_kQ_{ik}x_k+u'_{ik}u_{ik}+u_{jk}R_{ik}u_{jk}\right)\right]\qquad i\neq j,\quad i,j=1,2. \quad (3)$$

x_n, u_{in}, y_{in} take values in finite-dimensional Euclidean spaces of fixed dimensions, the matrices $A_n, B_{in}, Q_{ik}, R_{ik}$ are real, constant with appropriate dimensions and $Q_{ik}\geq 0$, $R_{ik}\geq 0$. x_0, w_k, v_{ik} are Gaussian, zero mean, independent random vectors and the v_{ik}'s have nonsingular covariance matrices. u_{ik} is chosen as a measurable function γ_{ik} of the elements of I_{ik}, where

$$I_{ik}=(I_k, Y_{ik}),\qquad i=1,2 \quad (4)$$

$$I_k=(y_{10},\cdots,y_{1,k-1},y_{20},\cdots,y_{2,k-1}) \quad (5)$$

so that $\gamma_{ik}(I_{ik})$ is a second-order random vector. Notice that the class of admissible γ_{ik}'s depends on $\gamma_{10},\cdots,\gamma_{1,k-1}\gamma_{20},\cdots,\gamma_{2,k-1}$ since each y_{il} depends on $\gamma_{10},\cdots,\gamma_{1,l-1},\gamma_{20},\cdots,\gamma_{2,l-1}$. Let $\gamma_i=(\gamma_{i0},\cdots,\gamma_{iN})$, so that J_i is a function of γ_1,γ_2, $J_i(\gamma_1,\gamma_2)$. We want to find γ_1^*,γ_2^* so that the Nash equilibrium conditions

$$J_1(\gamma_1^*,\gamma_2^*)\leq J_1(\gamma_1,\gamma_2^*),\qquad \forall \text{ admissible } \gamma_1 \quad (6)$$

$$J_2(\gamma_1^*,\gamma_2^*)\leq J_2(\gamma_1^*,\gamma_2),\qquad \forall \text{ admissible } \gamma_2 \quad (7)$$

are satisfied. A pair (γ_1^*,γ_2^*) satisfying (6) and (7) is called a Nash pair, N-pair for short.

Our presentation will be occasionally sketchy, but nonetheless completely rigorous.

III. Solution

Let us first introduce some notation and some quantities of importance to our analysis. Let $\bar{y}_{in}$ be defined as

$$\bar{y}_{in}=C_{in}(A_{n-1}A_{n-2}\cdots A_0x_0+A_{n-1}\cdots A_1w_0+\cdots+A_{n-1}w_{n-2}+w_{n-1})+v_{in}\qquad i=1,2,\quad n=0,1,2,\cdots,N. \quad (8)$$

$\bar{y}_{in}$ is the zero input response in (2) and is obtained from y_{in} by substituting in (2) recursively the x_n's using (1), thus expressing y_{in} as a linear function of $u_{i0},\cdots,u_{i,n-1},x_0,w_0,\cdots,w_{n-1},v_{in}$ and discarding the terms involving $u_{i0},\cdots,u_{i,n-1}$. Thus

$$y_{in}=\bar{y}_{in}+L^{in}(u_{ik},i=1,2,k=0,1,\cdots,n-1). \quad (9)$$

The symbol $L(\cdot)$ indicates a linear function of the arguments and it will be used repeatedly later. Let us also define the following quantities:

$$\bar{I}_n=(\bar{y}_{10},\cdots,\bar{y}_{1,n-1},\bar{y}_{20},\cdots,\bar{y}_{2,n-1}) \quad (10)$$

$$\bar{I}_{in}=(\bar{I}_n,\bar{y}_{in}) \quad (11)$$

$$z_{in}=\bar{y}_{in}-E(\bar{y}_{in}|\bar{I}_n), \quad (12)$$

$$z_{in}=L_{in}(y_{ik},u_{ik},i=1,2,k=0,1,\cdots,n-1)+y_{in}. \quad (13)$$

$$\tilde{I}_{in}=(\bar{I}_n,z_{in})=\tilde{L}_{in}(y_{ik},u_{ik},i=1,2,;k=0,1,\cdots,N-1;y_{in}). \quad (14)$$

z_{in} represents the essentially new information that $\bar{y}_{in}$ provides, if $\bar{I}_n$ is known and can be expressed as a known linear function of the observations and controls. We think of the I's as vectors with the $\bar{y}_{ik}$'s stuck one under the other.

It is clear that for fixed admissible $\gamma_{10},\cdots,\gamma_{1,n-1},\gamma_{20},\cdots,\gamma_{2,n-1}$, any admissible γ_{in} can be expressed as a function of $\bar{I}_{in}$, i.e., of the primitive random vectors $x_0,w_0,\cdots,w_{n-1},v_{i0},\cdots,v_{in}$, by recursive substitutions. The σ-fields generated by $\bar{I}_{in}$ and $\tilde{I}_{in}$ are obviously the same. Also, z_{in} is orthogonal to $\bar{I}_n$. $E[\cdot|\cdot]$ denotes conditional expectation and the L_{in}'s, $\tilde{L}_{in}$'s in (13), (14) denote linear functions of their arguments. The L_{in}'s and $\tilde{L}_{in}$'s can be found by recursive substitution and thus we consider them to be known. The following hold:

$$E[\cdot|I_{in}] = E[\cdot|\bar{I}_{in}] = E[\cdot|\tilde{I}_{in}] \tag{15}$$

$$u_{ik} = E[u_{ik}|I_n] = E[u_{ik}|I_{j,n+1}], \quad n \geq k, \quad i,j=1,2. \tag{16}$$

Notice that since we are dealing with second-order random vectors, the conditional expectations can be thought of as orthogonal projections in appropriately defined Hilbert spaces (where we consider that from each random vector we have subtracted its mean if this mean is nonzero). For easy reference we also introduce the following quantities (costs to go):

$$J_{in} = E\left[x'_{N+1}Q_{i,N+1}x_{N+1} + \sum_{k=n}^{N}\left(x'_kQ_{ik}x_k + u'_{ik}u_{ik} + u'_{jk}R_{ik}u_{jk}\right)\right] \cdot i \neq j, \; i,j=1;2, \; n=0,1,\cdots,N. \tag{17}$$

Before starting to work on our problem we need the following definition.

Definition: A pair (γ_1^*, γ_2^*), $\gamma_i^* = (\gamma_{i0}^*, \cdots, \gamma_{iN}^*)$, $i=1,2$, is called a stagewise Nash equilibrium pair for the game defined in (1)–(5) if instead of (6), (7) it holds that

$$J_1(\gamma_{10}^*, \cdots, \gamma_{1,k-1}^*, \gamma_{1k}^*, \gamma_{1,k+1}^*, \cdots, \gamma_{1N}^*, \gamma_2^*) \leq J_1(\gamma_{10}^*, \cdots, \gamma_{1,k-1}^*, \gamma_{1k}, \gamma_{1,k+1}^*, \cdots, \gamma_{1N}^*, \gamma_2^*) \quad \forall \text{ admissible } \gamma_{1k} \tag{18}$$

$$J_2(\gamma_1^*, \gamma_{20}^*, \cdots, \gamma_{2,k-1}^*, \gamma_{2k}^*, \gamma_{2,k+1}^*, \cdots, \gamma_{2N}^*) \leq J_2(\gamma_1^*, \gamma_{20}^*, \cdots, \gamma_{2,k-1}^*, \gamma_{2k}, \gamma_{2,k+1}^*, \cdots, \gamma_{2N}^*) \cdot \forall \text{ admissible } \gamma_{2k} \quad \forall k=0,1,2,\cdots,N. \tag{19}$$

A pair (γ_1^*, γ_2^*) satisfying (18), (19) will be referred to as an *SN*-pair.

It is clear that an *N*-pair is an *SN*-pair, but the converse is not true. Let us assume that (γ_1^*, γ_2^*) is an *N*-pair. Therefore it is an *SN*-pair and (18), (19) hold for $k=N$. Using (17) for $n=N$ and substituting x_{N+1} by its equal from (1) we obtain

$$\begin{aligned} J_{1N} = E\big[& u'_{iN}(I + B'_{iN}Q_{i,N+1}B_{iN})u_{iN} \\ &+ 2u'_{iN}B'_{iN}Q_{i,N-1}(A_Nx_N + B_{jN}u_{jN}) \\ &+ u'_{jN}(R_{iN} + B'_{jN}Q_{i,N+1}B_{jN})u_{jN} \\ &+ 2u'_{jN}B'_{jN}Q_{i,N+1}A_Nx_N \\ &+ x'_N(Q_{iN} + A'_NQ_{i,N+1}A_N)x_N\big] \\ &+ E(w'_NQ_{i,N+1}w_N) \qquad i \neq j, \; i,j=1,2 \end{aligned} \tag{20}$$

where we used the independence of w_N on the other quantities and the fact that $E(w_N)=0$. Considering now (18), (19) for $k=N$ and applying a standard minimization result for stochastic quadratic functions we obtain the following necessary and sufficient conditions in order that $\gamma_{1N}^*, \gamma_{2N}^*$ satisfy (18), (19):

$$(I + B'_{1N}Q_{1,N+1}B_{1N})u_{1N} + B'_{1N}Q_{1,N+1}B_{2N}P_{1N}u_{2N} + B'_{1N}Q_{1,N+1}A_NP_{1N}x_N = 0 \tag{21}$$

$$(I + B'_{2N}Q_{2,N+1}B_{2N})u_{2N} + B'_{2N}Q_{2,N+1}B_{1N}P_{2N}u_{1N} + B'_{2N}Q_{2,N+1}A_NP_{2N}x_N = 0 \tag{22}$$

where $E[\cdot|I_{iN}] = P_{iN}$. Although u_{iN} is a function of I_{iN}, it can be expressed as a function of $\tilde{I}_{iN}$, the functional form of which will depend on γ_{ik}^*, $i=1,2$; $k=0,1,\cdots,N-1$. Any such function can be expressed as

$$u_{iN} = \bar{u}_{iN} + \hat{u}_{iN} \tag{23}$$

$$\bar{u}_{iN} = \phi_0^i(\bar{I}_N) \tag{24}$$

$$\hat{u}_{iN} = \sum_{l \geq 1, k} p_k^i(\bar{I}_N)q_l^i(z_{iN})d_{kl}^i \tag{25}$$

where the ϕ_0^i's are some second-order random vectors, the p_k^i's, q_l^i's are real-valued Hermite polynomials so that the $\{p_k^iq_l^i\}_{k,l}$ constitutes an orthonormal complete set with respect to which any second-order random function of $\tilde{I}_{iN}$ can be expressed, and the constant vectors d_{kl}^i satisfy

$$\sum_{k,l}\|d_{kl}^i\|^2 < +\infty \tag{26}$$

($\|\cdot\|$ denotes the usual Euclidean norm). Notice that in the summation of (25) we consider $l \geq 1$, i.e., we exclude the term $q_0^i \equiv 1$, since the part of u_{iN} which depends only on $\bar{I}_N$ as well as the mean of u_{iN} are supposed to have been incorporated in $\bar{u}_{iN} = \phi_0^i(\bar{I}_N)$. It also holds that

$$P_{iN}x_N = L_N^i(u_{ik}, y_{ik}, i=1,2; \cdot k=0,1,\cdots,N-1) + \tilde{L}_N^i(z_{iN}) \tag{27}$$

where the L_N^i's, $\tilde{L}_N^i$'s denote linear functions which can be found by using recursively (1), (2), and (8)–(16) and thus we consider them to be known. Notice also that $L_N^i(u_{ik}, y_{ik}, i=1,2;\ k=0,1,\cdots,N-1)$ can be expressed as a function of $\bar{I}_N$. Thus, we can decompose (21), (22) into two sets of equations (28) and (29) as follows:

$$\begin{bmatrix} I + B'_{1N}Q_{1N}B_{1N} & B'_{1N}Q_{1,N+1}B_{2N} \\ B'_{2N}Q_{2,N+1}B_{1N} & I + B'_{2N}Q_{2,N+1}B_{2N}\end{bmatrix}\begin{bmatrix}\bar{u}_{1N} \\ \bar{u}_{2N}\end{bmatrix} = -\begin{bmatrix} B'_{1N}Q_{1,N+1}A_NL_N^1(u_{ik}, y_{ik}, i=1,2,; \; k=0,1,\cdots,N-1) \\ B'_{2N}Q_{2,N+1}A_NL_N^2(u_{ik}, y_{ik}, i=1,2,; \; k=0,1,\cdots,N-1)\end{bmatrix} \tag{28}$$

and

$$\begin{bmatrix} I + B'_{1N}Q_{2N}B_{1N} & B'_{1N}Q_{2,N+1}B_{2N} \\ B'_{2N}Q_{2,N+1}B_{1N} & I + B'_{2N}Q_{2,N+1}B_{2N}\end{bmatrix}\begin{bmatrix}\hat{u}_{1N} \\ \hat{u}_{2N}\end{bmatrix} = -\begin{bmatrix} B'_{1N}Q_{1,N+1}A_N\tilde{L}_N^1(z_{1N}) \\ B'_{2N}Q_{2,N+1}A_N\tilde{L}_N^2(z_{2N})\end{bmatrix}. \tag{29}$$

To find the $\hat{u}_{iN}$ as functions of $\tilde{I}_{iN}$ we have to solve (29). This can be done explicitly by using the results of [7]. The solution of (24) might in general not exist, be it nonunique or nonlinear. Nonetheless the following lemma holds (see [7]).

Lemma 1: The solution of (29) will exist, and will be unique if and only if the matrix

$$\begin{bmatrix} I + B'_{1N}Q_{1,N+1}B_{1N} & \mu B'_{1N}Q_{1,N+1}B_{2N} \\ \mu B'_{2N}Q_{2,N+1}B_{1N} & I + B'_{2N}Q_{2,N+1}B_{2N} \end{bmatrix} \tag{30}$$

is nonsingular for every μ equal to a finite product of powers of the canonical correlation coefficients on z_{1N}, z_{2N}. In this case the unique solution of (29) is given by

$$\hat{u}_{iN} = M_i z_{iN} \tag{31}$$

where the M_i's are the (unique) solutions of the system

$$(I + B'_{1N}Q_{1,N+1}B_{1N})M_1 + B'_{1N}Q_{1,N+1}B_{2N}M_2\Sigma_1 = -B'_{1N}Q_{1,N+1}A_N\tilde{L}^1_N \tag{32}$$

$$B'_{2N}Q_{2,N+1}B_{1N}M_1\Sigma_2 + (I + B'_{2N}Q_{2,N+1}B_{2N})M_2 = -B'_{2N}Q_{2,N+1}A_N\tilde{L}^2_N \tag{33}$$

and where Σ_1, Σ_2 are known matrices which satisfy

$$E\left[z_{jN}|z_{iN}\right] = \Sigma_i z_{iN}, \qquad i = 1,2 \tag{34}$$

(see [9] for the notion of the canonical correlation coefficients). It should be pointed out that the canonical correlation coefficients of $\tilde{I}_{1N}, \tilde{I}_{2N}$ are either 1 or the canonical correlation coefficients of z_{1N}, z_{2N} [see (11), (12)]. In (30) we do not consider $\mu = 1$ since that would correspond to a part of u_{iN} dependent only on $\bar{I}_N$. Also, the z_{1N}, z_{2N} have canonical correlation coefficients strictly less than 1 because v_{1N}, v_{2N} are independent. If the condition of Lemma 1 does not hold, then (29) has either no solution or it has infinitely many which may be nonlinear functions of $\bar{I}_N, z_{iN}$ (see [7]).

To find the $\bar{u}_{iN}$'s we have to study (28). Since we assume that an N-pair exists, (28) must have a solution. Thus $(\bar{u}'_{1N}, \bar{u}'_{2N})'$ must be equal to the product of the pseudoinverse of the left-hand side matrix of (28) by the right-hand side of (28) plus anything in the null space of the matrix on the left-hand side. To avoid this nonuniqueness we assume the invertibility of this matrix; i.e., we assume that the condition of Lemma 1 holds also for $\mu = 1$. Thus, the solution of both (28) and (29) becomes easy under the following assumption.

Assumption N: The matrix (30) is nonsingular for any $\mu = 1$ or equal to any finite product of powers of the canonical correlation coefficients of z_{1N}, z_{2N} (or equivalently $\mu =$ any finite product of powers of the canonical correlation coefficients of $\bar{I}_{1N}, \bar{I}_{2N}$).

In the following we assume that Assumption N holds. Since $u_{iN} = \gamma_{iN}(y_{10},\cdots,y_{1,N-1}, y_{20},\cdots,y_{2,N-1}, y_{iN})$, the $\bar{u}_{iN}$'s and $\hat{u}_{iN}$'s can be expressed as functions of the same y_{ik}'s. Considering (8)–(14) we obtain

$$\bar{u}_{iN} = \phi^i(y_{10},\cdots,y_{1,N-1}, y_{20},\cdots,y_{2,N-1}) \tag{35}$$

$$\hat{u}_{iN} = M_i y_{iN} + \psi^i(y_{10},\cdots,y_{1,N-1}, y_{20},\cdots,y_{2,N-1}) \tag{36}$$

for some ϕ^i's, ψ^i's which have to satisfy

$$\begin{bmatrix} \phi^1 \\ \phi^2 \end{bmatrix} = -\begin{bmatrix} I + B'_{1N}Q_{1,N+1}B_{1N} & B'_{1N}Q_{1,N+1}B_{2N} \\ B'_{2N}Q_{2,N+1}B_{1N} & I + B'_{2N}Q_{2,N+1}B_{2N} \end{bmatrix}^{-1} \cdot \begin{bmatrix} B'_{1N}Q_{1,N+1}A_N L^1_N(u_{ik}, y_{ik}, i=1,2;\ k=0,1,\cdots,N-1) \\ B'_{2N}Q_{2,N+1}A_N L^2_N(u_{ik}, y_{ik}, i=1,2;\ k=0,1,\cdots,N-1) \end{bmatrix} \tag{37}$$

$$\psi^i = L_{in}(y_{ik}, u_{ik}, i=1,2;\ k=0,1,\cdots,N-1). \tag{38}$$

The reason that the dependence of u_{iN} of y_{iN} is known [see (36)] is essentially due to the fact that z_{iN} contains v_{iN} as an additive term and so does y_{iN}, and v_{iN} is independent of all the other primitive random variables. Combining (35) and (36) we obtain

$$u_{iN} = M_i y_{iN} + \psi^i + \phi^i \tag{39}$$

where M_i is known and the ψ^i's, ϕ^i's are functions of I_N satisfying (37), (38).

We are now ready to go one step backwards and use (18), (19) for $x = N-1$. Using (17) for $n = N-1$, substituting x_{N+1}, x_N from (1) and using (39) to express y_{iN} in terms of $x_{N-1}, u_{1,N-1}, u_{2,N-1}$ we obtain

$$J_{i,N-1} = E\{\text{quadratic function of } (u_{1,N-1}, u_{2,N-1}, x_{N-1}, \psi^1 + \phi^1, \psi^2 + \phi^2, w_N, w_{N-1}, v_{1N}, v_{2N}.)\}. \tag{40}$$

The quadratic functions in (40) are positive definite in $u_{i,N-1}$ and their coefficient matrices depend on M_1, M_2. The $\psi^i + \phi^i$ terms are functions of $y_{10},\cdots,y_{1,N-1}, y_{20},\cdots, y_{2,N-1}$ and thus they do not depend on $u_{i,N-1}$. We are thus faced with a situation similar to (2). Applying the same standard result, we obtain the analogs of (21), (22), i.e.,

$$\hat{S}^{N-1}_{11}u_{1,N-1} + \hat{S}^{N-1}_{12}P_{1,N-1}u_{2,N-1} = P_{1,N-1}\hat{L}^1_{N-1}(x_{N-1}, \psi^1 + \phi^1, \psi^2 + \phi^2, w_N, w_{N-1}, v_{1N}, v_{2N}) \tag{41}$$

$$\hat{S}^{N-1}_{21}P_{2,N-1}u_{1,N-1} + \hat{S}^{N-1}_{22}u_{2,N-1} = P_{2,N-1}\hat{L}^2_{N-1}(x_{N-1}, \psi^1 + \phi^1, \psi^2 + \phi^2, w_N, w_{N-1}, v_{1N}, v_{2N}) \tag{42}$$

where the $\hat{S}^{N-1}_{ij}$'s are known matrices depending on M_1, M_2, the $\hat{L}^i_{N-1}$'s are known linear functions, and $P_{i,N-1} = E[\cdot|I_{i,N-1}]$. In (41) and (42) we now substitute the ψ^i, ϕ^i's with their equals from (37) and (38) which depend on $u_{1,N-1}, u_{2,N-1}$. We then group together the terms involving the $u_{i,N-1}$'s and end up with the equation

$$S^{N-1}_{11}u_{1,N-1} + S^{N-2}_{12}P_{1,N-1}u_{2,N-1} = P_{1,N-1}\tilde{L}^1_{N-1}(x_{N-1}, w_N, w_{N-1}, v_{1N}, v_{2N}, u_{10},\cdots,u_{1,N-2}, u_{20},\cdots,u_{2,N-2}, y_{10},\cdots,y_{1,N-1}, y_{20},\cdots,y_{2,N-1}) \tag{43}$$

$$S_{21}^{N-1}P_{2,N-1}u_{1,N-1}+S_{22}^{N-1}u_{2,N-1} = P_{2,N-1}\tilde{L}_{N-1}^{2}(x_{N-1},w_N,w_{N-1},v_{1N},v_{2N}, u_{10},\cdots,u_{1,N-2},u_{20},\cdots,u_{2,N-2}, y_{10},\cdots,y_{1,N-1},y_{20},\cdots,y_{2,N-1}) \quad (44)$$

where the $\tilde{L}_{N-1}^{i}$'s are known linear functions. In (43), (44) we can substitute $y_{i,N-1}$ with their equals from (2), use (10)–(16), and repeat the analysis we did for (21), (22) where the role of $\bar{I}_N$, z_{iN} will be assumed by $\bar{I}_{N-1}$, $z_{i,N-1}$, i.e., we will repeat the steps taken after (22) until (39). The counterpart of Assumption N is as follows.

Assumption $N-1$: The matrix

$$\begin{bmatrix} S_{11}^{N-1} & \mu S_{12}^{N-1} \\ \mu S_{21}^{N-1} & S_{22}^{N-1} \end{bmatrix} \quad (45)$$

is nonsingular for any μ = any finite product of powers of the canonical correlation coefficients of $\bar{I}_{1,N-1}$, $\bar{I}_{2,N-1}$.

Under Assumption $N-1$ we will obtain that $u_{i,N-1}$ is linear in $z_{i,N-1}$ and thus in $y_{i,N-1}$, i.e.,

$$u_{i,N-1}=\tilde{M}_i y_{i,N-1}+(\phi_{N-1}^{i}+\psi_{N-1}^{i}) \cdot(y_{10},\cdots,y_{1,N-2},y_{2,0},\cdots,y_{2,N-2}) \quad (46)$$

where the $\tilde{M}_i$'s are known matrices [recall the discussion between (35)–(39)]. Substituting $u_{1,N-1}$, $u_{2,N-1}$ from (46) into (37), (38) and using the fact that the $v_{1,N-1}$, $v_{2,N-1}$ are independent among themselves and independent of all the $y_{i0},\cdots,y_{i,n-2}$'s we learn the functional dependence of the ϕ^i, ψ^i's on $y_{1,N-1}$, $y_{2,N-1}$ and thus the functional dependence of $u_{1,N}$, u_{2N} on $y_{1,N-1}$, $y_{2,N-1}$. These functional dependencies are linear and the corresponding gains are known. In conclusion, at the end of the analysis at stage $N-1$ we have the following. Under Assumption N and Assumption $N-1$, the $u_{i,N}$'s, $u_{i,N-1}$'s are equal to

$$u_{i,N}=M_{iN}^{1N}y_{iN}+M_{iN}^{1,N-1}y_{1,N-1} + M_{iN}^{2,N-1}y_{2,N-1}+f_{iN}(I_{N-2}) \quad (47)$$

$$u_{i,N-1}=M_{i,N-1}^{i,N-1}y_{i,N-1}+f_{i,N-1}(I_{N-2}) \quad (48)$$

where the matrices M are known. The functions f_{iN}, $f_{i,N-1}$ in (47), (48) have to satisfy conditions corresponding to those that the ϕ^i's, ψ^i's had to satisfy at stage N [see (37), (38)].

We can now continue this process backwards. At stage k, for example, we will impose the following.

Assumption k: The matrix

$$\begin{bmatrix} S_{11}^{k} & \mu S_{12}^{k} \\ \mu S_{21}^{k} & S_{22}^{k} \end{bmatrix} \quad (49)$$

is nonsingular for any μ = any finite product of powers of the canonical correlation coefficients of $\bar{I}_{1,k}$, $\bar{I}_{2k}$ and we will obtain

$$u_{il}=M_{il}y_{il}+M_{il}^{1,l-1}y_{1,l-1}+\cdots+M_{il}^{1,k}y_{1k} + M_{il}^{2,l-1}y_{2,l-1}+\cdots+M_{il}^{2,k}y_{2k}+f_{il}(I_{k-1}) \qquad l=k,k+1,\cdots,N \quad (50)$$

where the M matrices are known and the f_{il}'s have to satisfy certain conditions [recall (37), (38)]. Also, the matrix in (49) depends on the M_{il}^{jm}'s with $l, m \geq k+1$, so that the nonsingularity of (49) can be checked recursively.

It should be clear by now that if Assumption k holds for $k=N, N-1,\cdots,0$, we will have a unique stagewise Nash equilibrium which will have to be linear in the information. It is an easy task to verify that it will also be a Nash equilibrium by checking directly that it satisfies Definition 1; in regard to this, the reader can consult [6]. Let us formalize the above analysis in the following theorem.

Theorem 1: If Assumption k holds for $k=N, N-1,\cdots,0$, then the Nash problem stated in Section II admits a unique solution which has to be linear in the information.

IV. Discussion

In this section we provide several remarks which indicate how one can generalize our analysis to other cases, how to weaken several assumptions, how to relate our results to the work of others and point out several open problems.

Remark 1: The essential aspect of our results is that the solution of the stochastic Nash game with one-step delay observation sharing pattern will amost always exist and be unique and linear in the information. Assumption k will hold for almost any choice of the parameter matrices A, B_{ik}, C_{ik}, Q_{ik}, R_{ik} and even if it does not, by perturbing a little the values of some parameter matrices it can be made to hold. Considering that in a real-life problem the values of the parameter matrices are known within a small error, one can conclude that when solving such a Nash game he can securely restrict himself to solutions linear in the information. If Assumption k fails for some k, then there exists no solution or there exist infinitely many which may be nonlinear functions of the information. This can be demonstrated easily by using the results of [7] where the static version is studied. An important result which holds for the static case is that if there exists a solution, then there will exist a solution linear in the information. The corresponding statement for the dynamic case considered here has not been established, although one would be tempted to conjecture it. The proof of such a conjecture for the dynamic case lies in showing that if Assumption N does not hold, but (29) is solvable for u_{iN} and u_{iN} is chosen to be any (perhaps nonlinear) function of z_{iN} (and thus of y_{iN}) which satisfies (29) and a solution for $u_{i,N-1}$ exists, then a solution for $u_{i,N-1}$ would also exist if u_{iN} had to be chosen linear in z_{iN}. (Using the results of [7] we can see that if Assumption N does not hold, but (29) is solvable for u_{iN}, then there are infinitely many u_{iN}'s which solve (29) and some of them—perhaps infinitely many—are linear in z_{iN}.) Showing this is quite difficult due to the nonlinear dependence of u_{iN} on z_{iN} and thus on $u_{i,N-1}$, which results in $J_{i,N-1}$ being nonlinear in $u_{i,N-1}$ (see also Remark 2).

Remark 2: Using the results of [7], one can analyze (29) completely; i.e., check whether or not it has a solution and if it has, construct all the solutions. If (29) is solvable, but

Assumption N does not hold, then $\hat{u}_{iN}$ will (in general) have the form (25). The assumption that v_{1N}, v_{2N} are independent yields that the canonical correlation coefficients of z_{1N}, z_{2N} are strictly less than 1 and thus the summation with respect to l in (25) will be finite; (inspection of (30) shows that Assumption N will hold as $\mu \to 0$). In (25) we can substitute z_{iN} from (12) and this will yield

$$\hat{u}_{iN} = \sum_{l=1}^{N_1} \sum_k \phi_k^i(y_{10}, \cdots, y_{1,N-1}, y_{20}, \cdots, y_{2,N-1}) \tilde{q}_l^i(y_{iN})$$

where the $\tilde{q}_l^i$'s are known polynomials of the components of y_{iN}. (The reason that the $\tilde{q}_l^i$'s are known is the same as the one stated after (22).) These $\hat{u}_{iN}$'s will assume the role of the $\hat{u}_{iN}$'s in (36) and the resulting $J_{i,N-1}$ (40) will be a polynomial function of $u_{i,N-1}$. (43) and (44) will no longer be linear in $u_{i,N-1}$. The procedure described above is the one that should be followed if one wants to come up with a complete analysis of the Nash game under consideration. For simple cases where N is small, it is feasible to carry it out, but as N increases the situation deteriorates rapidly. Nonetheless, it is the only way to go if one wants to prove or disprove the conjecture mentioned at the end of Remark 1. Also if Assumption N does not hold for $\mu = 1$, then the dependence of $(\bar{u}_{1N}, \bar{u}_{2N})$ on an arbitrary function of $(y_{ik}, i = 1,2,; k = 0,1,\cdots,N-1)$ with range in the null space of the left-hand side matrix in (28) will again create trouble when we try to find the functional dependence of the ϕ^i's [see (35), (37) and discussion following (46)] on $y_{1,N-1}, y_{2,N-1}$.

Remark 3: The calculation of the S_{ij}^k's in (49) can be done recursively, but it is a laborious task. We will not give their forms here, but the interested reader can consult [10], whose notation $\tilde{D}_{ij}(k-1)$ corresponds to our S_{ij}^k.

Remark 4: In order to make legitimate use of Hermite polynomials all we need to require from the primitive random variables x_0, w_k, v_{ik} is that they are Gaussian and such that the $\bar{I}_{iN}$'s have nonsingular covariance matrices. Also, in order to be able to extract the functional dependence of u_{iN} on y_{iN} what we really need is that $\bar{y}_{iN}$ and $\bar{I}_N$ have canonical correlation coefficients strictly less than 1 and in order to extract the functional dependence of u_{iN} on $y_{1,N-1}, y_{2,N-1}$ what we really need is that $\bar{y}_{1,N-1}$ and $\bar{y}_{2,N-1}$ have canonical correlation coefficients strictly less than one. All these are guaranteed by the independence of the x_0, w_k, v_{ik}'s and the nonsingular covariance matrix of each v_{ik}, but obviously we can relax these assumptions and allow interdependencies among the x_0, w_k, v_{ik}'s and demand only that what was stated as sufficient for stage N to hold in the beginning of this remark should hold for all stages. Under this relaxation, Theorem 1 still holds.

Remark 5: The validity of Assumption k is very difficult to check except if we are dealing with a scalar case with small N. A special case where they hold is when $\|(S_{11}^k)^{-1}S_{12}^k(S_{22}^k)^{-1}S_{21}^k\| < 1$($\|\cdot\|$ denotes the usual sup norm). This again is not an easy condition to check for all k's. It is trivial to see that if this norm is strictly less than 1, then the matrix (49) is nonsingular for any $|\mu| \leqslant 1$ and thus Assumption k holds (recall that a canonical correlation coefficient is always in $[0,1]$). This is the condition under which the results of [6] are proven (that $(S_{ii}^k)^{-1}$ exists is easily verifiable).

Remark 6: We mentioned in the Introduction an interpretation of Assumption k. According to this interpretation, the μ's represent the coupling of the information. That the canonical correlation coefficients and their products admit such an interpretation is clear. If Assumption k does not hold for some μ, this means equivalently that $(\mu)^{-2}$ is equal to some eigenvalue of the matrix $(S_{11}^k)^{-1}S_{12}^k(S_{22}^k)^{-1}S_{21}^k$. This matrix can be interpreted as the coupling of the costs to go at stage k, assuming that the players use their optimum strategies at stages $k, k+1, \cdots, N$. Thus, we can interpret Assumption k as demanding that at each stage the coupling of the optimal costs to go is not inversely proportional to some even power of the coupling of the information at this stage.

Remark 7: If we consider that $Q_{1k} = Q_{2k}$, $R_{ik} = I$, $k = 0,1,\cdots,N+1$, then it is easy to see that Assumption k holds for all k's and for all $\mu \in [0,1]$. This yields immediately that the solution of the team problem with one-step delay observation sharing pattern admits a unique solution which has to be linear in the information; see [8].

Remark 8: If $y_{in} = x_n$, $i = 1,2$, $n = 0,1,\cdots,N-1$, then Theorem 1 still holds if we assume that the $x_0, w_0, w_1, \cdots, w_N$ are independent Gaussian random vectors with nonsingular covariance matrices (see Remark 4 above concerning the relaxation of the assumptions on the primitive random variables). Actually, in this case Assumption k simplifies into asking that the matrix in (49) is nonsingular for $\mu = 1$. This result has been previously obtained by Basar in [5].

Remark 9: An obvious but useful extension of our results is to consider that at stage k player 1 knows the values of $u_{20}, \cdots, u_{2,k-1}$ with some noise and player 2 knows the values of $u_{10}, \cdots, u_{1,k-1}$ with some noise. This amounts to augmenting the state space into $\bar{x}_n = (x_n, x_{1n}, x_{2n})$ where $x_{in} = u_{i,n-1}$ and using

$$\bar{y}_{1n} = \begin{bmatrix} C_{1n} & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \bar{x}_n + \begin{bmatrix} v_{1n} \\ \bar{v}_{1n} \end{bmatrix}.$$

By assuming that the $\bar{v}_{in}$'s are independent among themselves and of the x_0, w_k's, v_{ik}'s and that they have nonsingular covariance matrices we are back to our original formulation. Using augmentation of the state space we can also consider cases where at stage k player 1 acquires $C_{ik}u_{j,k-m} + \bar{v}_{1n}$ where $m \geqslant 1$. We thus see that several cases where at each stage noisy versions of previous values of the controls are communicated can be reformulated by appropriate state-space augmentation into the form of the problem studied here.

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