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ROBUST CONTROL DESIGN VIA GAME THEORETIC METHODS

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ABSTRACT

The importance of game theoretic formulations in designing robust controllers has recently resurfaced in conjunction with advances in the H^∞ problem. In the present paper we survey several game theoretic results which are closely related and of potential value to robust control design. Using a game theoretic framework, we propose methods for robust control design in decentralized problems.

1. INTRODUCTION

The efficient design of control laws in the presence of unknown elements has been the subject of many theoretical and experimental studies for a variety of models and real-life problems. The unknown elements can be parameters which are constant or time varying, inputs to the system which are quite often modelled as stochastic processes, or unknown dynamics of the system. Such considerations have permeated control theory from its genesis. A basic characteristic of most work in control theory is that one considers a single objective and thus a single decision maker or controller. The theory of games, on the other hand, is concerned with decision making in the presence of many controllers, each one of which has his own objective and information, which are not identical with those of the others. Game theoretic problems have been studied by numerous researchers, mathematical economists being preeminent here. The multiobjective character of game problems introduced many intricacies, so that game theorists had their hands quite full with the seemingly simpler static models. Nonetheless, there are many theoretical contributions in game theory which dealt with bonafide dynamic problems (see [15-26,31-36,38-42,44-48,52-66]), many of which actually predated corresponding advances of control theory such as dynamic programming, Markovian formulations, adaptive schemes, etc. (see [18,52-53,59-63, 73]). The case of Rufus Isaacs stands out on its own. Isaacs studied dynamic game problems and introduced and used his "tenet of transition," which was popularized later to control theorists through the work of Bellman and is known as dynamic programming. When after years of being classified, Isaac's work was published, it sparked the interest of control theorists, since he was dealing with aerial dogfights, a topic to which control theorists had an interest. The seminal papers of [31-33] on dynamic zero sum and Nash games and [19-21] on dynamic Stackelberg games appeared, and several control theorists manifested an interest in these issues having as a main underlying motivation the control of large sacle decentralized systems.

2. LQ GAMES AND H^∞ CONTROL

A recent control problem which ignited new interest in game theory is the H^∞ robust control design theory. The H^∞ theory of control originated in the early 1980's as a technique for computing stabilizing linear time-invariant feedback control laws to satisfy inequality constraints involving the Bode plots of closed-loop frequency responses. The development of the H^∞ control theory was almost from the outset driven by the need for a theory for robustly-uncertainty-tolerant multivariable feedback control design in order to close the much-lamented "gap" of the 1970's between the theory and practice of control design which had results from the failure of the time-domain "modern" control theory of the era to adequately address robustness issues. In response to concern over this gap, the singular value Bode plot emerged as a useful indicator of multivariable feedback system performance (see, for example, Safonov, Laub and Hartmann [11]), ushering in a "post-modern" era for control theory research in which frequency-domain concepts again played a central role, as they had in the 1940's and 1950's. The origin of the H^∞ optimal control theory is usually traced to Zames [12], who formulated a SISO min-max disturbance sensitivity problem as a complex interpolation problem in the Hardy space H^∞. The MIMO H^∞ sensitivity problem was solved independently by several authors, each using different approaches [13,3,4]. Subsequently, the approach of Safonov and Verma [13], which involved reducing the H^∞ problem to a MIMO Hankel approximation problem, was extended to handle more general H^∞ problems via an embedding technique due to Doyle [7] involving spectral factorization which was closely related to the technique developed by Verma and Jonckheere [14], to handle "mixed sensitivity" H^∞ control problems. With the 1984 results of [7] as the solution to the H^∞ control problem was completely in hand from a theoretical standpoint, but the theory was inelegant and computer implementations of it proved to be both slow and unreliable. H^∞ control research over the next several years focused on reformulating the H^∞ solution equations in an effort to achieve simplficiations that would lead to a reliable computer implementation of the H^∞ control theory, the culmination of these efforts being the two-Riccati H_∞ controller formula derivations of Glover, Limebeer et al. [72,78], based on an extension of the embedding technique of Parrott [71].

The link between H^∞ optimal control and LQ game theory was first noted in the work of Peterson [67], Zhou and Khargonekar [68], Peterson and Zhou [69], but the concepts linking LQ games to robust stabilization in the face of gain-bounded uncertainty actually predate the early 1980's introduction of the term H^∞ into the control theorist's vocbulary by several years. Indeed, the details of the full-state feedback H^∞ control solution are so completely worked out in the 1977 paper of Mageirou and Ho [55], that one must regard much of the subsequent 1980's work concerning the full-state feedback case of H^∞ control as rediscovery.

The key observation in the state feedback H^∞ papers is that the control laws that emerge from LQ games with bounded cost also satisfy a frequency domain inequality which may be interpreted as a bound on the H^∞ norm of the closed-loop transfer function. The H^∞ state-feedback control laws obtained in the state-feedback literature require the solution of a single algebraic Riccati equation of the form

$$0 = PA + A^TP - P(BB' - 1/\gamma^2\, \Gamma\Gamma')P + C'C \qquad (2.1)$$

which is known to also arise in LQ game theory problems.

There are several papers in the literature which deal with LQ

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games. There are many intricate issues pertaining to these problems: Closed loop and open loop solutions are in general distinct even for deterministic formulations and lead to different costs and trajectories. The stochastic versions' solutions are not related to the solutions of the deterministic case via the separation principle. Finite and infinite duration games may differ much more drastically than their single objective counterparts. Nonlinearity or nonuniqueness of solutions depending on the information pattern, enter the picture in a drastic way. For an overall view of the subject, the reader is referred to the appropriate sections of the excellent book by Basar and Olsder [22]. A brief survey of papers dealing with linear quadratic games now follows.

Starr and Ho [31-33] introduces deterministic dynamic continuous time, zero-sum, Nash and Team problems. Both open loop and closed loop strategies are considered and the necessary conditions for the LQ case are carefully derived. This paper provides many insights. The analogous presentation for the Stackelberg case is one in [19-21] where an in-depth analysis of several issues is also provided. Ho, Bryson and Baron [36] study pursuit evasion game in the context of a zero-sum LQ formulation. Bellman and Ho [35] study a pursuit evasion game in a zero-sum LQ context where the one player has noise corrupted measurements. A similar problem where both players have noise corrupted measurements is studied in Ho [34]. Lukes, Ichikawa, and Minamide [38-41,46] study LQ Nash games in a Hilbert space framework. Meyer, Potter, Van Zwieten, Medanic, and Pachter [37,43,47,49-51] study a generalized Riccati equation related to the one arising in zero-sum LQ games of inifnite duration.

Rhodes, Laub [44,45] examine continuous time LQ Nash and zero-sum problems with noisy measurements.

Kriekelis [42] considers the numerical solution of the coupled matrix Riccati equations arriving in LQ Nash games via Newton's method.

The discrete time LQ Nash problem of finite duration under the one step delay observation sharing pattern is resolved in [15-17].

Papavassilopoulos [25-26] provides sufficient conditions for existence of stabilizing solutions of the coupled Riccati equations which appear in LQ deterministic continuous time Nash games.

Mageirou, Jacobson [48-56] study the interplay of solutions of the finite and infinite time zero-sum LQ continuous time games.

It is interesting to notice that a game theoretic approach to robust LQ design was first introduced in 1964 by Dorato and Drenick [30] in a generic framework. Minimax problems for deriving robust controllers were also considered in Witsenhausen, Bertsekas, Salmon [27,28,29], but these papers are remote from the H^∞ formulation. There are two particular papers nonetheless where the formulations used for deriving robust controllers are quite close to the one related to H^∞. They are the papers [54] and [55]. In this context, see also [57] for a theoretical study and algorithms pertaining to the zero-sum Riccati equation. (For further theoretical results on this equation, see in particular [49-51].)

In key breakthrough reported recently by Doyle, Glover et al. [9,8,74], an observer-like technique together with "change of variables" was found to permit the state-feedback H^∞ results to be extended to the case in which the state is not directly available for feedback. These "H^∞ observer" results, like those of [72,77,78], require the solution of <u>two</u> Riccati equations of the form (2.1). This approach has been extended and further developed by a number of authors (e.g., [10,75,76]).

3. <u>GAME THEORY, H^∞ CONTROL AND ε-OPTIMALITY</u>

In this section we briefly discuss the links between game theory, H^∞ optimal control theory and the concept of ε-optimality, originally introduced in the H^∞ control in the seminal paper of Zames [12] in order to deal with constraints requiring admissible control laws to have proper transfer functions.

Let Φ: $U \times W \rightarrow R$ be a function and consider the problems:

$$\inf_u \sup_w \Phi(u,w) = \gamma^* \qquad (3.1)$$

$$\sup_w \inf_u \Phi(u,w) = \gamma_* \qquad (3.2)$$

It always holds that

$$\gamma_* \le \gamma^* ,$$

i.e., the lower value (γ_*) is less or equal than the upper value (γ^*). A pair $(\overline{u}, \overline{w}) \in U \times W$ is called a saddle point if

$$\Phi(\overline{u},w) \le \Phi(\overline{w},\overline{u}) \le \Phi(w,\overline{u}), \quad \forall\ u \in U,\ \forall\ w \in W \qquad (3.3)$$

Such a point does not always exist. There are several conditions known as minimax theorems guaranteeing the existence of saddle points and most of them assume convexity of Φ in u, concavity of Φ in w, some type of continuity (upper, lower) of Φ and convexity compactness of U and W. It is clear that if $(\overline{u},\overline{w})$ is a saddle point, then $\overline{u}$ solves (3.1) and $\overline{y}$ solves (3.2). However, solutions $\tilde{u}$ of (3.1) and $\tilde{w}$ of (3.2), assuming they exist, do not provide a solution to (3.3). The following theorem holds (see [22, 79]):

<u>Theorem</u>. Let U,W be nonempty. A pair $(\overline{u},\overline{w})$ solves (3.3) if and only if $\overline{u}$ solves (3.1), (i.e., the infimum is attained in (3.1), $\overline{w}$ solves (3.2) (i.e., the supremum is attained in (3.2)) and the upper and lower values γ_*, γ^* are equal. The standard H^∞ problem is of the type (3.1) with

$$\Phi(u,w) = \|z\|^2 \qquad (3.4)$$

$$z = Hx + Ev \qquad (3.5)$$

$$\dot{x} = Ax + Bv + \Gamma w \qquad x(0) = 0 \qquad (3.6)$$

$$w \in h_2[0, +\infty) ,$$

$$\|w\|^2 = \int_0^\infty W(t)w(t) \le 1 \qquad (3.7)$$

$$\|z\|^2 = \int_0^\infty Z(t)\, z(t)\ dt . \qquad (3.8)$$

The H^∞ problem is

$$\inf_u \sup_w \|z\|^2 \qquad (3.9)$$

where $\|w\| \le 1$ and $u = u(y)$ is a possibly dynamical control law mapping the measurement $y(t) = Cx(t)+Du(t)$ into the control signal v(t). If one further supposes that the infimum and supremum in (3.9) are achieved, then one may reformulate the problem (3.9) with the aid of a Lagrange multiplier γ^2 as

$$\min_u \max_w \|z\|^2 - \gamma^2 (\|w\|^2 - 1) . \qquad (3.10)$$

In order to solve (3.10) we may try to solve for a saddle point of $||z||^2$. If we find a saddle point we have automatically solved (3.4). Solving for a saddle point of (3.4) leads to a linear quadratic differential game. Indeed, solving the linear quadratic game (3.10) in the usual fashion (e.g., [56,8] one finds that for any $\gamma, T > 0$

$$||z||_T^2 - \gamma^2(||w||_T^2 - 1) = ||v - v_0(x,\gamma)||_T^2$$

$$- \gamma^2(||w - w_0(x,\gamma)||_T^2 - 1) \quad (3.11)$$

where

$$||x||_T \triangleq \left(\int_0^T x'(t)x(t)dt\right)^{1/2} \quad (3.12)$$

and

$$v_0(x(t),t,\gamma) \triangleq -B' P(t,\gamma)x(t) \quad (3.13)$$

$$w_0(x(t),t,\gamma) \triangleq -1/\gamma^2 \Gamma' P(t,\gamma)x(t) \quad (3.14)$$

and $P(t,\gamma)$ is a solution to the Riccati equation (compare with (1.1))

$$-\dot{P} = PA + A'P - P(BB' - 1/\gamma^2 \Gamma\Gamma')P + H'H$$

$$P(T,\gamma) = 0 . \quad (3.15)$$

Evidently, when the admissible measurements include the full-state x, then $\overline{u}(x(t),t,\gamma) = v_0(x(t),t,\gamma)$, $\overline{w}(t) = w_0(x(t),t,\gamma)$ is the desired saddle point and, letting γ^* denote the optimal Lagrange multiplier one has that the optimal H^∞ full state feedback control law is

$$v(t) = \overline{u}(x(t),t,\gamma^*) = -B' P(t,\gamma^*) x(t) . \quad (3.16)$$

The solution to the orioginal H^∞ state feedback control problem may be recovered by examining the existence and behavior of the limiting solution $\lim_{T\to\infty} P(t,\gamma)$ for various values of the scalar parameter γ.

Of course, all of the foregoing supposes that the infimum and supremum are achieved, which in general will not be the case. However, even when a game does not admit an exact solution it may admit an approximate solution which is optimal to within some small ε. This may happen when the infimum or supremum in (3.1) cannot be achieved, but rather can only be approached by admissible controls $u \in U$, $w \in W$. In the case of H^∞ copntrol, one may often compute such ε-optimal solutions by taking γ to be only slightly greater than the optimal γ^*.

Let us examine the concept of ε-optimality further. Assume that there exist $(\tilde{u},\tilde{w} \in U,W)$ such that

$$-\varepsilon + \Phi(\tilde{u},w) \le \Phi(\tilde{u},\tilde{w}) \le \varepsilon + \Phi(u,\tilde{w}) \quad \forall u \in U, \forall w \in W \quad (3.17)$$

for some $\varepsilon > 0$. This means

$$-\varepsilon + \sup_w \Phi(u,w) \le \Phi(\tilde{u},\tilde{w}) \le \varepsilon + \inf_u \Phi(u,\tilde{w}) \quad (3.18)$$

Now,

$$\inf_u \Phi(u,\tilde{w}) \le \sup_w \inf_u \Phi(u,w) \quad (3.19)$$

$$\inf_u \sup_w \Phi(u,w) \le \sup_w \Phi(\tilde{u},w) \quad (3.20)$$

Using (3.19) and (3.20) in (3.18) yields

$$-\varepsilon + \inf_u \sup_w \Phi(u,w) \le \Phi(\tilde{x},\tilde{w}) \le \varepsilon + \sup_w \inf_u \Phi(u,w) \quad (3.21)$$

Using the fact that

$$\gamma_* = \sup_w \inf_u \Phi(u,w) \le \inf_u \sup_w \Phi(u,w) = \gamma^* , \quad (3.22)$$

relation (3.21) yields

$$-\varepsilon + \gamma^* \le \varepsilon \, \gamma_* \quad (3.23)$$

and thus

$$\gamma_* \le \gamma^* \le \gamma_* + 2\varepsilon \quad (3.24)$$

We are now ready to state the following lemma:

<u>Lemma</u>. A $\tilde{u} \in U$ which satisfies (3.17) with some $\tilde{w} \in W$ is within an error of 2ε a solution of

$$\inf_u \sup_w \Phi(u,w) \quad (3.25)$$

□

It is clear that a solution of (3.25) within an error 2ε is not necessarily a solution of (3.17). But if Φ is such that $\tilde{u}$ is a solution of (3.17) for some ε, it is necessarily a solution of (3.25) within an error of 2ε.

4. DECENTRALIZED, MULTIOBJECTIVE H^∞ CONTROL

Having seen the ease with which the full-state feedback H^∞ control problem is solved via linear quadratic game theory, and noting that Doyle et al. [74] have been able to readily extend this approach with H^∞ observers, it is natural to ask if game theory methods can also be successfully employed to solve decentralized and multiobjective H^∞ control problems. As we shall see, this leads to infinite systems of <u>coupled</u> Riccati equations.

<u>A Decentralized Multiobjective Suboptimal Scheme with Constant Output Feedback</u>

Consider the decentralized control system depicted in Fig. 4.1. Having closed-loop transfer function from w to $z = [z_1^T, z_1^T]^T$

$$T_{zw} \triangleq \begin{bmatrix} T_{z_1w} \\ T_{z_2w} \end{bmatrix} . \quad (4.1)$$

Our objective is to design feedbacks $u_1(y_1)$ and $u_2(y_2)$ such that

$$|| T_{z_1w} ||_\infty \le \delta_1 \quad (4.2)$$

$$|| T_{z_2w} ||_\infty \le \delta_2 \quad (4.3)$$

where δ_1 and δ_2 are as small as possible.

Let us suppose that the plant P(s) has state space realization

$$\begin{bmatrix} \dot{x} \\ z_1 \\ z_2 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} A & \Gamma & B_1 & B_2 \\ H_1 & 0 & D_1 & 0 \\ H_2 & 0 & 0 & D_2 \\ C_1 & E_1 & 0 & 0 \\ C_2 & E_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ w \\ u_1 \\ u_2 \end{bmatrix} \quad (4.4)$$

and denote

$$Q_1 \triangleq H_1^T H_1 , \qquad Q_2 \triangleq H_2^T H_2 . \quad (4.5a)$$

Further, we <u>assume</u>, for simplicity, that for i = 1,2

$$D_i^T H_i = 0\ , \qquad D_i^T D_i = I \tag{4.5b}$$

$$E_i^T C_i = 0\ , \qquad E_i^T E_i = I\ . \tag{4.5b}$$

Thus,

$$z_i^T z_i = x^T Q_i x + u^T u\ , \quad (i = 1,2)\ . \tag{4.6}$$

We restrict our attention to possibly dynamical control laws $u_i = u_i(y_i)$, $(i = 1,2)$.

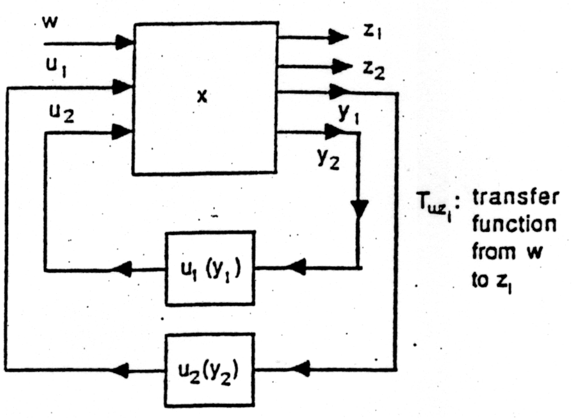

Figure 4.1. Decentralized Control System

If both controllers act as a team, they agree on some relative weights λ_1, λ_2, $0 \le \lambda_i$, $\lambda_1 + \lambda_2 = 1$ and we have the problem

$$\inf_{u_1,u_2} \sup_{w \neq 0} \frac{\lambda_1 \|z_1\|^2 + \|z_2\|^2 \lambda_2}{\|w\|^2} \tag{4.7}$$

(Alternatively, we may view the parameters λ_i as Lagrange multipliers.) If we restrict our attention to linear time invariant dynamic controllers, as long as y_1, y_2 are not equivalent (i.e., $T_1 C_1 = C_2$, $T_2 C_2 = C_1$ for some T_1, T_2) we will be led to infinite dimensional observers for the following reason: If u_2 is realized by using an observer of order K, then the control law u_1 solves

$$\inf_{u_1} \sup_{w} \frac{\lambda_1 \|z_1\|^2 + \lambda_2 \|z_2\|^2}{\|w\|^2}$$

with $u_2(y_2)$ fixed. Then u_1 will have to build an observer of order n+k where n = dim (x), i.e., to estimate both x and the estimate of the u_1 controller. Reversing the roles of u_1, u_2 we see that eventually they will have to use infinite dimensional observers. To alleviate this problem we can a priori restrict the observers of u_1, u_2 to be of fixed dimensions, i.e.,

$$\dot{p}_1 = F_1 p_1 + G_1 y_1,\ p_1(0) = 0 \qquad \dot{p}_2 = F_2 p_2 + G_2 y_2,\ p_2(0) = 0$$

$$u_1 = L_1 p_1 + M_1 y_1 \qquad u_2 = L_2 p_2 + M_2 y_2 \tag{4.8}$$

Let $K_1 = (F_1, G_1, L_1, M_1)$, $K_2 = (F_2, G_2, L_2, M_2)$ and consider the infimum in (4.7) to be with respect to K_1, K_2. To solve this problem we consider the associated game

$$\sup_{w} \ \lambda_1 \|z_1\|^2 + \lambda_2 \|z_2\|^2 - \gamma^2 \|w\|^2\ , \quad K_1, K_2 \text{ fixed} \tag{4.9}$$

$$\inf_{K_1,K_2} \ \lambda_1 \|z_1\|^2 + \lambda_2 \|z_2\|^2\ , \quad w \text{ fixed} \tag{4.10}$$

(4.9) is a standard LQ problem with state (x, P_1, P_2). We assume that the closed-loop $\tilde{A}$-matrix of this problem

$$\tilde{A} = \begin{bmatrix} A + B_1 M_1 C_1 + B_2 M_2 C_2 & B_1 L_1 & B_2 L_2 \\ G_1 C_1 & F_1 & 0 \\ G_2 C_2 & 0 & F_2 \end{bmatrix} \tag{4.11}$$

is asymptotically stable and that γ^2 is such that the Riccati equation associated with the maximization over w, viz.

$$0 = P\tilde{A} + \tilde{A} + \tilde{Q} + 1/\gamma^2 P\tilde{\Gamma}\tilde{\Gamma}' P\ , \tag{4.12}$$

has a positive definite solution, where

$$\tilde{\Gamma} = \begin{bmatrix} \Gamma \\ 0 \\ 0 \end{bmatrix}, \quad \tilde{Q} = \begin{bmatrix} \lambda_1 Q_1 + \lambda_2 Q_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} +$$

$$+ \begin{bmatrix} C'_1 M'_1 M_1 C_1 & C'_1 M'_1 L_1 & 0 \\ L'_1 M_1 C_1 & L'_1 L_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} +$$

$$+ \begin{bmatrix} C'_2 M'_2 M_2 C_2 & 0 & 0 \\ 0 & 0 & C'_2 M'_2 L_2 \\ 0 & L'_2 M'_2 C_2 & L'_2 L_2 \end{bmatrix}$$

For w fixed equal to $-1/\gamma^2 \tilde{\Gamma}' K$ we solve next

$$\inf_{K_1,K_2} \ \lambda_1 \|z_1\|^2 + \lambda_2 \|z_2\|^2 \tag{4.13}$$

by using the results of [82,83]. This leads to a set of coupled Riccati equations determining K_1, K_2. The study of these equations from a theoretical point of view and their algorithmic solution are obviously of central importance. We anticipate that the interplay of the controllability, observability spaces of the two controllers and the dimensions of the observers will be essential in studying these equations. The resulting γ^2 can be chosen as small as possible as to keep the solution P as (4.12) positive definite and it provides an upper bound on the optimal value of (4.7) which uses infinite dimensional observers.

An important feature of the procedure described above is that although each controller uses his own information y_i, in order to find his $K_i = (F_i, G_i, L_i, M_i)$ he has to know not only his own transfer function T_{wz_i}, i.e., his Q_i, but also that of the other. Essentially each controller, in order to find his K_i, has to know the K_i of the other. Although the development presented considers two controllers, the extension to the more than two case is straightforward.

5. A SUBOPTIMAL SCHEME FOR DECENTRALIZED H^∞ CONTROL

The approach outlined in the preceding section requires the solution of coupled Riccati equations. While the solution of such equations is possible (see Richter [84]), software for doing so is not yet commonplace and the available algorithms have not been independently validated. With this in mind, it may be desirable to consider suboptimal alternative approaches to decentralized H^∞ control design.

In what follows, we propose a scheme for creating dynamic feedback controllers for u_1 and u_2, which are of finite order and require very little coordination among u_1, u_2 in order to calculate them. Let us first consider u_1, J_1. Controller 1 thinks of both u_2 and w as a disturbance, i.e.,

$$\dot{x} = Ax + B_1u_1 + \tilde{\Gamma}\tilde{w} \qquad (5.1)$$

where

$$\tilde{\Gamma}(\alpha) \triangleq$$

$$\tilde{w}(\alpha) \triangleq \begin{bmatrix} \alpha_1 u_2 \\ w \end{bmatrix}$$

and α_1 is a constant to be specified later. Note that the augmentation of (Γ, w) to $\tilde{\Gamma}, \tilde{w}$ preserves the properties (4.5a,b,c).

Applying the 2-Riccati H^∞ control results (e.g., [8,10,72,75-78]) yields the result that there exists a control law for which $\|T_{z_1 w}\|_\infty < \gamma$, if and only if the two Riccati equations

$$0 = P_1A + A^TP_1 - P_1(B_1B_1^T - \alpha_1^{-2}B_2B_2^T - \Gamma\Gamma^T)P_1 + H_1^TH_1 \qquad (5.2)$$

$$0 = S_1A^T + AS_1 - S_1(C_1^TC_1 - H_1^TH_1)S_1 + \alpha_1^{-2}B_2B_2^T + \Gamma\Gamma^T \qquad (5.3)$$

have positive semidefinite solutions, say $P_1, S_1 \geq 0$, with $\lambda_{max}(P_1S_1) < 1$. Furthermore, since the optimal closed-loop transfer function satisfies $\|T_{z_1 w}\|_\infty < \gamma$, it follows that with control loop 1 closed with the H^∞ control law that

$$\|z_1\|^2 \leq \gamma_1^2 \left\| \begin{bmatrix} \alpha_1 u_2 \\ w \end{bmatrix} \right\|^2 = \gamma_1^2(\alpha_1^2 \|u_2\|^2 + \|w\|^2) \qquad (5.4)$$

irrespective of how loop 2 is closed.

Similarly, we pick constants α_2, γ_2 and solve the analogous H^∞ control for the second loop $u_2(y_2)$, treating u_1 as part of the disturbance. Letting P_2, S_2 denote the solutions to the appropriate Riccati equations, and provided $P_2, S_2 \geq 0$ and $\lambda_{max}(P_2S_2) < 1$, we have

$$\|z_2\|^2 \leq \gamma_2^2 (\alpha_2^2 \|u_1\|^2 + \|w\|^2) \,. \qquad (5.5)$$

Noting that (4.6) implies that $\|u_2\|^2 \leq \|z_2\|^2$ $(i = 1,2)$, it follows from (5.4)-(5.5) that

$$\begin{bmatrix} \|u_1\|^2 \\ \|u_2\|^2 \end{bmatrix} \leq \begin{bmatrix} \|z_1\|^2 \\ \|z_2\|^2 \end{bmatrix} \leq N \begin{bmatrix} \|u_1\|^2 \\ \|u_2\|^2 \end{bmatrix} + \begin{bmatrix} \gamma_1^2 \\ \gamma_2^2 \end{bmatrix} \|w\|^2 \qquad (5.6)$$

where the matrix N is given by

$$N = \begin{bmatrix} 0 & (\gamma_1\alpha_1)^2 \\ (\gamma_2\alpha_2)^2 & 0 \end{bmatrix}. \qquad (5.7)$$

Rearranging (5.6), we obtain the following lemma.

Lemma. If $\lambda_{max}(N) < 1$, then $(I-N)^{-1}$ exists and is nonnegative and

$$\begin{bmatrix} \|z_1\|^2 \\ \|z_2\|^2 \end{bmatrix} \leq (I-N)^{-1} \begin{bmatrix} \gamma_1^2 \\ \gamma_2^2 \end{bmatrix} \|w\|^2 \qquad (5.8)$$

Proof. Follows directly from (5.6) and the fact that, for any nonnegative matrix N, $(I-N)^{-1} = \sum_{k=0}^{\infty} N^k$ exists and is nonnegative whenever the Perron eigenvalue $\lambda_{max}(N) < 1$.

Q.E.D.

Notice that the only coupling condition is through (5.8), viz., i.e., $1 > \lambda_{max}(N) = (\alpha_1\gamma_1\alpha_2\gamma_2)^2$, which essentially requires a communication among u_1, u_2 about the relative to $\|w\|^2$ energies of u_1, u_2. Notice also that Controller u_1 needs to interact with u_2 only on the choice of the $\alpha_1, \alpha_2, \gamma_1, \gamma_2$ and does not need to know Q_2, C_2 as the Nash solution of Section 1 or the two previous approaches described in the beginning of this section require. Notice that instead of α_1, α_2 one can use nonsingular matrices. Finally, notice that the approach can be extended in a straightforward manner to the case of more than two controllers.

6. CONCLUSIONS

A survey of the linear quadratic game theory literature has revealed that the recent game theoretic formulation of H^∞ control problems has a great deal in common with earlier works of robust control from the game-theoretic framework, especially in the case of the full-state feedback H^∞ control theory. The preliminary studies presented herein indicate that there is also considerable potential for expanding the class of H^∞ robust control problems that can be successfully tackled within the game-theoretic setting. In particular, we have shown that decentralized H^∞ control problems and certain multiobjective H^∞ control problems fall within this class.

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