COOPERATIVE OUTCOMES OF DYNAMIC STOCHASTIC NASH GAMES*

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ABSTRACT

The attainment of cooperative outcomes in competitive environments is an important area of research in dynamic game theory. Most of the work on this topic has concentrated on repeated static games or deterministic dynamic ones. We examine a stochastic dynamic discrete time game and study the possibility of having approximate Nash equilibria which result in Pareto outcomes. The time horizon is infinite. In the context of a simple linear quadratic model we suggest strategies which may achieve the aforementioned aim and discuss several insights and results.

I. INTRODUCTION

The objective of this paper is to consider Nash equilibria for dynamic stochastic games, which result to Pareto costs. In other words, to find strategies which provide each player with the guarantee that his opponents will not deviate from an agreed decision which is a decision that could be realized if all players were behaving cooperatively. Schemes which achieve cooperative outcomes without assuming mutual trust among the decision makers are of obvious significance. They have been examined in the context of repeated games by several researchers, notably Radner see [1,2]. The dynamic case seems to be more complex, but there are already some results pertaining to the deterministic case (see [3-7]). The basic idea in all such attempts is that the Players behave cooperatively as long as they believe that their opponents also did so in the (recent) past. If a deviation is detected they resort to noncooperative behavior, i.e., they use a noncooperative (punishment) mode of behavior. The basic issues in designing such strategies are: i) the test which if passed or failed indicates that the opponent behaved or not cooperatively, ii) the period during which the cooperative or noncooperative mood are retained. A basic characteristic is that the period during which the game evolves should be large enough--in most such problems considered in literature the time horizon is infinite--so that a threat to resort to a punshing behavior will not be overlooked by the opponents since there is always a future during which these punishments will take effect. In this paper we consider an infinite time discrete time stochastic model and for reasons of simplicity we handle only the scalar case. We propose strategies and provide supporting evidence to show that they have the desired characteristics. Our analysis is not complete and occasionally sketchy. A basic feature of the proposed strategies is that the tests, according to which the cooperative or noncooperative behavior is checked, proceed very slowly, i.e., very cautiously.

II. PROBLEM STATEMENT

Consider a dynamical system evolving according to the scalar equation

$$x_{k+1} = ax_k + u_{1k} + u_{2k} + w_k , \qquad k = 0,1,2, \dots \tag{1}$$

where a is a real constant, $x_0, w_0, w_1, w_\ell, \dots$ are i.i.d. Gaussian with zero mean and unit variance. The u_{1k}, u_{2k} are chosen by two decision makers, P1 and P2 respectively as functions of $(x_k, x_{k-1}, \dots, x_0)$, i.e.,

$$u_{ik} : \gamma_{ik}(x_k, x_{k-1}, \dots, x_0) \qquad i = 1,2, \quad k = 0,1,2, \dots$$

where the functions

$$\gamma_{ik} : R^{k+1} \to R$$
$$\gamma_i = (\gamma_{i0}, \gamma_{i1}, \dots)$$

are Borel measurable. Let us also introduce two costs

$$J_i(\gamma_1,\gamma_2) = \limsup_{T\to+\infty} \frac{1}{T+1} E \sum_{k=0}^{T} [q_i x^2_{k+1} + u^2_{ik}] , \quad i = 1,2 . \tag{2}$$

A pair (γ_1^*, γ_2^*) is called a Nash equilibrium if

$$J_1(\gamma_1^*,\gamma_2^*) \le J_1(\gamma_1,\gamma_2^*) , \qquad \forall \text{ admissible } \gamma_1$$
$$J_2(\gamma_1^*,\gamma_2^*) \le J_2(\gamma_1^*,\gamma_2) , \qquad \forall \text{ admissible } \gamma_2$$

A pair $(\bar\gamma_1,\bar\gamma_2)$ is called a Pareto equilibrium if there exists no other pair $(\hat\gamma_1,\hat\gamma_2)$ so that $J_i(\hat\gamma_1,\hat\gamma_2) \le J_i(\bar\gamma_1,\bar\gamma_2)$ for i = 1,2, with strict inequality for at least one i. It is known that there exist constant $\ell_1^N, \ell_2^N, \ell_1^P, \ell_2^P$ so that the pairs (γ_1^N, γ_2^N), (γ_1^P, γ_2^P) with

$$\gamma_1^N = (\gamma_{10}^N, \gamma_{11}^N, \gamma_{12}^N, \dots), \quad \gamma_{1k}^N = \ell_1^N x_k, \quad \forall\, i = 0,1,2, \dots$$
$$\gamma_2^N = (\gamma_{20}^N, \gamma_{21}^N, \gamma_{22}^N, \dots), \quad \gamma_{2k}^N = \ell_2^N x_k, \quad \forall\, i = 0,1,2, \dots$$
$$\gamma_1^P = (\gamma_{10}^P, \gamma_{11}^P, \gamma_{12}^P, \dots), \quad \gamma_{1k}^P = \ell_1^P x, \quad \forall\, i = 0,1,2, \dots$$
$$\gamma_2^P = (\gamma_{20}^P, \gamma_{21}^P, \gamma_{22}^P, \dots), \quad \gamma_{2k}^P = \ell_2^P x_k, \quad \forall\, i = 0,1,2, \dots$$

are stationary Nash and Pareto equilibria, see [10]. The pair (ℓ_1^N, ℓ_2^N) is uniquely determined, see [10], whereas there are infinitely many pairs (ℓ_1^P, ℓ_2^P) determining stationary Pareto equilibria. Let (J_1^N, J_2^N), (J_1^P, J_2^P) denote the corresponding costs. Let us consider a pair (ℓ_1^P, ℓ_2^P) which determines a Pareto solution for which it also holds

$$J_1^P \le J_1^N \qquad J_2^P \le J_2^N .$$

We are interested in finding a pair $(\tilde\gamma_1, \tilde\gamma_2)$, $\tilde\gamma_i = (\tilde\gamma_{i0}, \tilde\gamma_{i1}, \tilde\gamma_{i2}, \dots)$ so that

$$J_i(\tilde\gamma_1, \tilde\gamma_2) \le J_i^P + \varepsilon \qquad i = 1,2$$

and

$$J_1(\tilde\gamma_1, \tilde\gamma_2) \le J_1(\tilde\gamma_1, \gamma_2) + \varepsilon , \quad \forall \text{ admissible } \gamma_2$$
$$J_2(\tilde\gamma_1, \tilde\gamma_2) \le J_2(\gamma_1, \tilde\gamma_2) + \varepsilon , \quad \forall \text{ admissible } \gamma_1 \tag{3}$$

where ε is some small nonnegative constant, i.e., we are interested in finding strategies which are ε-Nash equilibria, see [2], resulting in costs close to the Pareto set of costs. Obviously, if such a pair exists it will involve nonlinear $\tilde\gamma_{ik}$'s. In the following we are going to suggest such a pair.

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III. PROPOSED STRATEGIES

Consider the following equations

$$x_{k+1} = ax_k + u_{1k} + u_{2k} + w_k$$

$$c^1_{k+1} = c^1_k + \frac{\delta_1}{\sum_{i=0}^{k} x_i^2} (x_{k+1} - ax_k - u_{1k} - c^1_k x_k) \cdot x_k$$

$$c^2_{k+1} = c^2_k + \frac{\delta_2}{\sum_{i=0}^{k} x_i^2} (x_{k+1} - ax_k - u_{2k} - c^2_k x_k) \cdot x_k \qquad (4)$$

$$u_{1k} = \begin{cases} \ell_1^P x_k, & \text{if } |c_k^1 - \ell_2^P| \le \varepsilon_k^1 \\ \ell_1^N x_k, & \text{if } |c_k^1 - \ell_2^P| > \varepsilon_k^1 \end{cases} \qquad (5)$$

$$u_{2k} = \begin{cases} \ell_2^P x_k, & \text{if } |c_k^2 - \ell_1^P| \le \varepsilon_k^2 \\ \ell_2^N x_k, & \text{if } |c_k^2 - \ell_1^P| > \varepsilon_k^2 \end{cases} \qquad (6)$$

where $\delta_1 \delta_2$ are positive constants and $\varepsilon_k^1, \varepsilon_k^2$ are sequences monotonically decreasing towards zero. Later on we will specify the δ_i's and ε_k^i's, see (9), (10).

The idea underlying this choice is the following. P1 can think that if P2 uses a fixed strategy $u_{2k} = cx_k$, where c is unknown, then P1 is faced with the evolution equation

$$x_{k+1} = ax_k + u_{1k} + cx_k + w_k .$$

The least squares estimate of c is given recursively by

$$\hat{c}_{k+1} = \hat{c}_k + (1/\sum_{i=0}^{k} x_i^2)^{-1} (x_{k+1} - ax_k - u_{1k} - \hat{c}_k x_k) x_k$$

Instead of $(1/\sum_{i=0}^{k} x_i^2)^{-1}$, P1 uses $\delta_1 (\sum_{i=0}^{k} x_i^2)^{-1}$ so that a stochastic approximation type of c' results and this is (4). Then P1 compares the estimate of c to ℓ_2^P and if they are close, then P1 uses his Pareto strategy; if not, he uses his Nash strategy, see (5). If $u_{2k} = \ell_2^P x_2$, then $c_k - \ell_2^P$ is normally distributed with zero mean and variance of order 1/k. Thus, in order to diminish the probability of not passing the test of (5) when u_{2k} is the Pareto solution, the sequence ε_k^1 should have the property

$$\sqrt{k}\ \varepsilon_k^1 \to \infty, \qquad \text{as } k \to +\infty$$

Since $\varepsilon_k^1 \to 0$ we see that ε_k^1 should go to zero slower than $1/\sqrt{k}$. The ε_k^i's are specified in (9).

IV. ANALYSIS

In order to verify that the strategies proposed in (5-6) will be ε-Nash equilibria and result to costs close to the Pareto costs J_1^P, J_2^P we have to verify two things. First, that for u_{2k} fixed as in (6), the u_{1k} defined in (5) is ε-optimal for the problem he is faced with and the resulting cost is close to J_1^P. Similarly, when the roles of P1 and P2 are interchanged. Second, that the system (5-6) results to costs close to the Pareto ones.

For u_{2k} fixed as in (6), P1 is faced with the following control problem

$$x_{k+1} = ax_k + u_{1k} + u_{2k} + w_k$$

$$c^2_{k+1} = c_k^2 + \frac{\delta_2}{\sum_{i=0}^{k} x_i^2} (x_{k+1} - ax_k - u_{2k} - c_k^2 x_k) x_k$$

$$u_{2k} = \begin{cases} \ell_2^P x_k, & \text{if } |c_k^2 - \ell_1^P| \le \varepsilon_k^2 \\ \ell_1^N x_k, & \text{if } |c_k^2 - \ell_1^P| > \varepsilon_k^2 \end{cases}$$

$$J_1 = \limsup_{T\to+\infty} \frac{1}{T+1} E \sum_{k=0}^{T} [q_1 x_{k+1}^2 + u_{1k}^2] \qquad (7)$$

This is a difficult nonlinear, nonstationary stochastic control problem, the solution of which currently eludes us. We will nevertheless show that if u_{1k} is restricted to being linear in x_k with a gain which is time varying but periodic, then the best gain for u_{1k} is the constant ℓ_1^P. If such a periodic strategy is used by P1 it should be such that P2 will be creating -- by using (4) -- an estimate c_k^2 which converges to ℓ_1^P. If not, c_k^2 would converge to something different than ℓ_1^P; and thus P2 would end up using $u_{2k} = \ell_2^N x_k$, which would force P1 to use $\ell_1^N x_k$ with resulting cost to him equal to J_1^N. On the other hand, P1 can guarantee to himself the cost J_1^P by playing always $\ell_1^P x_k$ when faced with the problem (7). Thus, we have to show that out of all the linear time varying periodic control laws, which result in an estimate c_k^2 which converges to ℓ_1^P, the one that results to the best cost J_1 (see (7)) is the constant $u_{1k} = \ell_1^P x_k$. This is shown in Appendix A.

The second thing that has to be shown is that the system (4-6) will result to costs close to the Pareto costs. This means that the mutual tests of (5-6) will be met successfully or that the probability that although both P1 and P2 use the cooperative strategies ℓ_i^P for a certain period, the noise w_k causes failure of the tests, so that both P1, P2 are locked afterwards in a noncooperative mood of play (i.e., the ℓ_i^N's), is very small. These considerations will lead to further specifications of the δ_i's, ε_k^i's.

If the test is continuously passed then it will be that $u_{ik} = \ell_2^P x_k$. Setting $z_k^i = c_k^i - \ell_i^P$, we see that z_k^i satisfies:

$$z_{k+1}^i = z_k^i \left(1 - \frac{\gamma_i x_k^2}{\sum_{i=0}^{k} x_i^2}\right) + \frac{\delta_i}{\sum_{i=0}^{k} x_i^2} x_k w_k$$

The mean of z_k^i is zero and its variance $(\sigma_{i,k})^2 = E[(z_k^i)^2]$, satisfies

$$(\sigma_{i,k+1})^2 \cong (\sigma_{i,k})^2 + 1 - 2\gamma_i \frac{x_k^2}{\sum_{i=0}^{k} x_i^2} + \delta_i^2 \left(\frac{x_k^2}{\sum_{i=0}^{k} x_i^2}\right)^2 + \delta_i^2 \frac{x_k^2}{\left(\sum_{i=0}^{k} x_i^2\right)}$$

or

$$(\sigma_{i,k+1})^2 \cong (1 - \frac{2\delta_i}{k})\sigma_{ik}^2 + \frac{\delta_i^2}{k^2\sigma^2} + \frac{\delta_i^2}{k^2\sigma^2}$$

$$\cong (\sigma_{i,k})^2(1 - \frac{2\delta_i}{k}) + \frac{\delta_i}{k^2\sigma^2}$$

where $\sigma^2 = \lim E[x_k^2]$. Using Chung's Lemma, see [8, page 45], we have that if

$$2\delta_i > 1 \text{ then } (\sigma_{i,k})^2 \cong \frac{\delta_i^2}{\sigma_2}\frac{1}{2\delta_i - 1}\cdot\frac{1}{k} + o(\frac{1}{k}) \qquad (8)$$

$$1 > 2\delta_i > 0 \text{ then } (\sigma_{i,k})^2 \cong 0\left(\frac{1}{k^{\delta_i}}\right)$$

Using the fact that for any $m > 1$, there is a $B_m > 0$ so that

$$e^{-\frac{1}{2}y^2} < \frac{B_m}{y^m}, \quad y > 0$$

we have

$$\Pr[\,|z_i^k| \le \varepsilon_k^i] = 1 - \Pr[\,|z_i^k| > \varepsilon_k^i] =$$

$$= 1 - 2\frac{1}{\sqrt{2\pi}\,\sigma_{i,k}}\int_{\varepsilon_k^i}^{+\infty} e^{-\frac{1}{2}\frac{z^2}{\sigma_{ik}^2}}dz$$

$$= 1 - \frac{2B_m}{\sqrt{2\pi}}\int_{(\varepsilon_k^i/\sigma_{i,k})}^{+\infty} e^{-\frac{1}{2}z^2}dz > 1 - \frac{2}{\sqrt{2\pi}}\int_{(\varepsilon_k^i/\sigma_{ik})}^{+\infty}\frac{B_m}{y^m}dm$$

$$= 1 - \frac{2B_m}{\sqrt{2\pi}}\frac{1}{m-1}\left(\frac{\sigma_{i,k}}{\varepsilon_k^i}\right)^{m-1}$$

Thus, in order to have that the test will be continuously passed with probability close to 1 it suffices to have for some $p > 0$

$$\frac{2B_m}{\sqrt{2\pi}}\frac{1}{m-1}\left(\frac{\sigma_{i,k}}{\varepsilon_k^i}\right)^{m-1} < \frac{1}{k^p}$$

The larger p is the closer the product

$$\prod_{k=0}^{\infty} P(|z_k^i| < \varepsilon_k^i) > \prod_{k=0}^{\infty}(1 - \frac{1}{k^p})$$

is to one. Using (8), we see that it suffices to have

$$\varepsilon_k^i > {}^{m-1}\!\!\sqrt{\frac{2B_m}{(m-1)\sqrt{2\pi}}}\cdot\frac{\delta_i}{\sigma}\frac{1}{\sqrt{2\delta_i - 1}}\cdot k^{\frac{p}{m-1}-\frac{1}{2}} \qquad \text{if } 2\delta_i > 1$$

$$\varepsilon 2k > k^{\frac{p}{m-1}-\delta_i} \qquad \text{if } 0 < 2\delta_i < 1$$

Since we want $\varepsilon_k^i \to 0$ it suffices to have

$$\frac{p}{m-1} - \frac{1}{2} < 0 \qquad \text{if } 2\delta_i > 1$$

or

$$\frac{p}{m-1} < \delta_i \qquad \text{if } 0 < 2\delta_i < 1$$

Clearly, in either case, $\sqrt{k}\ \varepsilon_k^i = 0(k^{P/m-1}) \to +\infty$ in agreement with the comment at the end of Section 2. We can now take, for example, $p = 2$ and $m > 5$, for example, $m = 6$, which determines B_6 and ε_k^i can be taken to be any sequence which goes to zero and

$$\varepsilon_k^i > {}^{5}\!\sqrt{\frac{2B_6}{5\sqrt{2\pi}}}\ \frac{\delta_i}{\sigma}\ \frac{1}{\sqrt{2\delta_i - 1}}\ \frac{1}{\sqrt[10]{k}}$$

The basic conclusion is that the δ_1, δ_2 must be chosen as constants and the ε_k^1, ε_k^2 can be chosen as sequences which go to zero, while at the same time they satisfy (10) for some p and m. The larger the p, the closer to 1 becomes the probability that the test will be always passed. Obviously p affects the ε in tbe ε-Nash equilibrium that the proposed strategies hopefully satisfy.

APPENDIX A

Consider the equation

$$x_{\ell k+i+1} = a_{i+1}x_{\ell k+i} + w_{\ell k+i} \qquad (A-1)$$
$$k = 0,1,2,3,\ldots$$
$$i = 0,1,2,\ldots,\ell-1$$

where ℓ is a fixed positive integer. This is a dynamical equation where the coefficient a is periodic with period ℓ. The $x_0, w_0, w_0, \ldots$ are iid gaussian with zero mean and unit variance. The mean of x_n is zero for every n and the condition

$$|\,a_1 \ldots a_\ell\,| < 1 \qquad (A-2)$$

guarantees that the variances of the x_n's converge as follows:

$$E[x^2_{\ell k+i+1}] \to \sigma^2_{i+1} \text{ as } k \to +\infty \qquad (A-3)$$

where

$$\begin{bmatrix}\sigma_1^2\\ \cdot\\ \cdot\\ \cdot\\ \sigma_\ell^2\end{bmatrix} = \begin{bmatrix}0 & \cdots\cdots & 0 & a_1^2\\ a_2^2 & & & 0\\ 0 & 0 & & \vdots\\ \vdots & & & \\ 0 & \cdots 0 & a_\ell^2 & 0\end{bmatrix}\begin{bmatrix}\sigma_1^2\\ \cdot\\ \cdot\\ \cdot\\ \sigma_\ell^2\end{bmatrix} + \begin{bmatrix}1\\ 1\\ \cdot\\ \cdot\\ 1\end{bmatrix} \qquad (A-4)$$

If one assumes that the x_n's are generated by a linear time invariant system

$$x_{n+1} = ax_n + w_n \qquad (A-5)$$

then the least squares estimate of a, based on $x_0, \ldots, x_{n+1}$, is given by

$$\hat{a}_n = \frac{\sum_{k=0}^{n} x_{k+1}\cdot x_k}{\sum_{k=0}^{n} x_k^2} \qquad (A-6)$$

Thus, if the x_n's are generated by (A-1), but the estimate operates under the assumption that they are generated by (A-5), then the x_n's generated by (A-1) will be used in (A-6). It then holds

$$\hat{a}_n = \frac{a_1 \sum_k x^2_{\ell k} + a_2 \sum_k x^2_{\ell k+1} + \cdots + a_\ell \sum_k x^2_{\ell k+1\ell-1)}}{\sum_k x_k^2} + \sum_k w_k x_k \qquad \text{(A-7)}$$

where all the summations are done with respect to $k = 0,1,2,\ldots$ and do not include terms after $k=n$. As n goes to infinity, $\hat{a}_n$ converges to

$$a_0 = \frac{a_1 \sigma_\ell^2 + a_2 \sigma_1^2 + \cdots + a_\ell \sigma^2_{\ell-1}}{\sigma_\ell^2 + \sigma_1^2 + \cdots + \sigma^2_{\ell-1}} \qquad \text{(A-8)}$$

If the estimator assumes that the x_n's evolve as in (A-5), he uses the x_n's generated by (A-1) and $a_0 = a$, then his assumption that the a is constant will not be refuted.

Let us now assume that

$$\begin{aligned} a_i &= a_0 + x_i \qquad i = 1, \ldots, \ell \\ \ell_i &= \ell_p + x_i \qquad i = 1, \ldots, \ell \end{aligned} \qquad \text{(A-9)}$$

Consider also the cost

$$J = \lim_{N \to \infty} \frac{1}{N} E[\Sigma\, q\, x_i^2 + \ell_n^2 x_n^2] \qquad \text{(A-10)}$$

where

$$\ell_{\ell k+i} = \ell_{i+1}$$

Assuming that the x_n's evolve according to (A-1), we have

$$J = \frac{1}{\ell} \sum_{i=1}^{\ell} [q + (\ell_p - a_0 + a_i)^2]\, \sigma^2_{i-1} \qquad (\sigma_0 \triangleq \sigma_\ell)$$

$$= \frac{1}{\ell} \{[q + (\ell_p - a_0 + a_1)^2]\, \sigma_\ell^2 + [q + (\ell_p - a_0 + a_2)^2]\, \sigma_1^2 + [q + (\ell_p - a_0 + a_3)^2]\, \sigma_2^2 + \cdots + [q + (\ell_p - a_0 + a_\ell)^2]\, \sigma^2_{\ell-1}\} \qquad \text{(A-11)}$$

(A-11) can be also written as

$$\begin{aligned} J = {} & [q + (\ell_p - a_0)^2] (\sigma_1^2 + \cdots + \sigma_\ell^2) + \\ & + a_1^2 \sigma_\ell^2 + a_2^2 \sigma_1^2 + \cdots + a_\ell^2 \sigma^2_{\ell-1} + \\ & + 2 (\ell_p - a_0) (a_1 \sigma_\ell^2 + a_2 \sigma_1^2 + \cdots + a_\ell \sigma^2_{\ell-1}) \end{aligned} \qquad \text{(A-12)}$$

From (A-8) we have

$$a_1 \sigma_\ell^2 + a_2 \sigma_1^2 + \cdots + a_\ell \sigma^2_{\ell-1} = a_0 (\sigma_1^2 + \cdots + \sigma_\ell^2) \qquad \text{(A-13)}$$

From (A-4) we have

$$\begin{aligned} \sigma_1^2 &= a_1^2 \sigma_\ell^2 + 1 \\ \sigma_2^2 &= a_2^2 \sigma_1^2 + 1 \\ &\vdots \\ \sigma_\ell^2 &= a_\ell^2 \sigma^2_{\ell-1} + 1 \end{aligned} \qquad \text{(A-14)}$$

which we add and obtain

$$(a_1^2 \sigma_\ell^2 + a_2^2 \sigma_1^2 + \cdots + a_\ell^2 \sigma^2_{\ell-1}) = \sigma_1^2 + \cdots + \sigma^2_\ell - \ell \qquad \text{(A-15)}$$

Using (A-12), (A-14) in (A-12) yields

$$\begin{aligned} J &= [q + (\ell_p - a_0)^2 + 2 (\ell_p - a_0)\, a_0 + 1]\, (\sigma_1^2 + \cdots + \sigma_\ell^2) - \ell \\ &= [q + (\ell_p - a_0 + a_0)^2 - a_0^2 + 1]\, (\sigma_1^2 + \cdots + \sigma_\ell^2) - \ell \end{aligned}$$

$$J = [q + (\ell_p^2 + 1 - a_0^2]\, \frac{\sigma_1^2 + \cdots + \sigma_\ell^2}{\ell} - 1. \qquad \text{(A-16)}$$

We can now pose the following problem: Let a_0, ℓ_p, q be fixed numbers, $q \geq 0$, $|a_0| < 1$. Find $x_1, \ldots, x_\ell$, so that the a_i's defined by (A-9) satisfy (A-2), (i.e., stable dynamical system), (A-8) (i.e., the least squares estimate using the x_n's of the periodic system identifies a_0), and J as in (A-16) is minimum.

In Appendix B we solve this problem and show that the optimum is achieved for $x_1 = x_2 = \cdots = x_\ell$, $a_1 = \cdots = a_\ell = a_0$.

APPENDIX B

Consider the optimization problem

$$\min_{x_1, \ldots, x_n, a_1, \ldots, a_n} (x_1 + \cdots + x_n) \qquad \text{(B-1)}$$

subject to

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + e \qquad \text{(B-2)}$$

$$a_0 (x_1 + \cdots + x_n) = a_1 x_n + a_2 x_1 + \cdots + a_n x_{n-1} \qquad \text{(B-3)}$$

$$|a_1 \ldots a_n| < 1 \qquad \text{(B-4)}$$

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & a_1^2 \\ a_2^2 & 0 & & & 0 \\ & a_3^2 & \ddots & & \vdots \\ 0 & & \ddots & a_n^2 & 0 \end{bmatrix}, \qquad e = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \qquad \text{(B-5)}$$

a_0 is a given constant, $|a_0| < 1$. The unknowns are $x_1, \ldots, x_n$, $a_1, \ldots, a_n$. Let us first notice that (B-4) guarantees that the eigenvalues of A are strictly less than 1 in magnitude and thus (B-2) yields solvability for the x_i's in terms of the a_i^2's, i.e.,

$$x = (I - A)^{-1} e = e + Ae + A^2 e + \cdots$$

and that the x_i's will be greater or equal than 1.

Assuming that an optimum exists and applying the first order necessary conditions yields

$$\begin{aligned} 1 + \lambda_1 - \lambda_2 a_2^2 + \rho (a_0 - a_2) &= 0 \\ 1 + \lambda_2 - \lambda_3 a_3^2 + \rho (a_0 - a_3) &= 0 \\ &\vdots \\ 1 + \lambda_n - \lambda_1 a_1^2 + \rho (a_0 - a_1) &= 0 \end{aligned} \qquad \text{(B-6)}$$

$$\begin{aligned} x_n (\rho + 2\lambda_1 a_1) &= 0 \\ x_1 (\rho + 2\lambda_2 a_2) &= 0 \\ &\vdots \\ x_{n-1} (\rho + 2\lambda_n a_n) &= 0 \end{aligned} \qquad \text{(B-7)}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n, \rho$, append the equality constraints (2), (3).

In order to guarantee the existence of the Lagrange multipliers $\lambda_1, \dots, \lambda_n, \rho$ we demonstrate that if

$$\begin{aligned} \lambda_1 - \lambda_2 a_2^2 + \rho (a_0 - a_2) &= 0 \\ \lambda_2 - \lambda_3 a_3^2 + \rho (a_0 - a_3) &= 0 \\ &\vdots \\ \lambda_n - \lambda_1 a_1^2 + \rho (a_0 - a_1) &= 0 \end{aligned} \qquad \text{(B-8)}$$

$$\begin{aligned} x_n (\rho + 2\lambda_1 a_1) &= 0 \\ x_1 (\rho + 2\lambda_2 a_2) &= 0 \\ &\vdots \\ x_{n-1} (\rho + 2\lambda_n a_n) &= 0 \end{aligned} \qquad \text{(B-9)}$$

and (B-2)-(B-4) hold, then $\rho = \lambda_1 = \dots = \lambda_n = 0$. Since $x_i \geq 1$ (B-9) yields

$$\rho + 2\lambda_i a_i = 0, \quad i = 1, \dots, n .$$

If $\rho = 0$ then $\lambda_i a_i = 0$, $i = 1, \dots, n$, and (B-8) yields $\lambda_i = 0$, $i = 1, \dots, n$. If $\rho \neq 0$ then $\lambda_i a_i \neq 0$, $i = 1, \dots, n$, and thus

$$\lambda_i = -\frac{\rho}{2a_i} \qquad i = 1, \dots, n . \qquad \text{(B-10)}$$

(B-8) yields

$$a_{i+1} = 2a_0 - \frac{1}{a_i}, \qquad i = 1, \dots, n \quad (a_{n+1} = a_1) \qquad \text{(B-11)}$$

Using the results of Appendix A, with $\mu = 2a_0$, we conclude that since $|\mu| = 2|a_0| < 1$, we cannot have a periodic solution of (B-11) which also satisfies (B-4). Thus regularity holds and the Lagrange multiplier vector $(\lambda_1, \dots, \lambda_n, \rho)$ exists and is unique.

Let us consider now (B-6) ... (B-7). Since $x_i \geq 1$ (B-7) yield

$$\rho + 2\lambda_i a_i = 0, \quad i = 1, \dots, n .$$

If $\rho \neq 0$, then $\lambda_i a_i = 0$, $i = 1, \dots, n$, and (B-6) yield $\lambda_i = -1$, $i = 1, \dots, n$, and thus $a_1 = \dots = a_n = 0$. In this case it must be $a_0 = 0$. Then it is also true: $x_1 = \dots = x_n = 1$. If $\rho \neq 0$ then $\lambda_i a_i \neq 0$, $i = 1, \dots, n$, and thus

$$a_i = -\frac{\rho}{2\lambda_i}$$

In this case (B-6) yields

$$\left(\frac{2}{\rho}\lambda_i\right) + \left(\frac{2}{\rho}\lambda_{i+1}\right)^{-1} = -\left(\frac{2}{\rho} + \frac{a_0}{2}\right) \qquad \text{(B-12)}$$

Again using the results of Appendix C, since we want $|a_1 \dots a_n| < 1$, the only solution of (B-12) will be $\lambda_1 = \lambda_2 = \dots = \lambda_n$ in which case $a_1 = \dots = a_n = a_0$.

We thus conclude that the only candidate solution of the problem is

$$\begin{aligned} a_1 = \dots = a_n &= a_0 \\ x_1 = \dots = x_n &= (1 - a_0^2)^{-1} \end{aligned}$$

with

$$\lambda_i = -(1 - a_0^2), \quad \rho = 2a_0 (1 - a_0^2)$$

It is easy to see now that since the second order necessary conditions are satisfied the solution found is the global optimum.

APPENDIX C

Consider the difference equation

$$a_{n+1} = \mu - \frac{1}{a_n}, \qquad n = 1,2,3, \dots \qquad \text{(C-1)}$$

where μ and $a_1 \neq 0$ are given cosntants. If $a_n = 0$ for some n, the evolution of (1) stops.

Lemma. The only periodic solution of (1) with period n, which satisfies

$$|a_1 \cdot a_2 \dots a_n| < 1 \qquad \text{(C-2)}$$

is the constant solution $a_1 = a_2 = \dots = a_n$, in which case it must also hold

$$0 < |a_1| < 1 \quad \text{and} \quad \mu = a_1 + \frac{1}{a_1} \neq 0, \; |\mu| > 2 \qquad \text{(C-3)}$$

Proof. The study of (C-1) is equivalent to the study of the linear equation

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = M \begin{bmatrix} x_n \\ y_n \end{bmatrix}, \quad M = \begin{bmatrix} \mu & -1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} a_1 \\ 1 \end{bmatrix} \qquad \text{(C-4)}$$

$n = 1,2,3, \dots$

where

$$a_n = \frac{x_n}{y_n} \qquad \text{(C-5)}$$

The eigenvalues of M are the solutions of

$$\lambda^2 - \mu\lambda + 1 = 0 \qquad \text{(C-6)}$$

and they are

$$\lambda_1 = \frac{\mu + \sqrt{\Delta}}{2}, \qquad \lambda_2 = \frac{\mu + \sqrt{\Delta}}{2}, \qquad \Delta = \mu^2 - 4 . \qquad \text{(C-7)}$$

Clearly $\lambda_1, \lambda_2 \neq 0$.

Case i. $\Delta = 0$, i.e., $\mu^2 = 4$ and $\lambda_1 = \lambda_2 = \mu/2$. Let $\mu = 2$. Then

$$M = T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} T^{-1}, \qquad T = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

$$M^n \cdot T \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} T^{-1} = \begin{bmatrix} n+1 & -n \\ n & -n+1 \end{bmatrix}$$

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = M^n \begin{bmatrix} a_1 \\ 1 \end{bmatrix} = \begin{bmatrix} a_1 (n+1) - n \\ a_1 n - (n-1) \end{bmatrix} \qquad \text{(C-8)}$$

If the solution of (C-1) is periodic, i.e., $a_{n+1} = a_1$ for some n, then $a_1 = x_{n+1}/y_{n+1}$ which yields

$$a_1 = \frac{a_1(n+1) - n}{a_1 n - (n-1)}$$

or

$$n(a_1 - 1)^2 = 0$$

i.e., $a_1 = 1$. With $a_1 = 1$ and $\mu = 2$, (C-1) has the constant solution $a_1 = a_2 = \cdots = a_n = 1$. Similarly, if $\mu = -2$, then the only periodic solution of (C-1) is the constant solution $a_1 = \cdots = a_n = -1$. In either case,

$$| a_1 \dots a_1 | = 1 .$$

<u>Case ii</u>. $\Delta \neq 0$. It holds

$$M = T \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} T^{-1}, \quad T = \begin{bmatrix} 1 & 1 \\ \lambda_2 & \lambda_1 \end{bmatrix}$$

$$M^n = T \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} T^{-1}$$

If we want a_n to be periodic for some n, i.e., $a_{n+1} = a_1$, then

$$a_1 = \frac{x_{n+1}}{y_{n+1}}$$

Using

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = T \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} T^{-1} \begin{bmatrix} a_1 \\ 1 \end{bmatrix} \qquad \text{(C-9)}$$

yields that (C-9) is equivalent to

$$(\lambda_1^n - \lambda_2^n)\,[a_1^2 - \mu a_1 + 1] = 0 \qquad \text{(C-10)}$$

If $\lambda_1^n = \lambda_2^n$ then $\left(\frac{\lambda_1}{\lambda_2}\right)^n = 1$, i.e.,

$$\frac{\mu + \sqrt{\Delta}}{\mu - \sqrt{\Delta}} = e^{i\frac{2k\pi}{n}}, \quad k = 1,2, \dots ,n-1 \qquad \text{(C-11)}$$

The cases k = 0, k = n are excluded since they yield $\lambda_1 = \lambda_2$, i.e., $\Delta = 0$. After some calculations (C-11) is seen to yield

$$\mu = \pm\, 2\cos\left(\frac{k\pi}{n}\right), \quad k = 1,2, \dots ,n-1 .$$

It also holds

$$a_1 a_2 \dots a_n = \frac{x_1}{y_1} \cdot \frac{x_2}{y_2} \cdot \dots \cdot \frac{x_1}{y_n} = x_n$$

since $y_{n+1} = x_n$ and $y_1 = 1$. x_n can be explicitly calculated from (C-9) and it holds

$$x_n = (\lambda_1^n + \lambda_2^n)1/2$$

If $\lambda_1^n = \lambda_2^n$, then $x_n = \lambda_1^n$, $\mu^2 = 4\,[\cos(k\pi/n)]^2$ and thus

$$\lambda_1 = \frac{\mu}{2} + i\sin\frac{k\pi}{n}, \quad |\lambda_1|^2 = \frac{\mu^2}{4} + \sin\left(\frac{k\pi}{n}\right)^2 = 1$$

We thus conclude that if $\Delta \neq 0$ the solution of (C-1) is periodic, and $\lambda_1^n = \lambda_1^n$, then

$$| a_1 a_2 \dots a_n | = 1$$

(C-10) can also be satisfied if $a_1^2 - \mu a_1 + 1 = 0$, i.e.,

$$\mu = a_1 + 1/a_1$$

Then the solution of (C-1) is a constant $a_1 = a_2 = \cdots = a_n$. Thus, the only case where (C-1) has a periodic solution which may satisfy (C-2) is the case where the solution is constant and $\mu = a_1 + 1/a_1$ and $\Delta = \mu^2 - 4 \neq 0$, which is equivalent to

$$(a_1 + 1/a_1)^2 - 4 \neq 0 \quad \text{or} \quad (a_1 - 1/a_1)^2 \neq 0 \quad \text{or} \quad a_1 \neq 1 ..$$

If we want (C-2) to be satisfied we have to have $|a_1| < 1$.

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