ON THE OPTIMAL CHOICE OF MEASUREMENTS IN LINEAR QUADRATIC GAUSSIAN TEAM PROBLEMS

George P. Papavassilopoulos
Department of Electrical Engineering - Systems
SAL 300
University of Southern California
Los Angeles, California 90089

Abstract

The problem of optimal choice of information for some simple linear quadratic gaussian team problems is considered. The unknowns to be chosen subject to constraints are the matrices involved in the linear measurements available to the decision makers. For several types of such problems, characterizations of the best choices of these matrices are given and several results illustrating the meaning of these characterizations and ways for finding the optimal choices are also presented.

1. Introduction

The purpose of this paper is to examine the problem of optimal choice of information in a simple stochastic team set-up. Most of the papers which consider stochastic optimization problems, assume that the information used is provided by given measurement devices and try to characterize or find the optimal control laws. Nonetheless, the information to be used--or the measuring devices--is quite often subject to the choice of the decision makers, and although the decision makers would wish to have all the possible information available, it could oftentimes be an impossible burden to collect and (or) process it. Thus, a decision maker is obliged to choose, subject to constraints, what information he will use. The choice of information might be a nontrivial problem, because the objective to be achieved does not always reveal in a straightforward manner what the essentially needed information is. Having such considerations in mind we formulate some problems related to the choice of information. Our objective functions are quadratic, the measurements linear and the random variables involved gaussian. We try to characterize the best choice of information subject to restrictions which usually assume the form of upper bounds on the rank of some matrices and can be interpreted as restrictions on the number of linearly independent measurements available to the decision makers. We also impose occasionally the condition that the information of two different decision makers are orthogonal.

Problems related to the optimal choice of information have been previously considered in several papers as for example in [1, 2, 3, 4, 5]. In [1] the problem of finding the best measurement in order to achieve the minimum possible value of a quadratic cost is considered for a static problem. In contrast with [2, 3, 4] where the cost is the covariance of the error of the Kalman estimate, [1] considers an arbitrary quadratic cost. Our framework is very similar to the one of [1], but our attention is directed to the case where there are more than one decision makers, i.e., we deal with a team problem. In [1], an algorithm for finding the optimal choice of information for the case of two or more decision makers is suggested but as is pointed out in [1] and also demonstrated by an example in Section 3, this algorithm might fail to converge to the global solution; this is essentially due to the nonconvex character of the underlying optimization problem. In the present paper we characterize the optimal choice of information in terms of "generalized type eigenvalue" problems, which at present seem quite nontrivial to solve in their generality. Some examples are also considered in order to elaborate on the difficulties associated with solving such problems. Some of the work presented have appeared in a preliminary form in [5].

2. Problem Statement

Let x be a Gaussian random vector in R^n with zero mean and unit variance. The measurements y_1, y_2 are defined by

$$\begin{aligned} y_1 &= C_1 x \\ y_2 &= C_2 x \end{aligned} \tag{1}$$

where C_1, C_2 are real constant matrices of dimensions $r_1 \times n$, $r_2 \times n$ respectively. Let $\gamma_1 : R^{r_1} \to R^{m_1}$, $\gamma_2 : R^{r_2} \to R^{m_2}$ be two functions and set

$$\begin{aligned} u_1 &= \gamma_1(y_1) \\ u_2 &= \gamma_2(y_2) \ . \end{aligned} \tag{2}$$

γ_1, γ_2 are chosen as to minimize the cost

$$J(\gamma_1, \gamma_2) = E\Big[\tfrac{1}{2}u_1'u_1 + \tfrac{1}{2}u_2'u_2 + u_1'Ru_2 + u_1'S_1x + u_2'S_2x\Big] \tag{3}$$

The matrices R, S_1, S_2 are real, constant, with appropriate dimensions and it holds

$$\begin{bmatrix} I & R \\ R' & I \end{bmatrix} > 0 \tag{4}$$

i.e., J is strictly convex in (u_1, u_2). Of course, γ_1, γ_2 have to satisfy the appropriate measurability assumptions. It is known that the pair (γ_1^*, γ_2^*) which minimizes (3) exists and is of the form

$$\gamma_1^*(y_1) = L_1 y_1 \ , \qquad \gamma_2^*(y_2) = L_2 y_2 \tag{5}$$

where L_1, L_2 are two matrices satisfying the system of equations

$$L_1C_1 + RL_2C_2C_1'(C_1C_1')^+C_1 + S_1C_1'(C_1C_1')^+C_1 = 0 \tag{6}$$

$$L_2C_2 + R'L_1C_1C_2'(C_2C_2')^+C_2 + S_2C_2'(C_2C_2')^+C_2 = 0 \tag{7}$$

(+ denotes the pseudoinverse). Equations (6) and (7) can be solved uniquely for L_1C_1, L_2C_2. The optimal cost J^* is uniquely determined and if we consider that R, S_1, S_2 are fixed, J^* can be considered as a function of C_1 and C_2, i.e.,:

$$J^* = \hat{J}(C_1, C_2) \tag{8}$$

Thus one is motivated to consider problems of the form

$$\text{minimize } \hat{J}(C_1, C_2)$$
$$\text{subject to restrictions on } C_1, C_2 \tag{9}$$

i.e., to consider what is the best choice of information in the sense that the resulting optimal cost is as small as possible.

3. Case 1: One Decision Maker

The results presented here overlap to some extent with those in [1], but we include them for reasons of completeness. In the case of one decision maker, the cost and the measurements are given by

$$J = E[\tfrac{1}{2}u'u + u'Sx] \qquad y = Cx \qquad u = \gamma(y) \tag{10}$$

The optimal u is given by

$$u = -SPx = Ly \tag{11}$$

where P is the projection matrix which projects on the range of C', i.e.,

$$P = C'(CC')^{+}C', \qquad L = -SC'(CC')^{+} \tag{12}$$

The optimal cost is given by

$$J^* = \hat{J}(C) = E[-\tfrac{1}{2}x'PS'SPx] = -\tfrac{1}{2}\text{tr}[PS'SP] = -\tfrac{1}{2}\text{tr}[S'SP] \tag{13}$$

Let us first consider the problem of minimizing $\hat{J}(C)$ subject to the restriction that no more than ρ linearly independent measurements should be available, or equivalently rank $C \le \rho$, or equivalently rank $P \le \rho$. Formally:

$$\max \text{tr}[S'SP]$$
$$\text{subject to: rank } P \le \rho \tag{14}$$

where $\rho \le n$. The solution of Problem (15) is known and can be obtained by taking P to be the projection matrix which projects on the space spanned by the ρ eigenvectors which correspond to the ρ largest eigenvalues of S'S.

A slight generalization of the problem considered above is the following: C can be chosen as any matrix with maximal rank ρ but it has to be of the form

$$C = TC_0 \tag{15}$$

for some matrix T, where C_0 is a given matrix with rank greater or equal than ρ; i.e., we essentially have to choose the best ρ dimensional subspace lying within the space range C_0'. It is easy to see that if P_0 is the projection matrix on the range of C_0, this problem can be equivalently stated as

$$\max \text{tr}[P_0S'SP_0 \cdot P] \tag{16}$$
$$\text{subject to rank } P \le \rho$$

and can be solved in a similar fashion.

In the formulations given above, we assumed that at most ρ linearly independent measurements of x can be obtained, each one of them with perfect accuracy. If we want to consider the case where we can make at most ρ linearly independent measurements, but there is a fixed measurement noise, we need a different formulation. Namely, let

$$x = \begin{bmatrix} x_1 \\ --- \\ v \end{bmatrix}, \quad S = [S_1 \mid 0], \quad Sx = S_1x_1 \tag{17}$$

$$C = [C_1 \mid I] \qquad y = C_1x_1 + v \tag{18}$$

Only x_1 appears in the cost and v represents the noise in the measurements. For fixed C_1, the optimal cost is given by (see also (15))

$$\hat{J}(C) = -\tfrac{1}{2}\text{tr}[S_1'S_1C_1'(I + C_1C_1')^{-1}C_1] \tag{19}$$

We are going to impose two constraints on C_1. The first one is rank $C_1 \le \rho$. The second one is $\|C_1\| \le a$, i.e., a magnitude constraint on C_1, where a is some fixed positive constant. Such a magnitude constraint would represent no restriction to the previously considered cases, since what matters there is only the range of C'. Thus we want to solve

$$\max \text{tr}[S_1'S_1 C_1'(I + C_1C_1')^{-1}C_1]$$
$$\text{subject to: rank } C_1 \le \rho \tag{20}$$
$$\|C_1\| \le a$$

(We employ the equal Euclidean norm for vectors and the sup norm for matrices.) To solve (20) we proceed as follows. Let

$$C_1 = V_1'\Sigma V_1', \quad \Sigma = \begin{bmatrix} \sigma_1 & & & & & & 0 \\ & \sigma_2 & & & & & \\ & & \ddots & & & & \\ & & & \sigma_\rho & & & \\ & & & & 0 & & \\ 0 & & & & & \ddots & \\ & & & & & & 0 \end{bmatrix},$$

$$\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_\rho \ge 0 \tag{21}$$

be the singular value decomposition of C_1 where U_1, V_1 are square unitary matrices. (20) can be written equivalently

$$\max \text{tr}\, V_1'S_1S_1'V_1 \,\text{diag}\left(\frac{\sigma_1^2}{1+\sigma_1^2}, \ldots, \frac{\sigma_\rho^2}{1+\sigma_\rho^2}, 0, \ldots, 0\right) \tag{22}$$

$$\text{subject to: } V: \text{unitary}$$
$$0 \le \sigma_1 < a, \; 0 \le \sigma_2 \le a, \ldots, 0 \le \sigma_\rho \le a$$

We obviously choose $\sigma_1 = \sigma_2 = \cdots = \sigma_\rho = a$. We also set

$$V_1 = [V_{11} \mid V_{12}]$$
$$V_{11} = [v_1 \, v_2 \cdots v_\rho] \tag{23}$$

and we thus obtain

$$\max \text{tr}\, V_{11}'S_1'S_1V_{11}$$
$$\text{subject to: } V_{11}'V_{11} = I \tag{24}$$

To solve (24), we just choose $v_1, v_2, \ldots, v_\rho$ to be the (orthonormalized) eigenvectors of $S_1'S_1$ corresponding to its ρ largest eigenvalues $\lambda_1, \ldots, \lambda_\rho$. Having found

V_{11} and Σ we can go back to (21) and construct C, using any arbitrary U_1. V_{12} can be taken to be any matrix which adjoined to V_{11} yields V_1 unitary, but its choice does not matter as V'_{12} will be nullified when V'_1 is multiplied by Σ. Notice that the optimal C is

$$C_1^* = \frac{a^2}{1+a^2} U V'_1 \tag{25}$$

and the optimal cost

$$\begin{aligned} J^*(C_1^*) &= -\tfrac{1}{2} \frac{a^2}{1+a^2} \operatorname{tr} V'_{11} S_1 S'_1 V_{11} \\ &= -\tfrac{1}{2} \frac{a^2}{1+a^2} (\lambda_1 + \lambda_2 + \cdots + \lambda_\rho) \end{aligned} \tag{26}$$

If there is no constraint on C, i.e., $a \to +\infty$ we obtain

$$C_1^* = V V'_1$$

$$J^*(C^*) = -\tfrac{1}{2}(\lambda_1 + \lambda_2 + \cdots + \lambda_\rho)$$

in agreement with the results concerning the problem (15), (16) where C_1, S_1 play now the role of C, S.

Finally notice that the interesting feature of this problem is the separation between the magnitude and rank constraints on C_1.

4. Case 2: Two Decision Makers with Restricted Number of Measurements

In this section we consider the problem

$$\begin{aligned} &\text{minimize } \hat{J}(C_1, C_2) \\ &\text{subject to: } \operatorname{rank}(C_1) \le \rho_1 \\ &\qquad\qquad\quad \operatorname{rank}(C_2) \le \rho_2 \end{aligned} \tag{27}$$

The rank condition on C_i represents the inability of the decision maker i to acquire or process more than ρ_i measurements. If $m_i \le \rho_i$, then the constraint rank $(C_i) \le \rho_i$ can be deleted, since the decision maker i does not really need more than m_i measurements. We can thus assume that $\rho_i \le n$, $\rho_i \le m_i - 1$. We can also substitute the inequality constraints in (35) with equality constraints, since more information does not hurt. For fixed C_1, C_2, the optimal γ_1^*, γ_2^* which minimize $J(\gamma_1, \gamma_2)$ are given by (5)-(7). Multiplying (6) and (7) from the left by $C'_1(C_1C')^{-1}$, $C'_2(C_2C'_2)^{-1}$ respectively, yields

$$L_1 + RL_2C_2C'_1(C_1C'_1)^{-1} + S_1C'_1(C_1C'_1)^{-1} = 0$$

$$L_2 + R'L_1C_1C'_2(C_2C'_2)^{-1} + S_2C'_2(C_2C'_2) = 0$$

We can also assume, without loss of generality, that $C_iC'_i$ = unit, since we can premultiply each y_i by some matrix. Let

$$C_1C'_2 = \Sigma\,(\rho_1 \times \rho_2) \tag{28}$$

We thus have the system

$$L_1 + RL_2\Sigma' + S_1C'_1 = 0 \tag{29}$$

$$L_2 + R'L_1\Sigma + S_2C'_2 = 0 \tag{30}$$

A similar system can be obtained in the case of more than two decision makers. In the case of two decision makers, we can also assume without loss of generality that

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \cdots 0 & 0 \cdots 0 \\ 0 & \sigma_2 & 0 \cdots & \vdots \\ & & \vdots & \vdots \\ & & 0 & \\ 0 & & 0 \;\; \sigma_{\rho_1} & 0 \cdots 0 \end{bmatrix}$$

$$\begin{aligned} &\rho_1 \le \rho_2 \\ &1 \ge \sigma_1 \ge \cdots \ge \sigma_{\rho_1} \ge 0 \end{aligned} \tag{31}$$

(see [6]), so that the system (29), (30) can be easily solved explicitly for L_1, L_2. (Unfortunately, such a simplification is not in general possible for the case of three or more decision makers.)

Let

$$\begin{aligned} L_1 &= [\ell_1, \ldots, \ell_{\rho_1}] \\ L_2 &= [\bar{\ell}_1 \ldots \bar{\ell}_{\rho_2}] \\ C'_1 &= [v_1 \ldots v_{\rho_1}] \\ C'_2 &= [\bar{v}_1 \ldots \bar{v}_{\rho_2}]\,. \end{aligned} \tag{32}$$

(29) and (30) can be written equivalently as

$$\begin{aligned} &\left.\begin{aligned} &\ell_i + \sigma_i R \bar{\ell}_i + S_1 v_i = 0 \\ &\bar{\ell}_i + \sigma_i R' \ell_i + S_2 \bar{v}_i = 0 \end{aligned}\right\} \quad i = 1, 2, \ldots, \rho_1 \\ &\bar{\ell}_j + S_2 \bar{v}_j = 0\,, \qquad j = \rho_1 + 1, \ldots, \rho_2 \end{aligned} \tag{33}$$

We can thus solve for $\ell_i, \bar{\ell}_i$: substitute $u_i = L_i y_i$ in the cost and find

$$\begin{aligned} \hat{J}(C_1, C_2) = &-\tfrac{1}{2} \sum_{i=1}^{\rho_1} \begin{bmatrix} S_1 & v_i \\ S_2 & \bar{v}_i \end{bmatrix}' \begin{bmatrix} I & \sigma_i R \\ \sigma_i R' & I \end{bmatrix}^{-1} \begin{bmatrix} S_1 & v_i \\ S_2 & \bar{v}_i \end{bmatrix} \\ &-\tfrac{1}{2} \sum_{j=\rho_1+1}^{\rho_2} \bar{v}'_j S'_2 S_2 \bar{v}_j \end{aligned} \tag{34}$$

Thus, solving (27) is equivalent to solving the following problem:

$$\underset{\substack{v_1, \ldots, v_{\rho_1} \\ \bar{v}_1, \ldots, \bar{v}_{\rho_2}}}{\text{maximize}} \; \hat{J}(C_1, C_2) \tag{35}$$

$$\begin{aligned} \text{subject to: } &\|v_i\| = 1, \quad i = 1, \ldots, \rho_1 \\ &\|\bar{v}_i\| = 1, \quad i = 1, \ldots, \rho_2 \\ &v'_i v_j = v'_i \bar{v}_j = \bar{v}'_j \bar{v}_j = 0, \quad i \ne j \\ &v'_i \bar{v}_i = \sigma_i, \quad i = 1, 2, \ldots, \rho_1 \end{aligned}$$

The general solution of (35) seems, at present, hard to come by. As an example, let us consider the case where $\rho_1 = \rho_2$, $R = \mu I$, $S_1 = s_1 I$, $S_2 = s_2 I$. Then (43) assumes the form

$$\max \sum_{i=1}^{\rho_1} \frac{s_1^2 + s_2^2 - 2\mu s_1 s_2 \sigma_i^2}{1-\mu^2\sigma_i^2} \tag{36}$$

subject to: $0 \le \sigma_i \le 1, \quad i = 1, \ldots, \rho_1$

Since each term in the summation (36), for $0 \le \sigma_i^2 \le 1$, is a piece of hyperbola, to maximize (36) we have to have:

$$\sigma_i = 1 \text{ if } \frac{s_1^2 + s_2^2 - 2\mu s_1 s_2}{1-\mu^2} \ge s_1^2 + s_2^2$$

Thus if $\mu^2(s_1^2+s_2^2) - 2\mu s_1 s_2 \ge 0$ we can choose $C_1 = C_2$ = any matrix with rank ρ_1.

If $\mu^2(s_1^2+s_2^2) - 2\mu s_1 s_2 < 0$ and $\rho_1+\rho_2 = 2\rho_1 \le n$ we can choose $\sigma_i = 0$, $i = 1, \ldots, \rho_1$ and thus choose C_1, C_2 to be any two matrices of rank $\rho_1 = \rho_2$ such that $C_1 C_2' = 0$. The only difficulty appears if $\rho_1+\rho_2 = 2\rho_1 > n$ and $\rho^2(s_1^2+s_2^2) - 2\rho s_1 s_2 < 0$, since we cannot now have $C_1 C_2' = 0$ and rank C_1 = rank $C_2 = \rho_1 = \rho_2$. A little reflection will persuade the reader that in this case we can choose $C_1' = [v_1, \ldots, v_\rho]$ $C_2' = [\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_\rho]$, where $[v_1, v_2, \ldots, v_{\rho_1}, \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_{n-\rho_1}]$ is any orthonormal basis of R^n and $\bar{v}_{n-\rho_1+1}, \ldots, \bar{v}_{\rho_1}$ have the same span as $v_{n-\rho_1+1}, \ldots, v_{\rho_1}$. As another example, consider the case where R, S_1, S_2 are 2x2 matrices, and $\rho_1 = \rho_2 = 1$. By transforming u_i to $V_i u_i$ where V_1, V_2 are unitary matrices and $R = V_1' \operatorname{diag}(\mu_1, \mu_2) V_2$ is the singular value decomposition of R, with $0 \le \mu_i < 1$ we can consider without loss of generality that $R = \operatorname{diag}(\mu_1, \mu_2)$. Since $v_1, \bar{v}_1$ are vectors on the plane we can set

$$v_1 = \begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix}, \quad \bar{v}_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \quad \sigma_1 = \cos(\varphi - \theta)$$

and solve an unconstrained problem with unknowns φ, θ. (We can actually restrict our attention to the square $-\pi/2 \le \varphi \le \pi/2$.) The objective function of this problem is the sum of two terms, each one of which is a quotient with nominator sums of powers of $\cos\theta$, $\sin\theta$ and denominator $1-\mu_2^2 \cos^2(\varphi-\theta)$.

Let us now consider a different approach for problem (27). Since for given C_i, u_i will be linear in y_i and thus in x (recall also (19)-(21)) we set

$$u_1 = X_1 x \qquad u_2 = X_2 x \tag{37}$$

where X_i is an $m_i \times n$ matrix and consider the equivalent to (35)

$$\min \tilde{J}(X_1, X_2) = \operatorname{tr}\left[\tfrac{1}{2} X_1'X_1 + \tfrac{1}{2} X_2'X_2 + X_1'RX_2 + X_1'S_1 + X_2'S_2\right] \tag{38}$$

subject to: $\operatorname{rank} X_1 \le \rho_1$

$\operatorname{rank} X_2 \le \rho_2$

If X_2 is fixed, the minimization in (38) with respect to X_1 can be carried out by solving

$$\min \operatorname{tr}\left[\tfrac{1}{2}(X_1+RX_2+S_1)'(X_1+RX_2+S_1) - \tfrac{1}{2}(RX_2+S_1)'(RX_2+S_1) + X_2'S_2\right] \tag{39}$$

subject to: $\operatorname{rank} X_1 \le \rho_1$

To solve (39) for X_1 we consider the singular value decomposition of RX_2+S_1:

$$RX_2+S_1 = U_1' \Sigma_1 V_1 \tag{40}$$

where U_1, V_1 are square unitary matrices, and Σ_1 is of the form

$$\Sigma_1 = \begin{bmatrix} \sigma_1^1 & & & O \\ & \sigma_2^1 & & \\ & & \sigma_3^1 & \\ O & & & \ddots \end{bmatrix}_{(m_1 \times n)}, \quad \sigma_1^1 \ge \sigma_2^1 > \cdots > 0 \tag{41}$$

The X_1 that solves (48) is given by

$$\bar{X}_1 = U_1' \bar{\Sigma}_1 V_1 \tag{42}$$

where

$$\bar{\Sigma}_1 = \begin{bmatrix} -\sigma_1^1 & & & & & O \\ & -\sigma_2^1 & & & & \\ & & \ddots & & & \\ & & & -\sigma_{\rho_1}^1 & & \\ & O & & & 0 & \\ & & & & & \ddots \; 0 \end{bmatrix} \tag{43}$$

Similarly, for fixed X_1, to find X_2 we consider the singular value decomposition

$$R'X_1 + S_2 = U_2' \Sigma_2 V_2$$

$$\Sigma_2 = \begin{bmatrix} \sigma_1^2 & & O \\ & \sigma_2^2 & \\ & & \sigma_3^2 \\ O & & & \ddots \end{bmatrix}_{(m_2 \times n)} \qquad \sigma_1^2 \ge \sigma_2^2 \ge \sigma_3^2 > \cdots > 0 \tag{44}$$

and choose

$$\bar{X}_2 = U_2' \Sigma_2 V', \quad \bar{\Sigma}_2 = \begin{bmatrix} -\sigma_1^2 & & & O \\ & -\sigma_2^2 & & \\ & & \ddots & \\ & & & -\sigma_{\rho_2}^2 \\ & O & & & 0 \\ & & & & & \ddots \end{bmatrix} \tag{45}$$

Thus, the problem is reduced to the following: Out of all pairs X_1, X_2 which satisfy (90)-(95) choose the one that results to the minimum value of the objective function in (38). If this pair is (X_1^*, X_2^*) we can choose $C_1 = X_1^*$, $C_2 = X_2^*$ and the L_1, L_2 of (6), (7) can be taken to be unit matrices.

This last way of tackling the problem, although is interesting, does not facilitate very much the solution of the problem, as long as we do not at present know how to find explicitly all the pairs X_1, X_2 which satisfy (40)-(45). It nonetheless suggests an algorithm for generating pairs X_1, X_2

which satisfy (40)-(45), namely the following: For fixed $X_2 = X_2^0$ solve

$$\min \bar{J}(X_1, X_2^0)$$
$$\text{rank } (X_1) \leq \rho_1$$

according to (40)-(43). Let X_1^0 be the solution. Next for fixed $X_1 = X_1^0$, solve

$$\min \bar{J}(X_1^0, X_2)$$
$$\text{rank } X_2 \leq \rho_2$$

according to (44)-(45). Let X_2^1 be the solution. Fix $X_2 = X_2^1$ and generate X_1^1 and so on. This algorithm is essentially the same with the one suggested in [1]. It obviously holds:

$$\bar{J}(X_1^{k+1}, X_2^{k+1}) \leq \bar{J}(X_1^{k+1}, X_2^k) \leq \bar{J}(X_1^k, X_2^k) \leq \cdots \leq \bar{J}(X_1^0, X_2^0)$$

and $\bar{J}$ is bounded from below by the best cost J^{**} corresponding to the case $y_1 = y_2 = x$. It is easy to verify that because $\bar{J}(X_1, X_2)$ is a quadratic and strictly convex function of X_1, X_2, the set of (X_1, X_2) which satisfy $\bar{J}(X_1, X_2) \leq \bar{J}(X_1^0, X_2^0)$ = constant is a compact set, so that the sequence (X_1^k, X_2^k) has necessarily at least one convergent subsequence. Thus the algorithm just described is guaranteed to provide in the limit a pair (X_1, X_2) which satisfies the relations (40)-(43). Unfortunately, this limit is not guaranteed to be the solution of problem (27) or equivalently of (38). [Notice that problem (27) is guaranteed to have a solution as $\bar{J}(C_1, C_2)$ is a continuous function of C_1, C_2, which C_1, C_2 are assumed to be $\rho_i \times n$ matrices, on which a magnitude constraint of the form $\|C_i\| \leq a_i$ $(a_1, a_2 > 0)$ can be imposed without loss of generality, since what matters is only the ranges of C_1', C_2'. Thus (27) can be considered as a problem of minimizing a continuous function subject to compact constraints and thus has a global solution.]

The following example demonstrates that this algorithm might fail to converge to the global solution of the problem.

Example

Let $n = m_1 = m_2 = 2$, $\rho_1 = \rho_2 = 1$

$$R = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}, \quad S_1 = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix}, \quad S_2 = \begin{bmatrix} \bar{s}_1 & 0 \\ 0 & \bar{s}_2 \end{bmatrix}$$

We will consider pairs X_1, X_2 which satisfy (40)-(43) and we will restrict our attention to X_1, X_2 diagonal, i.e.,

$$X_1 = \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}, \quad X_2 = \begin{bmatrix} y_1 & 0 \\ 0 & y_2 \end{bmatrix}$$

(Notice that although nondiagonal X_1, X_2 which satisfy (40)-(43) might exist, if we choose X_2 diagonal, and find X_1 according to (40)-(43), X_1 will also be diagonal.) It holds

$$RX_2 + S_1 = \begin{bmatrix} \mu_1 y_1 + s_1 & 0 \\ 0 & \mu_2 y_2 + s_2 \end{bmatrix}$$

$$R'X_1 + S_2 = \begin{bmatrix} \mu_1 x_1 + \bar{s}_1 & 0 \\ 0 & \mu_2 x_2 + \bar{s}_2 \end{bmatrix}$$

We have the following four cases:

(α) $x_1 = (\mu_1 \bar{s}_1 - s_1)/(1-\mu_1^2)$ $\quad x_2 = 0$

$y_1 = (\mu_1 s_1 - \bar{s}_1)/(1-\mu_1^2)$ $\quad y_2 = 0$

$$\text{if } \begin{cases} |x_1| \geq |s_2| \\ |y_1| \geq |\bar{s}_2| \end{cases}$$

(β) $x_1 = 0$ $\quad x_2 = (\mu_2 \bar{s}_2 - s_2)/(1-\mu_2^2)$

$y_1 = 0$ $\quad y_2 = (\mu_2 s_2 - \bar{s}_2)/(1-\mu_2^2)$

$$\text{if } \begin{cases} |s_1| \leq |x_2| \\ |\bar{s}_1| \leq |y_2| \end{cases}$$

(γ) $x_1 = -s_1$ $\quad x_2 = 0$

$y_1 = 0$ $\quad y_2 = -\bar{s}_2$

$$\text{if } \begin{cases} |s_1| \geq |\mu_2 \bar{s}_2 - s_2) \\ |\mu_1 s_1 - \bar{s}_1| \leq |\bar{s}_2| \end{cases}$$

(δ) $x_1 = 0$ $\quad x_2 = -s_2$

$y_1 = -\bar{s}_1$ $\quad y_2 = 0$

$$\text{if } \begin{cases} |\mu_1 \bar{s}_1 - s_1| \leq |s_2| \\ |\bar{s}_1| \geq |\mu_2 s_2 - \bar{s}_2| \end{cases}$$

It is easy to see that there are choices for $\mu_1, \mu_2, s_1, s_2, \bar{s}_1, \bar{s}_2$ so that more than one of the cases $\alpha, \beta, \gamma, \delta$ are acceptable. For example, if $\mu_1 = \mu_2 = 1/2$, $s_1 = -7/2$, $s_2 = 4$, $\bar{s}_1 = 0$, $\bar{s}_2 = 2$, then both cases γ and δ are acceptable.

Thus the algorithm described might fail to reach the global optimum of (27). Actually, to solve the above mentioned example, within the class of diagonal X_1, X_2, we have to check which out of the four cases $\alpha, \beta, \gamma, \delta$ are acceptable and if more than one is, to calculate the values of $\hat{J}$ at each one of them and choose the one which results to the smallest.

There is a third, slightly different, approach that one could follow for solving the problem. Problem (27) can be written equivalently as follows:

$$\min \text{tr } [\tfrac{1}{2} C_1' L_1' L_1 C_1 + \tfrac{1}{2} C_2' L_2' L_2 C_2 + C_1' L_1' R L_2 C_2 + C_1' L_1' S_1 + C_2' L_2' S_2] \tag{46}$$

$$\text{subject to: } L_1 + R L_2 C_2 C_1' + S_1 C_1' = 0 \tag{46-1}$$

$$L_2 + R' L_1 C_1 C_2' + S_2 C_2' = 0 \tag{46-2}$$

$$C_1C_1' = I(\rho_1 \times \rho_1) \quad (46\text{-}3)$$

$$C_2C_2' = I(\rho_1 \times \rho_2) \quad (46\text{-}4)$$

Notice that L_1, L_2, C_1, C_2 are considered as unknowns. It can be easily verified that a Lagrange multiplier by which we can append the constraints exists, by the following argument: (46-1) and (46-2) are always uniquely solvable for L_1, L_2, given C_1 and C_2 as in (46-3) and (46-4); one can also verify that the full rank condition with respect to the unknowns C_1, C_2 is always fulfilled by the gradient of (46-3) and (46-1); thus the full rank condition is satisfied by the constraints and consequently a Lagrange multiplier exists. We now append the constraints to the objective, take the gradient to be zero and after some calculations we end up with the following necessary conditions that have to be satisfied by L_1, L_2, C_1, C_2:

$$L_1 + RL_2C_2C_1' + S_1C_1' = 0 \quad (47\text{-}1)$$

$$L_2 + R'L_1C_1C_2' + S_2C_2' = 0 \quad (47\text{-}2)$$

$$L_1'(L_1C_1 + RL_2C_2 + S_1) = 0 \quad (47\text{-}3)$$

$$L_2'(L_2'C_2 + R'L_1C_1 + S_2) = 0 \quad (47\text{-}4)$$

$$C_1C_1' = I \quad (47\text{-}5)$$

$$C_2C_2' = I \;. \quad (47\text{-}6)$$

(47-3) can be multiplied from the left by C_1' to yield equivalently

$$C_1'L_1'(L_1C_1 + RL_2C_2 + S_1) = 0 \;. \quad (48)$$

Notice that multiplying (48) from the left by C_1 yields (47-3) because of (47-5). It is obvious now that by setting $X_i = L_iC_i$, (47) can be written equivalently

$$X_1 + RX_2C_1'C_1 + S_1C_1'C_1 = 0 \quad (49\text{-}1)$$

$$X_2 + R'X_1C_2'C_2 + S_2C_2'C_2 = 0 \quad (49\text{-}2)$$

$$X_1'(X_1 + RX_2 + S_1) = 0 \quad (49\text{-}3)$$

$$X_2'(X_2 + R'X_1 + S_2) = 0 \quad (49\text{-}4)$$

$$C_1C_1' = I(\rho_1 \times \rho_1) \quad (49\text{-}5)$$

$$C_2C_2' = I(\rho_2 \times \rho_2) \quad (49\text{-}6)$$

A little reflection will persuade the reader that (49-3) could have been directly derived from (40)-(43) and similarly (49-4) from (44)-(45). One could in principle solve (47-1), (47-2) explicitly for L_1, L_2, plug their values in (47-3), (47-4) and have a system of equations that should be satisfied by $v_1, \ldots, v_{\rho_1}, \bar{v}_1, \ldots, \bar{v}_{\rho_2}$. One could do the same thing with (49-1)-(49-4). (49-1)-(49-2) are easy to solve explicitly for X_1, X_2 under the assumption that the R matrix is square and $R = \mu I$ for some μ: $|\mu| < 1$, in which case (49-1)-(49-6) can be simplified (after some calculations) to the equivalent:

$$[\mu C_1C_2'C_2S_2'S_2C_2' - C_1S_1'S_2C_2'][I - C_2C_1'C_1C_2] = 0 \quad (50)$$

$$[\mu C_2C_1'C_1S_1'S_1C_1' - C_2S_2'S_1C_1'][I - C_1C_2'C_2C_1'] = 0 \;. \quad (51)$$

(50) and (51) characterize the optimal C_1, C_2.

Example

Let x be two-dimensional, $\rho_1 = \rho_2 = 1$ and $R = \mu I$. The unknowns C_1, C_2 can be taken to be

$$C_1 = [\cos\varphi, \sin\varphi], \qquad C_2 = [\cos\theta, \sin\theta] \;.$$

It holds

$$C_1C_2' = \cos(\varphi - \theta)$$

If

$$\cos(\varphi - \theta) \neq \pm 1,$$

(50) and (51) yield a system of two equations with unknowns, φ and θ, which can be simplified to the form

$$\mu \cos(\varphi - \theta) \left\| S_2 \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \right\|^2 = [\cos\varphi, \sin\varphi]\, S_1'S_2 \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \quad (52)$$

$$\left\| S_2 \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \right\|^2 = \left\| S_1 \begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix} \right\|^2 \quad (53)$$

The geometrical meaning of these two conditions is that the vectors

$$S_1 \begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix}, \qquad S_2 \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

have equal lengths and their angle w satisfies

$$\cos w = \mu \cos(\varphi - \theta) \;.$$

The case $\cos(\varphi - \theta) = \pm 1$, i.e., $\varphi = \theta +$ integer multiple of π --or without loss of generality: $\varphi - \theta$ -- can be easily examined separately.

5. Case iii. Two Decision Makers with Restricted Number of Measurements and No Common Information

In this section we consider the problem

$$\min \hat{J}(C_1, C_2)$$

$$\text{subject to:} \quad \operatorname{rank}(C_1) \leq \rho_1, \quad \operatorname{rank}(C_2) \leq \rho_2, \quad C_1C_2' = 0 \quad (54)$$

The meaning of the additional condition $C_1C_2' = 0$ (compare with (27) is that there is no common information between the two decision makers. In this case the term $u_1'Ru_2$ is of no importance, since its expectation is obviously zero. Use of (6)-(7) yields that (54) is equivalent to solving

$$\min \operatorname{tr}[-\tfrac{1}{2} S_1'S_1P_1 - \tfrac{1}{2} S_2'S_2P_2]$$

subject to:

$$P_1, P_2 \text{ projection matrices} \quad (55)$$

$$\operatorname{rank}(P_1) \leq \rho_1, \quad \operatorname{rank}(P_2) \leq \rho_2$$

$$P_1P_2 = 0$$

Recall that:

$$P_i = C_i'(C_iC_i')^+C_i, \quad i = 1,2 \;.$$

(55) is the obvious generalization of (16).

Two main cases can now be considered. The first case is $\rho_1 + \rho_2 \geq n$ and the second one $\rho_1 + \rho_n < n$. The first case is quite easy to solve as the following argument shows: we can take rank (C_1) + rank (C_2) = n in which case $P_2 = I - P_1$, i.e., we have only one unknown. In particular, let

$$\text{rank } (P_1) = n-\ell \ , \quad \text{rank } (P_2) = n - \rho_1 + \ell \ ,$$

where ℓ is an integer satisfying:

$$0 \leq \ell \leq \rho_1 + \rho_2 - n \ .$$

For fixed ℓ (55) assumes the form

$$\max \operatorname{tr} [(S_1'S_1 - S_2'S_2)P_1]$$

$$\text{subject to:} \quad \text{rank } P_1 = n-\ell \leq P_1$$

We obviously choose $\ell = (n-\rho_1)$ and we solve

$$\max \operatorname{tr} [(S_1'S_1 - S_2'S_2)P_1]$$

$$\text{subject to:} \quad \text{rank } (P_1) = \rho_1$$

which falls within the class of problems solved in Section 2 and can be solved, by taking P_1 to be the projection on the space spanned by the ρ_1 eigenvectors of $S_1'S_1 - S_2'S_2$ corresponding to the ρ_1 largest eigenvalues. So, we only need to concentrate on the case $\rho_1 + \rho_2 < n$. P_i can be taken to be

$$P_i = U_i U_i' \ , \qquad i = 1, 2$$

where U_i is an $n \times \rho_i$ matrix with $U_i'U_i$ = unit $\rho_i \times \rho_i$ and $U_1'U_2$ = zero matrix. Problem (55) assumes the form

$$\max U_1'S_1'S_1U_1 + U_2'S_2'S_2U_2$$

$$\begin{aligned} \text{subject to:} \quad U_1'U_1 &= I(\rho_1 \times \rho_1) \\ U_2'U_2 &= I(\rho_2 \times \rho_2) \\ U_1'U_2 &= O(\rho_1 \times \rho_2) \end{aligned} \tag{56}$$

Because of the compactness of the constraint set and the continuity of the objective function, problem (54) has obviously a solution. Appending the constraints with Lagrange multipliers--which exist because the constraints satisfy the full rank condition as can be easily verified--we obtain the following necessary conditions for (54):

$$S_1'S_1U_1 = U_1\Lambda_1 + U_2\Lambda \tag{57}$$

$$S_2'S_2U_2 = U_2\Lambda_2 + U_1\Lambda' \tag{58}$$

$$\begin{aligned} U_1'U_1 &= I \\ U_2'U_2 &= I \\ U_1'U_2 &= 0 \end{aligned} \tag{59}$$

where Λ_1 and Λ_2 are symmetric matrices. Out of all the $U_1, U_2, \Lambda_1, \Lambda_2, \Lambda$ that satisfy (57)-(59) we want the one that yields the maximum falue for

$$\operatorname{tr} (\Lambda_1 + \Lambda_2) \ .$$

(57)-(59) is a "generalized" eigenvalue type of problem. As an example, let us consider the case where $\rho_1 = \rho_2 = 1$, and $S_1'S_1, S_2'S_2$ are 3x3 matrices. Then (57)-(59) reduce to finding vectors v_1, v_2 such that

$$\begin{aligned} S_1'S_1v_1 &= \lambda_1 v_1 + \lambda v_2 \\ S_2'S_2v_2 &= \lambda_2 v_2 + \lambda v_1 \\ v_1'v_1 &= v_2'v_2 = 1 \\ v_1'v_2 &= 0 \end{aligned} \tag{60}$$

For the case where $S_1'S_1, S_2'S_2$ are 3x3 matrices, and $\rho_1 = \rho_2 = 1$, we can solve the problem in the following way. If we knew the vector v_3 which spans the space perpendicular to v_1 and v_2 (i.e., $v_2', v_1 = v_3'v_2 = 0$, $v_3'v_3 = 1$) then we could solve the problem

$$\begin{aligned} &\max \operatorname{tr} (I - P_3)S_1S_1'(I - P_3)P_1 + (I-P_3)S_2S_2'(I-P_3)P_2 \\ &P_1 = v_1v_1', \quad P_2 = v_2v_2' \\ &P_1P_2 = 0 \end{aligned} \tag{61}$$

where the unknowns are v_1, v_2 and $P_3 = v_3v_3'$ is known. This problem can be solved since rank P_1+ rank P_2 = 2 which is equal to the dimension of the space where we are working, i.e., the space where $I - P_3$ project; (recall case with $\rho_1 + \rho_2 = n$). Thus it suffices to be able to find v_3. It holds

$$\operatorname{tr} [S_1'S_1P_1 + S_2'S_2P_2] =$$

$$= \operatorname{tr}[(I-P_3)(S_1'S_1 - S_2'S_2)(I-P_3)P_1 + S_2'S_2(I-P_3)]$$

Thus, we can consider equivalently to (61) the problem:

$$\max_{P_3} \left\{ \operatorname{tr} (S_2'S_2(I-P_3)) + \text{maximum eigenvalue } [(I-P_3)(S_1'S_1 - S_2'S_2)(I-P_3)] \right\} \tag{62}$$

The maximum eigenvalue of $(I-P_3)(S_1'S_1 - S_2'S_2)(I-P_3) = \overline{\lambda}(v_3)$ can be found explicitly, for we are working with 3x3 matrices. Without loss of generality let

$$S_1'S_1 - S_2'S_2 = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}, \quad a_1 \geq a_2 \geq a_3 = 0 \tag{63}$$

$$v_3 = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}, \quad n_1^2 + n_2^2 + n_3^2 = 1 \ .$$

To find $\overline{\lambda}(v_3)$ we have to solve:

$$\det \left[(I - v_3v_3') \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} (I - v_3v_3') - \lambda I \right] = 0$$

or,

$$\lambda \left\{ [a_1(1-n_1^2) - \lambda][a_2(1-n_2^2) - \lambda] - a_1a_2n_1^2n_2^2 \right\} = 0$$

thus

$$\overline{\lambda}(v_3) = \frac{a_1(1-n_1^2) + a_2(1-n_2^2) + \sqrt{[a_1(1-n_1^2) + a_2(1-n_2^2)]^2 - 4a_1a_2n_3^2}}{2}$$

and (86) can be written equivalently as

$$\max_{n_1,n_2,n_3} \left\{ -2\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}' S_2'S_2 \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} + a_1(1-n_1^2)+a_2(1-n_2^2) + \sqrt{[a_1(1-n_1^2)+a_2(1-n_2^2)]^2-4a_1a_2n_3^2} \right\} \quad (64)$$

$$\text{s. to: } n_1^2+n_2^2+n_3^2=1$$

If $a_1=a_2$, then (64) assumes the form

$$\min v_3'S_2'S_2v_3$$

and v_3 will be chosen as the eigenvectors corresponding to the smallest eigenvalue of $S_2'S_2$. If $a_2=0$, then (64) assumes the form

$$\min v_3'S_1'S_1v_3$$

and v_3 will be chosen as the eigenvector corresponding to the smallest eigenvalue of $S_1'S_1$. If $a_1>a_2>a_3$, there is no obvious immediate simplification of (64), which has to be solved as a classical nonlinear programming problem. The only obvious conclusion is that if two of the eigenvalues of $S_1'S_1 - S_2'S_2$ coincide, then it is very easy to solve (64). If this is not the case, we can still exploit this previous conclusion in order to give upper and lower bounds for the optimal value. Let $a_1>a_2>a_3$. Then

$$S_1'S_1 = S_2'S_2+\begin{bmatrix} a_1 & & \\ & a_1 & \\ & & a_3 \end{bmatrix} < S_2'S_2+\begin{bmatrix} a_1 & & \\ & a_1 & \\ & & a_3 \end{bmatrix} = M_1 ,$$

$$S_2'S_2+\begin{bmatrix} a_1 & & \\ & a_2 & \\ & & a_2 \end{bmatrix} = M_2$$

$$S_1'S_1= S_2'S_2+\begin{bmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{bmatrix} > S_2'S_2+\begin{bmatrix} a_2 & & \\ & a_2 & \\ & & a_3 \end{bmatrix} = N_1 ,$$

$$S_2'S_2+\begin{bmatrix} a_1 & & \\ & a_3 & \\ & & a_3 \end{bmatrix} = N_2$$

and

$$\max \operatorname{tr}(S_1'S_1P_1+S_2'S_2P_2) \leq \max \operatorname{tr}(M_iP_1+S_2'S_2P_2) , \quad i = 1,2$$

$$\max \operatorname{tr}(S_1'S_1P_1+S_2'S_2P_2) \geq \max \operatorname{tr}(N_iP_1+S_2'S_2P_2) , \quad i = 1,2 . \quad (65)$$

The maximization problems on the right hand side of (65) can be easily solved to provide upper and lower bounds.

For the case where $S_i'S_i$ has dimension larger than 3 and P_i larger than 1, we can easily generalize several of the results presented and we can similarly create upper and lower bounds for the optimal values of the objective function.

Before completing this section let us consider another class of problems which can be reduced to those considered in this section. Consider the problem

$$\begin{aligned} &\min \hat{J}(C_1, C_2) \\ &\text{subject to:} \quad \operatorname{rank}(C_1) \leq \rho_1 \\ &\qquad \operatorname{rank}(C_2) \leq \rho_2 \\ &\qquad \operatorname{range}(C_1) \subseteq \operatorname{range}(C_2) \end{aligned} \quad (66)$$

In this case

$$P_1P_2 = P_2P_1 = P_1$$

($P_i = C_i (C_iC_i')^+C_i'$) and (6), (7) can be explicitly solved for u_1, u_2, so that $\hat{J}(C_1, C_2)$ can be explicitly calculated and is found to be of the form

$$\hat{J}(C_1, C_2) = -\operatorname{tr}(A_1P_1+ A_2P_2)$$

where A_1, A_2 are known symmetric matrices which depend on R, S_1, S_2. Let

$$\hat{P}_2 = P_2 - P_1 .$$

$\hat{J}$ assumes the form

$$\hat{J} = -\operatorname{tr}[(A_1+ A_2)P_1+ A_2\hat{P}_2]$$

where $P_1\hat{P}_2 = 0$ which shows that if we consider $P_1, \hat{P}_2$ as unknowns we have reduced (66) to the form (55). It should be pointed out that the problem (66) is important in its own, since it is exactly the problem that has to be solved in a two stage dynamic linear quadratic gaussian problem, with no measurement noise, which is characterized by nested information.

6. Case iv. Three Decision Makers with Independent Measurements

Here we consider the case where the cost is given by

$$J = \tfrac{1}{2}(u_1^2+u_2^2+u_3^2) + u_1S_1'x+u_2S_2'x+u_3S_3'x$$

u_1, u_2, u_3 are scalar valued, S_1, S_2, S_3 are fixed fectors in R^3, and x is a gaussian random vector in R^3 with zero mean and unit variance. The measurement available to u_i is y_i where

$$y_i = v_i'x , \quad v_i'v_j = \delta_{ij} , \quad i, j = 1, 2, 3 .$$

Notice that each decision maker's information is independent of the information of the others. The problem of optimal choice of information reduces to solving the following problem

$$\max v_1'S_1S_1'v_1+ v_2'S_2S_2'v_2+ v_3'S_3S_3'v_3 = (S_1'v_1)^2+ (S_2'v_2)^2+ (S_3'v_3)^2 \quad (67)$$

$$\text{subject to: } v_i'v_i = \delta_{ij}$$

The geometrical interpretation of (67) is the following: Given three vectors, S_1, S_2, S_3, find the orthogonal parallelepipedon with maximal diagonal, whose one corner is at the origin and the opposite corner's three adjacent sides pass from the end points of the vectors S_1, S_2, S_3. The corresponding problem on the two dimensional plane is the following: Given two vectors S_1, S_2, find the orthogonal parallelogrammon with maximal diagonal, whose one corner is at the origin and the opposite corner's two adjacent sides pass from the end points of the vectors S_1, S_2. The two dimensional case is easy to solve: If A_i is the end point of S_i, let M be the middle of A_1, A_2. Consider the circle with diameter A_1A_2 and let N_1, N_2 be the points where the line defined by OM meets the circle. Let OB_i be the perpendicular to N_1A_i from O. The parallelogrammon O $B_1N_1B_2$ O is the one with maximal diagonal (N_2 corresponds to the minimum). That the two dimensional case is easy to solve is not surprising, because it actually corresponds to the case where rank P_1 + rank P_2 = n, $P_1P_2 = 0$ of Section 3. The three dimensional case that we are interested in corresponds to rank P_1 + rank P_2 = 1 + 1 < 3 = n.

Using the Lagrange multiplier rule for (67), yields the following necessary condition:

$$
\begin{aligned}
S_1S_1'v_1 &= \lambda_{11}v_1+\lambda_{12}v_2+\lambda_{13}v_3 & \lambda_{12} &= \lambda_{21} \\
S_2S_2'v_2 &= \lambda_{21}v_1+\lambda_{22}v_2+\lambda_{23}v_3 & \lambda_{31} &= \lambda_{13} \qquad (68)\\
S_3S_3'v_3 &= \lambda_{31}v_1+\lambda_{32}v_2+\lambda_{33}v_3 & \lambda_{23} &= \lambda_{32}
\end{aligned}
$$

(It is easy to justify the existence of a Lagrangle multiplier for this problem.) Out of all v_i, λ_{ij} which satisfy (68) we are interested in the one which maximizes $\lambda_{11}+\lambda_{22}+\lambda_{33}$.

If $S_1'v_1 = S_2'v_2 = 0$, then (68) yields $\lambda_{ij} = 0$ and thus $\lambda_{11}+\lambda_{22}+\lambda_{33} = 0$; if at least one of the S_i's is different than zero, then this contradicts the fact that the maximum in (67) has to be strictly positive. If $S_1'v_1 = S_2'v_2 = 0$, and $S_3'v_3 \neq 0$, then (68) yields $\lambda_{11} = \lambda_{12} = \lambda_{13} = \lambda_{21} = \lambda_{22} = \lambda_{23} = \lambda_{31} = \lambda_{32} - 0$ and $S_3S_3'v_3 = \lambda_{33}v_3$ and thus $v_3 = S_3/\|S_3\|$. It can now be easily checked whether v_1, v_2 exist with $v_1'S_3 = v_2'S_3 = v_1'S_1 = v_2'S_2 = 0$. If $S_3'v_3 = 0$ and $S_2'v_2 \neq 0$, $S_1'v_1 \neq 0$ then (68) yields

$$
\begin{aligned}
S_1S_1'v_1 &= \lambda_{11}v_1+\lambda_{12}v_2\\
S_2S_2'v_2 &= \lambda_{12}v_1+\lambda_{22}v_2
\end{aligned}
$$

which actually means that we have to solve the two-dimensional analog discussed above. Similarly we can examine all the cases where at least one of the $S_i'v_i$'s is zero. Thus we can concentrate on the case where $S_1'v_1$, $S_2'v_2$, $S_3'v_3 \neq 0$, in which case the conditions (68) can be written as

$$
S = (S_1 : S_2 : S_3] = [v_1v_2v_3] \begin{bmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{bmatrix} \begin{bmatrix} 1 & a & b \\ \alpha & 1 & c \\ b & c & 1 \end{bmatrix}
$$

$$
= UMR \qquad (69)
$$

where $\mu_i^2 = \lambda_{ii}$. If S_1, S_2, S_3 are linearly independent we can replace (69) equivalently by:

$$
R(S'S)^{-1}R = \text{diagonal}
$$

which yields a system of three equations with three unknowns. For each solution (α, β, γ) of this system we can find the diagonal elements of $R(S'S)^{-1}R$ which are actually equal to $1/(\mu_i)^2 = 1/\lambda_{ii}$ and pick the solution (α, β, γ) that yields the maximum for $\lambda_{11}+\lambda_{22}+\lambda_{33}$. Having thus found μ_1, μ_2, μ_3, a, b, c, we find $U = SR^{-1}M^{-1}$. The only difficulty in the above procedure lies in solving the system

$$
[1 \quad a \quad b](S'S)^{-1}\begin{bmatrix} a \\ 1 \\ c \end{bmatrix} = 0
$$

$$
[1 \quad a \quad b](S'S)^{-1}\begin{bmatrix} b \\ c \\ 1 \end{bmatrix} = 0
$$

$$
[a \quad 1 \quad c](S'S)^{-1}\begin{bmatrix} b \\ c \\ 1 \end{bmatrix} = 0
$$

for a, b, c.

7. Conclusions

Our main objective in this paper was to formulate some problems related to the optimal choice of information in a team problem. Some partial results were also presented, which suggest several possible ways of handling these problems. As it turned out, several "generalized type eigenvalue" problems have to be solved and their geometric meaning to be connected in a simple and intuitive way with the matrices involved in describing the cost and information. We believe that the importance of the topic asks for further investigation of these issues.

8. References

1. K. C. Chu, "Designing Information Structures for Quadratic Decision Problems," JOTA, Vol. 25, No. 1, May 1978, pp. 139-160.

2. M. Athans, "On the Determination of Optimal Costly Measurement Strategies for Linear Stochastic Systems," Automatica, Vol. 8, 1972, pp. 397-412.

3. K. D. Herring and J. L. Melsa, "Optimum Measurements for Estimation," IEEE-AC, Vol. AC-19, No. 3, June 1974, pp. 264-266.

4. R. K. Mehra, "Optimization of Measurement Schedules and Sensor Designs for Linear Dynamical Systems," IEEE-AC, Vol. AC-21, No. 1, Feb. 1976, pp. 55-64.

5. G. P. Papavassilopoulos, "Optimal Allocation of Information in Team Problems," Optimization Days, Montreal, May 1981.

6. T. W. Anderson, "An Introduction to Multivariate Statistical Analysis," John Wiley, New York, 1958.