A Control Problem with Structural Choices†

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ABSTRACT: *The case is considered in which, during the operation of an optimal control system, the optimizer, in addition to applying his usual control, may switch structures. Necessary and sufficient conditions are derived and emphasis is placed on the special characteristics of this problem. Continuous and discrete time set-ups are considered and the separation principle is shown not to hold for the linear quadratic case in the presence of noise.*

I. Introduction

In the usual optimal control problem it is assumed that the structure of the plant is fixed and that the control variable is the only way in which the evolution of the plant can be influenced. In many problems in practice however, the structure of the plant may be amenable to changes which are at the decision maker's disposal. This paper considers the case in which, during the operation of the system, the decision maker, in addition to applying his usual control, may switch from one structure to another at instants of time that he chooses.

As an introductory example, consider the linear system

$$\dot{x}(t)=A(t)x(t)+B(t)u(t), \quad x(t_0)=x_0, \quad t\in[t_0,t_f]=\text{fixed},$$

and the cost functional

$$J(u)=\tfrac{1}{2}\left\{x'(t_f)Fx(t_f)+\int_{t_0}^{t_f}[x'(t)Q(t)x(t)+u'(t)u(t)]\,\mathrm{d}t\right\},$$

where $u(t)\in R^4$ so that the decision maker has four single input positions available. In addition, consider that, although four input positions are available, only two can be utilized at a time, i.e. $u(t)$ will be equal to $(u_1(t), u_2(t), 0, 0)'$ or $(u_1(t), 0, u_3(t), 0)'$ or $(u_1(t), 0, 0, u_4(t))'$ or $(0, u_2(t), u_3(t), 0))'$ or $(0, u_2(t), 0, u_4(t))'$ or $(0, 0, u_3(t), u_4(t))'$, where $u_i(t)\in R$. The decision maker can use any of these six configurations and can switch from one to another during the operation. His final objective is to minimize $J(u)$.

We can formulate this situation as follows. Let C_{ij}, where $i\neq j$ and $i, j=1, 2, 3, 4$, be 4×2 matrices with ith row equal to $(1, 0)$, jth row equal to $(0, 1)$, and

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the other two rows equal to (0, 0). Let $\tilde{B}_p = BC_{ij}$, where $p = p(i, j)$ and $p = 1, \ldots, 6$, and let $v(t) \in R^2$. Then the problem is equivalent to the following: given

$$\dot{x}(t) = A(t)x(t) + \tilde{B}_p(t)v(t), \quad x(t_0) = x_0, \quad t \in [t_0, t_f],$$

$$J'(v) = \tfrac{1}{2}\left\{x'(t_f)Fx(t_f) + \int_{t_0}^{t_f} [x'(t)Q(t)x(t) + v'(t)v(t)]\,\mathrm{d}t\right\},$$

find a control $v(t)$ and a switching strategy for $\{\tilde{B}_p\}_{p=1}^{6}$ determined by a partition P of $[t_0, t_f]$, also find a rule for which one out of the six $\tilde{B}_p$ terms will be in operation during each subinterval in P so as to minimize $J'(v)$. The extension of this problem to the nonlinear case where N structures $f_1, \ldots, f_N[\dot{x} = f_i(x, u, t)]$ are available is obvious. It can also be extended to the case where M structures $L_1, \ldots, L_M$ are possible for the cost functional $[J = \int_{t_0}^{t_f} L_i(x, u, t)\,\mathrm{d}t]$ with the cost calculated by using the chosen L_i on each subinterval in P.

Cases where changes of structure are used for several purposes have been reported previously in the literature. In (**1, 2**) the transfer function of a second-order system is studied where the changes of structure result in a piecewise constant linear system. In (**3, 4**) a stochastic control problem, where there is a cost for using the measurements is studied. This amounts to allowing changes in the structure of the cost functional and in the measurement equation simultaneously. In (**5**) a discrete linear deterministic system is studied when the "B" matrix (actuator), which multiplies the input, is allowed to take certain finite-in-number values at each instant of time. In (**6**) a deterministic linear control problem is considered, and relaxed controls are introduced to prove the existence of a solution when the B matrix is allowed to take certain finite-in-number values at each instant of time. In (**7, 8**) the stabilization of a system by switching structures is studied, the main concern being the study of the motion of the system along preassigned surfaces in the state space at which the switching occurs. One very nice and sufficient-for-motivation property is that often (even if two structures result individually in unstable systems) by switching back and forth from one structure to the other we can stabilize the resulting system (**8**). There are many situations where switching of structures is allowed, see for example (**9**) for flight applications.

In this paper we introduce switching of structures to minimize a cost functional rather than to provide stabilization. This problem will be studied by reducing it to a control problem; it will be shown how singular surfaces arise in this case also, though we will not dwell on this topic since our intentions are to introduce an interesting class of problems, which includes as special cases most of the previously considered ones, to find the solutions for some cases, and to gain certain insights. We will consider the case where only two structures with linear state equations and quadratic cost functionals are available. We will derive necessary and sufficient conditions for optimality for the continuous case (Section II) and for the discrete time case (Section III) with particular emphasis on the first. The stochastic counterpart will also be considered (Section III) and the separation principle will be shown not to hold. Suboptimal procedures for finding the optimal switching strategy are also introduced.

II. Continous Time Case

Assume that we are given $A_i(t)$, $B_i(t)$, $Q_i(t)=Q_i(t)'$, F, $[t_0, t_f]$, and x_0, where (a) the time interval $[t_0, t_f]$ is fixed, (b) A_i, B_i, Q_i are piecewise continuous functions of $t\in[t_0, t_f]$ whose values are real matrices of dimensions $n\times n$, $n\times m$, $n\times n$, respectively, (c) F is an $n\times n$ real matrix which is constant, and (d) $x_0\in R^n$. Given $P\subseteq[t_0, t_f]$, with P Lebesgue measureable and any bounded measurable function of time u: $[t_0, t_f]\rightarrow R^m$, consider the system whose state $x(t)$ evolves in accordance with the differential equation

$$\dot{x}(t)=A_i(t)x(t)+B_i(t)u(t), \quad x(t_0)=x_0, \quad t\in[t_0, t_g], \tag{1}$$

and the cost functional

$$J(P, u)=\tfrac{1}{2}\left\{x'(t_f)Fx(t_f)+\int_{t_0}^{t_f}[x'(t)Q_i(t)x(t)+u'(t)u(t)]\,\mathrm{d}t\right\}, \tag{2}$$

where

$$\begin{aligned} i&=1 \quad \text{if} \quad t\in P,\\ i&=2 \quad \text{if} \quad t\in P. \end{aligned} \tag{3}$$

The solution $x(t)$ of (1) is assumed to be absolutely continuous. Notice that for given P and u, the solution of (1) exists over $[t_0, t_f]$ since (1) with (3) defines a linear differential equation. The problem that we intend to solve is

$$\underset{P,u}{\text{minimize}}\; J(P, u). \tag{4}$$

Let us now consider the following problem: Given the state equation

$$\begin{aligned} \dot{x}(t)=&\left[\frac{1+s(t)}{2}A_1(t)+\frac{1-s(t)}{2}A_2(t)\right]x(t)\\ &+\left[\frac{1+s(t)}{2}B_1(t)+\frac{1-s(t)}{2}B_2(t)\right]u(t), \quad x(t_0)=x_0, \end{aligned} \tag{5}$$

and the cost functional

$$\begin{aligned} J(s, u)=\tfrac{1}{2}\bigg\{&x'(t_f)Fx(t_f)+\int_{t_0}^{t_f}\left[x'(t)\left(\frac{1+s(t)}{2}Q_1(t)+\frac{1-s(t)}{2}Q_2(t)\right)x(t)\right.\\ &\left.+u'(t)u(t)\right]\mathrm{d}t\bigg\}, \end{aligned} \tag{6}$$

where u and s are bounded measurable functions of time $u:[t_0, t_f]\rightarrow R^m$, $s:[t_0, t_f]\rightarrow\{-1, +1\}$, the problem

$$\underset{s,u}{\text{minimize}}\; J(s, u) \tag{7}$$

is equivalent to (4). A straightforward application of the maximum principle to problem (7) gives the following necessary conditions for problem (4).

Proposition 1.

If the pair (P^*, u^*) solves (4), then there exists an n-dimensional vector function $p(t)$ such that for $t \in [t_0, t_f]$

$$\dot{x}(t) = A_i(t)x(t) - B_i(t)B_i'(t)p(t), \quad x(t_0) = x_0, \tag{8}$$

$$-\dot{p}(t) = A_i'(t)p(t) + Q_i(t)x(t), \quad p(t_f) = Fx(t_f)\dagger, \tag{9}$$

$$u^*(t) = -B_i'(t)p(t), \tag{10}$$

$$i = 1 \quad \text{if} \quad p'(t)[A_1(t) - A_2(t)]x(t) - p'(t)[B_1(t) - B_2(t)]B_1'(t)p(t) + \tfrac{1}{2}x'(t)[Q_1(t) - Q_2(t)]x(t) \le 0, \tag{11}$$

$$i = 2 \quad \text{if} \quad P'(t)[A_2(t) - A_1(t)]x(t) - p'(t)[B_2(t) - B_1(t)]B_2'(t)p(t) + \tfrac{1}{2}x'(t)[Q_2(t) - Q_1(t)]x(t) \le 0. \tag{12}$$

Clearly, (11) and (12) partially characterize P^*. The case where both quantities in (11) and (12) are ≥ 0 and at least one of them is $\geqq 0$ cannot arise, since if this were the case adding (11) and (12) would give $-p'(B_1 - B_2)(B_1 - B_2)'p \geqq 0$ which is impossible. Therefore, at each instant of time either (11) or (12) is satisfied and if one of them is not satisfied then the other one is satisfied with strict inequality. If both (11) and (12) are satisfied with equality, by adding them we obtain $B_1'p = B_2'p$, i.e. the two possible control values are equal.

In general, additional analysis is needed to determine whether $i = 1$ or $i = 2$ is optimal, in case both (11) and (12) are satisfied. It is worth pointing out that if at a certain point both (11) and (12) are satisfied, it may very well happen that both the trajectories with structures 1 and 2 are optimal, thus there may be more than one optimal trajectory emanating from x_0, and that at certain points an optimal trajectory might split into two trajectories both of which are optimal. (The same phenomenon will be noticed in the discrete case treated in Section III). This phenomenon is basically due to the singular character of the problem (7) since s enters the Hamiltonian linearly. Lastly, notice that it is not possible to have infinitely fast switchings from one structure to the other over a whole nonempty open interval $I \subseteq [t_0, t_f]$, since if this were the case there would exist Lebesgue measurable sets P_1 and P_2 such that $I = P_1 \cup P_2$, $P_1 \cap P_2 = \phi$, $s(t) = 1$ if $t \in P_1$, $s(t) = -1$ if $t \in P_2$, and $\bar{P}_1 = \bar{P}_2 = I$; since $\text{meas}(P_i) = \inf\{\text{meas}(V),\ P_i \subseteq V,\ V \text{ open}\}$, and since $P_i \subseteq V$, V open, $\bar{P}_i = I \doteq V = I$, we conclude that $\text{meas}(P_i) = \text{meas}(I)$, $i = 1, 2$, and thus $\text{meas}(I) = 0$, i.e. $I = \phi$.

Let us now try to find a solution of $p(t)$ of the form

$$p(t) = K(t)x(t), \tag{13}$$

† If in (2) we had F_1 for $i = 1$ and F_2 for $i = 2$ instead of F, where $F_1 \ne F_2$, then (9) changes to $P(t_f) = F_i x(t_f)$. This can be shown by transforming first the costs (2) to the Lagrange form, (i.e. no explicit terminal penalty).

where $K(t)$ is an $n \times n$ real matrix to be determined. Substituting $p(t)$ from (13) into (8)–(12), we obtain

$$\dot{x}(t) = [A_i(t) - B_i(t)B_i'(t)K(t)]x(t), \quad x(t_0) = x_0, \tag{14}$$

$$-\{\dot{K}(t) + K(t)[A_i(t) - B_i(t)B_i'(t)K(t)]\}x(t) = [A_i'(t)K(t) + Q_i(t)]x(t),$$
$$K(t_f)x(t_f) = Fx(t_f), \tag{15}$$

$$u^*(t) = -B_i'(t)K(t)x(t), \tag{16}$$

$$i=1 \quad \text{if} \quad x'(t)\{K(t)[A_1(t) - A_2(t)] - K'(t)[B_1(t) - B_2(t)]B_1'(t)K(t) + \tfrac{1}{2}[Q_1(t) - Q_2(t)]\}x(t) \le 0, \tag{17}$$

$$i=2 \quad \text{if} \quad x'(t)\{K(t)[A_2(t) - A_1(t)] - K'(t)[B_2(t) - B_1(t)]B_2'(t)K(t) \; \tfrac{1}{2}[Q_2(t) - Q_1(t)]\}x(t) \le 0. \tag{18}$$

Notice that as (17) and (18) indicate, the switching of structures depends on $x(t)$ and t, and not on t alone. The following proposition gives sufficient conditions under which a control law of the form (16) is optimal. Its proof is based on a direct application of Dynamic Programming where one assumes a value function of the form $V(x, t) = \frac{1}{2}x'K(t)x$.

Proposition 2.

Consider the following system of one differential equation and two matrix inequalities which are solved backwards in time $t \in [t_0, t_f]$:

$$-\dot{K}(t) = K(t)A_i(t) + A_i'(t)K(t) + Q_i(t) - KB_i(t)B_i'(t)K, \quad K(t_f) = F, \tag{19}$$

$$i=1 \quad \text{if} \quad [A_1(t) - A_2(t)]'K + K[A_1(t) - A_2(t)] - K[B_1(t) - B_2(t)]B_1'(t) + B_1(t)[B_1(t) - B_2(t)]'K + [Q_1(t) - Q_2(t)] \le 0, \tag{20}$$

$$i=2 \quad \text{if} \quad [A_2(t) - A_1(t)]'K + K[A_2(t) - A_1(t)] - K[B_2(t) - B_1(t)]B_2'(t) + B_2(t)[B_2(t) - B_1(t)]'K + [Q_2(t) - Q_1(t)] \le 0, \tag{21}$$

and assume that their solution exists on $[t_0, t_f]$. Then the control law

$$u(t) = -B_i'(t)K(t)x$$

is optimal, where $i = 1$ or 2, in accordance with (20) and (21).

Notice that if the conditions of Proposition 2 hold, then the switching of structures depends on t and not on $x(t)$. Therefore, K is a function of t only and the switching points are the same in time for any initial point x_0. The matrix K is symmetric and positive semidefinite if $F \geq 0$. The cost to go at a point (x, t) is $V(x, t) = \frac{1}{2}x'K(t)x$. Notice also that at least one of (20) or, (21) has to hold at each t, for the same reasons that this happens for (11), and (12).

The conditions (20) and (21) are restrictive and in general will not hold. Nonetheless there are important cases where they do hold. We will consider three special cases.

Case I. Let $A_1 = A_2$ and $Q_1 = Q_2$. Then (20) and (21) yield

$$i=1 \quad \text{if} \quad KB_2B_1'K + KB_1B_2'K \leq 2KB_1B_1'K, \tag{20-1}$$

$$i=2 \quad \text{if} \quad KB_2B_1'K + KB_2B_1'K \leq 2KB_2B_2'K. \tag{21-1}$$

If $B_2(t) = b_2(t)B_1(t)$, $b_2(t) \in R$, then (20-1) holds when $b_2(t) \leq 1$, and (21-1) holds when $b_2(t) \leq b_2^2(t)$. So, when (a) $b_2(t) \geq 0$, structure 2 is optimal, (b) $0 \leq b_2(t) \leq 1$, structure 1 is optimal, (c) $b_2(t) \leq 0$, both structures are optimal. Therefore, if $b_2(t) \leq 0$ on an interval we may very well have two optimal trajectories.

Case II. Let $A_1 = A_2$ and $B_1 = B_2$. Then (20) and (21) yield

$$i=1 \quad \text{if} \quad Q_1(t) \leq Q_2(t), \tag{20-2}$$

$$i=2 \quad \text{if} \quad Q_2(t) \leq Q_1(t), \tag{21-2}$$

which is an intuitively acceptable conclusion. If $Q_2(t) = q_2(t)Q_1(t)$, $Q_1(t) \geq 0$, $q(t) \in R$, then $i=1$ if $q_2(t) \geq 1$, and $i=2$ if $q_2(t) \leq 1$.

Case III. Let $B_1 = B_2$ and $Q_1 = Q_2$. Then (20) and (21) yield

$$i=1 \quad \text{if} \quad KA_1 + A_1'K \leq A_2K + KA_2, \tag{20-3}$$

$$i=2 \quad \text{if} \quad KA_2 + A_2'K \leq A_1'K + KA. \tag{21-3}$$

For these three special cases it is easy to find subcases where (20) and (21) both fail or one at least holds. For example, in Case II if Q_1 and Q_2 cannot be ordered on some interval, then both (20) and (21) fail. We can also give conditions under which the solution $K(t)$ exists over $[t_0, t_f]$; such a condition is that both pairs (A_i, B_i) are controllable and $Q_1 \geq 0$, $Q_2 \geq 0$. The reason for the nonapplicability of Proposition 2 in general lies in the fact that this proposition concerns cases where the switching in time will be the same for any optimal trajectory and independent on x, while in general the switching should depend on both the current time and state values.

III. Discrete and Stochastic Cases

(i) *Discrete time deterministic case*

In this section we will consider the discrete version of the problem of Section II and the stochastic analogs for the discrete and the continuous time cases.

Assume that we are given real constant matrices A_k^i, B_k^i, $Q_k^i = Q_k^{i\prime}$, Q_N^i, $k = 0, 1, \ldots, N-1$, $i = 1, 2$. Given the set Ω of all ordered $(N+1)$-tuples of zeros and ones [e.g. (0, 1, 1, 0, 1, 0, 0, . . . , 1, 0)], we want to solve the problem

$$\underset{P \in \Omega, u}{\text{minimize}} \; J(P, u), \tag{22}$$

where

$$x_{k+1} = A_k^{i(k)} x_k + B_k^{i(k)} u_k, \quad x_0 \text{ given}, \ k = 0, 1, \ldots, N-1, \tag{23}$$

$$J(P, u) = \tfrac{1}{2}\left[x_N' Q_N^{i(N)} x_N + \sum_{k=0}^{N-1} (x_k' Q_k^{i(k)} x_k + u_k' u_k)\right], \tag{24}$$

$$P = [i(0), \ i(1), \ldots, i(N)], \tag{25}$$

$$u = (u_0, \ldots, u_{N-1}). \tag{26}$$

Since the set Ω is finite (2^{N+1} elements), one can solve 2^{N+1} linear quadratic problems and then pick out the one which results in the minimum cost. Therefore, the solution of (22) will be a linear function of the current state of the form $u_k = L_k x_k$. Notice that, since the comparison of the 2^{N+1} problems is based on the comparison of the costs $J^P = \frac{1}{2} x_0 K_0^P x_0$, where K_0^P is the Riccati gain corresponding to some $P \in \Omega$, the choice of the optimum P is not generally independent of x_0. Therefore, in the optimal control law $u_k = L_k x_k$, L_k depends on x_0. A more efficient way of solving problem (22) is to consider the equivalent problem where A_k^i, B_k^i, Q_k^i are replaced by

$$\left(\frac{1+s_k}{2} A_k^1 + \frac{1-s_k}{2} A_k^2\right), \quad \left(\frac{1+s_k}{2} B_k^1 + \frac{1-s_k}{2} B_k^2\right), \quad \left(\frac{1+s_k}{2} Q_k^1 + \frac{1-s_k}{2} Q_k^2\right),$$

respectively, where $s_k = +1$ or -1, $S = (s_0, s_1, \ldots, s_N)$, and solve for the best control law (S^*, u^*) by Dynamic Programming. Notice that out of the 2^{N+1} problems mentioned above, several might attain the best cost and the optimal trajectories of some of them might have common points. This phenomenon was mentioned as a possibility for the continuous time case, but in the discrete time case it is very easy to construct examples where it does occur.

(ii) *Continuous time stochastic case*

Let us now consider the stochastic case. First we will look at the continuous time case and show that the problem can be solved in two steps, the first step involving the solution of a classical linear quadratic Gaussian problem and the second step involving the solution of a deterministic singular control problem. The system's state evolves in accordance with

$$dx(t) = [A_i(t)x(t) + B_i(t)u(t)]\,dt + G(t)\,dw(t), \quad t \in [t_0, t_f], \tag{27}$$

and the information available at time t is $y_t = \{y(\theta), \ t_0 \le \theta \le t\}$, where

$$dy(t) = C_i(t)x(t)\,dt + R(t)v(t), \quad t \in [t_0, t_f], \quad y(t_0) = 0. \tag{28}$$

The matrices A_i, B_i, G, C_i, R are piecewise continuous functions of time, $RR' > 0$, $y(t) \in R^q$, $w(t)$ and $v(t)$ are standard independent Wiener processes with zero mean and covariances equal to unit matrices, and $x(t_0)$ is a Gaussian random variable independent of w and v with mean $\bar{x}_0$ and covariance L_0. The objective is to minimize the expected value of J, is given by equation (2). The optimizer chooses P and u as measurable functions of y_t. This problem, although nonlinear, can be solved as follows: Since

$$\inf_{P,u} E(J) = \inf_P [\inf_u E(J)], \tag{29}$$

we first solve a classical linear quadratic Gaussian problem for fixed P [i.e. $s(t)$] and finally we have to solve

$$\inf_s \bar{x}_0 K(t_0)\bar{x}_0 + \int_{t_0}^{t_f} t_r\Big[L\Big(\frac{1+s}{2}C_1+\frac{1-s}{2}C_2\Big)'(RR')^{-1}\Big(\frac{1+s}{2}C_1+\frac{1-s}{2}C_2\Big)LK\Big]\,dt$$
$$+t_rFL(t_f)+\int_{t_0}^{t_f} t_r\Big[\Big(\frac{1+s}{2}Q_1+\frac{1-s}{2}Q_2\Big)L\Big]\,dt, \quad (30)$$

subject to

$$-\dot{K}=K\Big(\frac{1+s}{2}A_1+\frac{1-s}{2}A_2\Big)+\Big(\frac{1+s}{2}A_1+\frac{1-s}{2}A_2\Big)'K+\Big(\frac{1+s}{2}Q_1+\frac{1-s}{2}Q_2\Big)$$
$$+K\Big(\frac{1+s}{2}B_1+\frac{1-s}{2}B_2\Big)\Big(\frac{1+s}{2}B_1+\frac{1-s}{2}B_2\Big)'K, \quad K(t_f)=F,$$
$$\dot{L}=L\Big(\frac{1+s}{2}A_1+\frac{1-s}{2}A_2\Big)'+\Big(\frac{1+s}{2}A_1+\frac{1-s}{2}A_2\Big)L+GG'$$
$$+L\Big(\frac{1+s}{2}C_1+\frac{1-s}{2}C_2\Big)'RR'\Big(\frac{1+s}{2}C_1+\frac{1-s}{2}C_2\Big)L, \quad L(t_0)=L_0.$$

If we introduce $v=(1-s)/2$, then $v(t)=0$ or 1 and $v^2=v$. Thus problem (30) is a control problem with control v, where v enters linearly in the state equations and in the cost, and so is singular. Notice that although in (30), it seems that no information y_t is used for finding the optimal s^*, this is not really the case since s^* will depend on C_1, C_2 and R.

(iii) *Discrete time stochastic case*

Let us now consider the discrete stochastic case. The cost is given by (24), but noise w_k is added to the right-hand side of (23). The control u_k should be composed as a measurable function of $y^k=(y_0, y_1, \ldots, y_k)$, where

$$y_k = C_k^{i(k)}x_k + v_k \quad (31)$$

with C_k^1, C_k^2 $l\times n$ real constant matrices. We also assume that $\{w_k\}$, $\{v_k\}$, x_0 are independently distributed Gaussian random variables with $E(w_k)=0$, $E(v_k)=0$, $E(x_0)=\bar{x}_0$. By considering again the 2^{N+1} possible $(N+1)$-tuples P, we can solve 2^{N+1} linear quadratic Gaussian problems and pick the P which corresponds to the problem with the minimum cost. Thus the control law will be of the form $u_k=L_k\hat{x}_k$, where $\hat{x}_k$ is the minimum mean-square estimate of x_k, and L_k depends on $\bar{x}_0$ for the same reasons as mentioned in the deterministic case. Therefore the separation principle does not hold, except in a restricted way, i.e. the optimal control value depends on $\bar{x}_0$ and on $\hat{x}_k$ and the dependence on $\hat{x}_k$ is linear. Finally notice that the solution can also be found by introducing the control $s_k=\pm 1$ as in the deterministic case and applying Dynamic Programming.

The nonlinear character of the equivalent problem with control (s_k, u_k) with respect to s_k is intimately related to the failure of the separation principle. The following example demonstrates the failure of the separation principle for the problem (22)–(26), (27). Consider the following, one-step, scalar version of the problem (22)–(26), (27):

$$\begin{aligned} x_1 &= a_i x_0 + u,\\ y &= x_0 + v, \\ J &= q_i x_1^2 + u^2, \end{aligned} \tag{32}$$

where $i = 1, 2,;$ a_1, a_2, q_1, $q_2 > 0$. We want to choose a control law u as a function of y, and $i = 1$ or 2 to minimize J. The terms x_0 and v are independent Gaussian random variables with $E(x_0) = x_0$, $E(v) = 0$, $E(x^2) = \sigma^2$ and, $E(v^2) = \delta^2$. The solution is as follows:

$$\begin{aligned} u^*(y) &= l^* y, \\ l &= -\frac{q_i a_i}{q_i + 1}\frac{\sigma^2}{\sigma^2 + \delta^2}, \end{aligned} \tag{33}$$

$$J^* = q_i a_i^2 \sigma^2 \left[1 + \frac{q_i^2}{(q_i+1)^2}\frac{\sigma_2}{\sigma^2+\delta^2} - 2\frac{q_i}{q_i+1}\frac{\sigma^2}{\sigma^2+\delta^2} \right.$$

$$\left. \frac{q_i^2}{(q_i+1)^2}\frac{\delta^2}{\sigma^2+\delta^2} + \frac{\sigma^2}{\sigma^2+\delta^2}\frac{q_i}{(q_i+1)^2}\right] \triangleq a_i^2 \sigma^2 \Phi(q_i, \sigma, \delta), \tag{34}$$

$$\begin{aligned} i = 1 \quad &\text{if} \quad a_1^2 \Phi(q_1, \sigma, \delta) \le a_2^2 \Phi(q_2, \sigma, \delta), \\ i = 2 \quad &\text{if} \quad a_2^2 \Phi(q_2, \sigma, \delta) \le a_1^2 \Phi(q_1, \sigma, \delta). \end{aligned} \tag{35}$$

Assume that

$$0 < a_1^2 \Phi(q_1, \bar{x}_0, 0) < a_1^2 \Phi(q_1, \bar{x}_0, 0), \tag{36}$$

i.e. the deterministic version of (32) has solution $i = 1$. To demonstrate that separation does not hold, it suffices to show that for some $\delta \neq \sigma$, the solution is $i = 2$, i.e.

$$a_2^2 \Phi(q_2, \sigma, \delta) < a_1^2 \Phi(q_1, \sigma, \delta) \tag{37}$$

It therefore suffices to find a_1, a_2, q_1, q_2, δ, $\bar{x}_0$, such that

$$\frac{\Phi(q_2, \sigma, \delta)}{\Phi(q_1, \sigma, \delta)} < \frac{a_1^2}{a_2^2} < \frac{\Phi(q_2, \bar{x}_0, 0)}{\Phi(q_1, \bar{x}_0, 0)}, \tag{38}$$

and hence it suffices to find q_1, q_2, σ, δ, $\bar{x}_0$, such that

$$\frac{\Phi(q_2, \sigma, \delta)}{\Phi(q_2, \bar{x}_0, 0)} < \frac{\Phi(q_1, \sigma, \delta)}{\Phi(q_1, \bar{x}_0, 0)}. \tag{39}$$

For any fixed σ, $\delta \neq 0$ and we can calculate the value of $\Phi(z, \sigma, \delta)/\Phi(z, \bar{x}_0, 0)$. We can find z', $z^\circ > 0$ such that

$$\frac{\Phi(z', \sigma, \delta)}{\Phi(z', \bar{x}_0, 0)} < \frac{\Phi(z'', \sigma, \delta)}{\Phi(z'', \bar{x}_0, 0)}.$$

We set $q_2 = z'$ and $q_1 = z''$, and choose a_1, $a_2 > 0$ so as to satisfy (38). Obviously now we have values of q_1, q_2, a_1, a_2, and δ so the separation principle does not hold.

The comments about nonuniqueness and bifurcation of the optimal trajectory made for deterministic cases apply to the stochastic cases as well.

(iv) *Suboptimal procedures*

It should be clear by now that although the discrete version of the problem (deterministic or stochastic) considered in this paper will have a solution, the continuous version might not. This is primarily due to the fact that values of $s(t)$ in $(-1, 1)$ are not permitted. We are therefore obliged either to introduce relaxed controls, see (**6**) or to introduce suboptimal schemes. Since the solution of our problem eventually reduces to the solution of (30) (where the deterministic case is included for $G \equiv 0$, $L \equiv 0$), one should consider suboptimal schemes for (30). A first suboptimal scheme is to pick up fixed t_i, $i = 1, 2, \ldots, n-1$, $t_0 < t_1 < t_{n-1} < t_y = t_f$, and then try to determine $\tilde{s}$, where $\tilde{s} = -1$ or $+1$ on each $(t_i\, t_{i+1})$, so as to solve (30) in the class of these $\tilde{s}$. A second way is to try to find t_i, $i = 1, \ldots, n-1$, $t_0 \le t_1 \le \ldots \le t_{n-1} \le t_n = t_f$, where n is fixed, which determines an $\tilde{s}$ with $\tilde{s}(t) = -1$ on (t_0, t_1), $\tilde{s}(t) = t_1$ on (t_1, t_2) and so on, and solve (30) in the class of these $\tilde{s}$. Let us consider the second procedure. Introducing $\tilde{v} = (1-\tilde{s})/2$, the state equations of (30) assume the form

$$\dot{E} = f(E)\tilde{v} + g(E), \left(E = \begin{bmatrix} K \\ L \end{bmatrix}\right),$$

which is linear in $\tilde{v}$ [and so is the cost functional of (30)]. For illustration purposes let $n = 2$, so that only t_1 needs to be found, and

$$\tilde{v} = \frac{1 - \text{sgn}(t - t_1)}{2}.$$

We have therefore to solve a parameter optimization problem which nonetheless has a discontinuous dependence on the parameter t_1. To use the known techniques of parameter optimization, [see (**10**) and the references therein], we can approximate sign $(t - t_1)$ with a smooth function in t_1 and solve a sequence of parameter optimization problems as this approximation increases.

IV. Conclusions

In this paper we have an analyzed some aspects of a special class of control problems, of which the basic characteristic is that during the operation the decision maker may switch structures, in addition to applying the usual control function of time. Potential applications of such problem formulations are, for example, in the area of economic organization where the organizational structure is easy to change in time in a way to be chosen by a manager, in the area of power systems where many input positions might be available but only a subset of them can be utilized at a time, and in flight control. Another important area where structural choices might exist is that of Leader–Follower hierarchical games where the follower minimizes his cost for a given leader's strategy of switching structures, while the leader's only control to minimize his cost is exactly the strategy of switching structures.

It should be obvious how to generalize the procedures presented here in order to analyze problems where more than two, say N structures, are available. Simply, if $\dot{x} = f_i(x, u, t)$, $i = 1, \ldots, N$ are the N structures, one can consider $\dot{x} = f(x, u, s, t)$ where the range of $s(t) = \{1, 2, \ldots, N\}$ and $f(x, u, i, t) = f_i(x, u, t)$, and find the optimal (u^*, s^*). Restrictions concerning the switching of structures during certain intervals of time can be taken into account by imposing proper restrictions on s. Notice that $f(x, u, s, t)$ should be at least piecewise continuous in s and that not restricting f to be affine in s, as we did in Sections II and III, might facilitate the study of the singularities. The singular aspects of the problem presented remain open for further investigation.

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