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ALGORITHMS FOR LEADER-FOLLOWER GAMES*

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ABSTRACT

The purpose of this short paper is to describe some algorithms for solving finite dimensional static Leader-Follower (L-F) problems.

Introduction

The purpose of this short paper is to describe some algorithms for solving finite dimensional static Leader-Follower (L-F) problems. The L-F (or Stackelberg) concept has become of increasing importance during the last years [1-10], but despite the many interesting results available, very little attention has been paid to related algorithmic procedures. The only attempt in this direction that we know of is in [11], where a penalty type method is used. The algorithms that we describe are different from the one of [11]. We do not report here any theoretical results concerning convergence, rate of convergence or computational experience; it is our aim rather to outline certain sensible procedures, indicate their connections with existing optimization algorithms and delineate directions for further investigation.

1. Review of Definitions

The general definition of the finite dimensional static L-F problem is the following. Let $f_1, f_2: R^{n_1+n_2} \to R$ be two functions and $X_1, X_2 \subseteq R^{n_1+n_2}$ given subsets of $R^{n_1+n_2}$. $x_i \in R^{n_i}$. $f_1(f_2)$ represents the cost of the Leader (Follower) and X_1 his constraint set (X_2 for the Follower). For given x_1, the Follower solves

$$\min_{x_2} f_2(x_1,x_2) \quad \text{subject to: } (x_1,x_2) \in X_2 \tag{1}$$

This problem defines a mapping $T: R^{n_1} \to$ {subset of R^{n_2}} where $T(x_1)$ is the set of x_2's which solve (1). The Leader solves

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$$\min_{x_1,x_2} f_1(x_1,x_2)$$
$$\text{subject to:} \quad (x_1,x_2) \in X_1 \qquad (2)$$
$$x_2 \in T(x_1).$$

The final aim of the L-F problem is to solve (2). To avoid the situation where $(\bar{x}_1,\bar{x}_2)$ solves (2), and the Follower chooses $\tilde{x}_2 \in T(\bar{x}_1)$ with $(\bar{x}_1,\tilde{x}_2) \notin X_1$, we will assume that $X_2 \subseteq X_1$. X_1 can be a subset of $R^{n_1+n_2}$ determined through equality or inequality constraints.

2. Algorithms

(i) First Class: Let $X_1 = X_2$ be determined by $h(x_1,x_2) \leq 0$, where h is a function $R^{n_1+n_2} \to R$. Then under (smoothness, regularity or convexity) assumptions, (2) is equivalent to solving

$$\min f_1(x_1,x_2)$$
$$\text{subject to:} \quad \frac{\partial f_2(x_1,x_2)}{\partial x_2} + \mu \frac{\partial h(x_1,x_2)}{\partial x_2} = 0 \qquad (3)$$
$$\mu \geq 0$$
$$h(x_1,x_2) \leq 0.$$

(3) is a classical nonlinear programming problem with unknowns x_1,x_2,μ. Application of the known methods will (usually) require that f_2 and h are twice continuously differentiable. Solving (3) might not be so appealing, since we have increased the number of unknowns and constraints. This is not so bad since we cannot get rid of the constraint $x_2 \in T(x_1)$ without any punishment. The important issue is to employ a nonlinear programming algorithm for (3) which will take advantage of the underlying L-F problem philosophy. Such algorithms are generated in the second class below.

(ii) Second Class: Let $X_1 = X_2$ be determined by $h(x_1,x_2) = 0$. Since for given x_1, the Follower solves (2), it is natural to generate an algorithm based on the following rational: For $x_1 = x_1^k$, the Follower solves (2) and finds $x_2 = x_2^k$. The Leader, knowing x_1^k, x_2^k, moves to $x_1^{k+1} = x_1^k + d_k$ and the Follower solves (2) with $x_1 = x_1^{k+1}$ and finds $x_2^{k+1} = x_2^k + c_k$. The Leader would like to choose d_k, so that $f_1(x_1^k, x_2^k) > f_1(x_1^{k+1}, x_2^{k+1})$, i.e. to improve his cost in the next iteraction. It is reasonable to assume that for d_k small, so is c_k. (In what follows appropriate regularity or convexity conditions and sufficient smoothness of f_1,f_2,h are tacitly assumed. Also for simplicity reasons let

$n_1 = n_2 = 1$.) Since x_2^k solves (2) when $x_1 = x_1^k$, it will hold

$$\frac{\partial f_2(x_1^k, x_2^k)}{\partial x_2} + \mu^k \frac{\partial h(x_1^k, x_2^k)}{\partial x_2} = 0 \tag{4}$$

$$h(x_1^k, x_2^k) = 0$$

Similarly

$$\frac{\partial f_2(x_1^k + d_k, x_2^k + c_k)}{\partial x_2} + \mu^{k+1} \frac{\partial h(x_1^k + d_k, x_2^k + c_k)}{\partial x_2} = 0 \tag{5}$$

$$h(x_1^k + d_k, x_2^k + c_k) = 0.$$

μ^k, μ^{k+1} are Lagrange multipliers. A first order approximate expansion of (5), around x_1^k, x_2^k and use of (4) yields

$$0 = \frac{\partial^2 f_2(x_1^k, x_2^k)}{\partial x_1 \partial x_2} d_k + \frac{\partial^2 f_2(x_1^k, x_2^k)}{\partial x_2^2} c_k$$

$$+ \mu^{k+1} \left[\frac{\partial^2 h(x_1^k, x_2^k)}{\partial x_1 \partial x_2} d_k + \frac{\partial^2 h(x_1^k, x_2^k)}{\partial x_2^2} c_k \right] + (\mu^{k+1} - \mu^k) \frac{\partial h(x_1^k, x_2^k)}{\partial x_2} \tag{6a}$$

$$\frac{\partial h(x_1^k, x_2^k)}{\partial x_1} d_k + \frac{\partial h(x_1^k, x_2^k)}{\partial x_2} c_k = 0 \tag{6b}$$

Since we want $f_1(x_1^{k+1}, x_2^{k+1}) < f_1(x_1^k, x_2^k)$, expanding $f_1(x_1^k + d_k, x_2^k + c_k)$ around x_1^k, x_2^k we obtain

$$\frac{\partial f_1(x_1^k, x_2^k)}{\partial x_1} d_k + \frac{\partial f_1(x_1^k, x_2^k)}{\partial x_2} c_k \quad 0 \tag{7}$$

If one has an efficient way of solving the system (6a), (6b), (7) for d_k, c_k, μ^{k+1} then one can determine d_k. The bad thing with this is that (6a) is nonlinear since μ^{k+1} multiplies d_k and c_k. Of course one could choose μ^{k+1} to be equal to μ^k, which μ^k is known presumably from the previous minimization of (2) and try to find d_k, c_k as to make the left hand side of (7) as small as possible. (Conditions like $\|d_k\| \leq 1$ $\|c_k\| \leq 1$ should be imposed in doing that.) If the second order sufficiency conditions for (2) are satisfied at (x_1^k, x_2^k) and μ^{k+1} is close to μ^k, it might hopefully hold

$$\frac{\partial^2 f_2(x_1^k, x_2^k)}{\partial x_2^2} + \mu^{k+1} \frac{\partial^2 h(x_1^k, x_2^k)}{\partial x_2^2} = A_k > 0 \tag{8}$$

(since μ^{k+1} will be close to μ^k) and thus solving (6) for c_k we obtain

$$c_k = -A_k^{-1}\left[\frac{\partial^2 f_2(x_1{}^k,x_2{}^k)}{\partial x_1 \partial x_2} + \mu^{k+1}\frac{\partial^2 h(x_1{}^k,x_2{}^k)}{\partial x_1\partial x_2}\right] d_k +$$

$$-A_k^{-1}(\mu^{k+1}-\mu^k)\frac{\partial h(x_1{}^k,x_2{}^k)}{\partial x_2} \qquad (9)$$

Substituting c_k from (9) into (7) and (6b) yields

$$\left\{\frac{\partial f_1}{\partial x_2} - A_k^{-1}\frac{\partial f_1}{\partial x_1}\left[\frac{\partial^2 f_2}{\partial x_1\partial x_2} + \mu^{k+1}\frac{\partial^2 h}{\partial x_1 \partial x_2}\right]\right\} d - \frac{\partial f_1}{\partial x_2} A_k^{-1}(\mu^{k+1}-\mu^k)\frac{\partial h}{\partial x_2} < 0 \qquad (10a)$$

$$\left\{\frac{\partial h}{\partial x_1} - \frac{\partial h}{\partial x_2} A_k^{-1}\left[\frac{\partial^2 f_2}{\partial x_1\partial x_2} + \mu^{k+1}\frac{\partial^2 h}{\partial x_1\partial x_2}\right]\right\} d - \frac{\partial h}{\partial x_2} A_k^{-1}(\mu^{k+1}-\mu^k)\frac{\partial h}{\mu x_2} = 0 \qquad (10b)$$

A full rank assumption on $\frac{\partial h}{\partial x_2}$ in conjunction with the invertibility of A_k makes (10b) solvable in $\mu^{k+1}-\mu^k$, in which case substitution of $\mu^{k+1}-\mu^k$ in (10a) results in a single inequality condition for d_k. Of course A_k depends on μ^{k+1}, but we can set μ^{k+1} equal to μ^k while calculating A_k and use also μ^k in place of μ^{k+1} in the first term of (10a). Thus we end up with a single inequality for d_k through which d_k can be determined, by using $x_1{}^k, x_2{}^k, \mu^k$ only.

A little reflection will persuade the reader that the method described above (after (8)) is a variation of the reduced gradient projection method for the problem (3), where the unknowns are x_1, x_2, μ. The difference between this method and the reduced gradient method as described in [13] is essentially the following. In [13] the problem

$$\min f(y_1,y_2) \qquad \text{subject to: } h(y_1,y_2) = 0 \qquad (11)$$

is considered and $\frac{\partial h}{\partial y_2}$ (mxm matrix) is assumed nonsingular. y_1 is considered as the independent variable and then $y_2 = g(y_1)$ (since $\frac{\partial h}{\partial y_2}$ nonsingular). (g is not found explicitly.) The procedure of [13] solves essentially:

$$\min f(y_1,g(y_2))$$

In the algorithm we described, the situation is as follows. Starting from

$$\min f(y_1,y_2) \qquad \text{subject to: } h_1(y_1,y_2,y_3) = 0, \quad h_2(y_1,y_2) = 0 \qquad (12)$$

((12) corresponds to (3) and (y_1, y_2, y_3) corresponds to (x_1, x_2, u) we consider $\frac{\partial h_1}{\partial y_2}$ (m×m) nonsingular (see (8)) and thus $y_2 = g(y_1, y_3)$ for some g. So, we solve essentially

$$\min f(y_1, g(y_1, y_3), y_3)$$
$$\text{subject to: } h_2(y_1, g(y_1, y_3), y_3) \tag{13}$$

In other words whereas in (11) ([13]) the constraint is implicitly eliminated, the algorithm we described for the L-F problem eliminates only part of the constraints of (3) (see (12),(13)).

Considering the difficulty of handling gradient projection methods and the intricasies concerning proofs about their convergence, rate of convergence and so on, one might feel not optimistic concerning the algorithm we described. It should not be forgotten though that the partial elimination of constraints method that we essentially described, might very well be suited to several Leader-Follower problems, because perhaps of the type of functions involved and because it springs up from the philosophy underlying L-F games as such.

It should also be noticed that if (7) had been substituted by

$$\underset{c_k,\ d_k}{\text{minimize}}\ \frac{\partial f_1}{\partial x_1} d_k + \frac{\partial f_1}{\partial x_2} c_k + \tfrac{1}{2}\begin{bmatrix} d_k \\ c_k \end{bmatrix}' [\nabla^2_{x_1 x_2} f_1] \begin{bmatrix} d_k \\ c_k \end{bmatrix} \tag{14}$$
$$\text{Subject to: (6a),(6b)}$$

we would have ended up with a Newton-type iteration on the submanifold of the constraints.

<u>A Special Case of the Second Class: Linear Case</u>: In case f_1, f_2 are linear and the constraint set $X_1 = X_2$ is a polyhedron described by linear inequality constraints, the philosophy underlying the algorithm presented yields a particularly attractive procedure. Let $f_1(x_1 x_2) = c_1' x_1 + c_2' x_2$, $f_2(x_1, x_2) = d_1' x_1 + d_2' x_2$ and $X_1 = X_2$ be determined by $A_1 x_1 + A_2 x_2 \le b$. c_i, d_i, b are constant vectors and A_1, A_2 constant matrices. Let us assume that the constraint set is bounded. (2) assumes the form

$$\min c_1' x_1 + c_2' x_2$$
$$\text{subject to: } A_1 x_1 + A_2 x_2 \le b \tag{15}$$
$$d_2' x_2 \le d_2' \bar{x}_2,\ \forall \bar{x}_2 \text{ with } A_1 x_1 + A_2 \bar{x}_2 \le b$$

$T(x_1)$ lies on the closure of the constraint polyhedron and thus the whole reaction set $\mathcal{R}$ of the Follower (i.e., the union of $T(x_1)$'s for all x_1's) lies on the surface of the polyhedron. Thus we can be moving on the outer surface of the polyhedron. Our aim is to find a point in $\mathcal{R}$ which admits a supporting hyperplane perpendicular to (c_1,c_2) leaving $\mathcal{R}$ on the opposite direction of the one where $(-c_1,-c_2)$ points. These geometric considerations suggest the following algorithm.

<u>Step 1</u>: For given x_1 we solve the Followers problem and find a solution x_2.

<u>Step 2</u>: At x_2, we examine all the edges passing from it (finite in number) as to find those having a negative angle with (c_1,c_2). Call $(x_1^1,x_2^1)\ldots(x_1^e,x_2^e)$ the extreme points which are the ends of those directions of decent for f_1, emanating from (x_1,x_2).

<u>Step 3</u>: Find which ones out of $(x_1^1,x_2^1)\ldots,(x_1^e,x_2^e)$ are stationary points for d_2x_2 when we restrict our attention to the hyperplane $x_1 = x_1^1$(or...or x_1^e) (By stationary we mean that it solves min $d_2'x_2$ subject to $A_1x_1 + A_2x_2 \le b$ where $x_1 = x_1^1$ (or...or x_1^e)). If x_2^m is stationary, let $x_1 = x_1^m$ and go to Step 1. If none is stationary stop.

This algorithm will terminate effectively in a finite number of iterations, but it might converge to a point which is a local minimum of (2), because the reaction set $\mathcal{R}$ is not necessarily convex. The nonconvexity of $\mathcal{R}$ appears although f_1,f_2 and the constraints are linear (convex). The nonconvex character of $\mathcal{R}$ is not easy to deal with. Of course under conditions which guarantee convexity (or even less a kind of monotonicity) of $\mathcal{R}$ this is taken care of. One such case is the one where $d_2 = c_2$, where the Leaders problem can be treated as a team problem; i.e. to solve the L-F game we just minimize the Leaders cost subject to x_1,x_2 in the constraint set and forget about the Followers cost. (This can be done also if $f_1(x_1,x_2) = f_2(x_1,x_2) + g(x_1)$ with f_1,f_2,g some nonlinear functions, i.e., when the Leaders cost is the sum of the Followers cost and of another cost which depends on the Leader's decision only.)

(iii) <u>Third Class</u>: This algorithm is motivated by the algorithms presented in [12]. The algorithm of [12] deals with the problem

$$\begin{aligned} &\min f(x) \\ &\text{subject to: } g(x,y) \le 0 \quad \forall y \in Y \qquad (16) \\ &\qquad\qquad\quad x \in X \end{aligned}$$

where Y is an infinite (compact) subset of $R^n (x \in R^n, y \in R^m)$. The idea in [12] is to choose a <u>finite</u> subset $Y^k \subseteq Y$ and substitute $g(x,y) \le 0 \ \forall y \in Y$ with $g(x,y) \le 0 \ \forall y \in Y^k$; after solving the new problem which has a finite number of constraints, alter Y^k to Y^{k+1} and solve a new problem with finite number of constraints again. An intelligent construction of the Y^k's leads to the solution of the initial problem by solving a sequence of problems with finite number of constraints. This scheme applies directly to the L-F problem if the constraints for x_1, x_2 are not coupled; i.e., $x_1 \in X_1 \subseteq R^{n_1}$, $x_2 \in X_2 \subseteq R^{n_2}$, where X_2 will assume the role of Y in the algorithm of [12]. Nonetheless, if the L-F problem is subjected to a constraint of the form $h(x_1,x_2) \le 0$, then (2) becomes

$$\begin{aligned} &\min f_1(x_1,x_2) \\ &\text{subject to: } h(x_1,x_2) \le 0 \qquad (17) \\ &\qquad f_2(x_1,x_2) \le f_2(x_1,x_3) \quad \forall x_3 \ni: \ h(x_1,x_3) \le 0 \end{aligned}$$

and thus the set Y depends on x_1, i.e., $Y(x_1) = \{x_2 : h(x_1,x_2) \le 0\}$. Thus, one should generalize the scheme of [12] as to handle the problem

$$\begin{aligned} &\min f(x) \\ &\text{subject to: } g(x,y) \le 0 \quad \forall y \in Y(x) \qquad (18) \\ &\qquad\qquad\quad x \in X \end{aligned}$$

if he wishes to employ the rational of [12]. The choice of Y^k is quite difficult since Y^k will depend on x. It is not obvious to us how one can do that. In any case, if the constraints of the L-F problem are of the form $x_1 \in X_1 \subseteq R^{n_1}$, $x_2 \in Y_2 \subseteq R^{n_2}$, then use of [12]'s algorithm is possible. Essentially the same discussion applies to the algorithm suggested in [14] for handling problems with an infinite number of constraints.

Conclusions

The three types of classes of algorithms that we described qualitatively above, seem to provide promising directions along which a detailed construction and study of L-F algorithms might go. A quite difficult task concerning the L-F algorithms is due to the nonconvex character of the Followers reaction set. Of course difficulties due to nonconvexity appear in many optimization problems and one cannot expect the researchers interested in L-F algorithms to resolve the "nonconvex" issues. What is reasonable though is to consider subclasses of L-F problems (such as those where the Leaders problem is essentially a team problem, see end of discussion on the Second Class of Algorithms) and prove global convergence of the algorithms under appropriate (convexity ?) assumptions on the constraints and cost which yield a nicely behaved reaction set. The study of local convergence of the second class algorithms can benefit by knowledge concerning gradient projection methods [13].

As far as it concerns the algorithms of the third class it would be worthy to try to bypass the dependence of Y on x keeping still in force the ideas of [12] or [14].

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