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SINGLE SIDEBAND HYBRID MODULATION

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Summary

The signal in a hybrid modulation scheme is of the form,

$$x(t)=a(t)\cos[\omega_c t+\varphi(t)] \qquad (1)$$

The signal a(t) corresponds to amplitude modulation (AM), and the signal $\varphi(t)$ to phase modulation (PM). The problem which we consider in the present paper is the finding of necessary and sufficient conditions for the hybrid modulation signal x(t) in (1) to be a single-sideband (SSB) signal. I.e. we are interested in determining under what general conditions the spectrum of x(t) is zero in the frequncy interval $(-f_c, f_c)$, where $\omega_c=2\pi f_c$.

A number of publications exist about the specific subject of single--sideband frequency modulation (SSB-FM), or generaly SSB angle modulation. Most characteristic are the papers by Powers [1], Bedrosian [2], Barnard [3], Voelker [4] and Werner [5]. Bedrosian [2] defines the SSB-FM signal via the analytic signal.

$$s(t)=m(t)c(t) \qquad (2)$$

Where:

$$c(t)=e^{j\omega_c t} \qquad \text{and} \qquad m(t)=e^{j[\varphi(t)+j\Phi(t)]} \qquad (3)$$

Where $\Phi(t)$ is the Hilbert transform of $\varphi(t)$, i.e.

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$$\hat{\Phi}(t)=\mathcal{H}[\varphi(t)]=\varphi(t)\circledast \frac{1}{\pi t}$$

$$\text{or} \quad \Phi(t) = \frac{1}{\pi} \text{ P.V. } \int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau} d\tau \tag{4}$$

Where $\circledast$ is the symbol for convolution and P.V. denotes the principal value of the integral. Thus Bedrosian deduces the SSB-FM signal as follows:

$$x(t)=\text{Re}[s(t)]=e^{-\hat{\Phi}(t)} \cos[\omega_c t+\varphi(t)] \tag{5}$$

Voelcker[4], using the definitions of Bedrosian, represented all known analog modulation schemes to the form: $A(t)\cos[\theta(t)]$ and presented a unified theory of modulation via the real and complex zeros of the corresponding signals. Werner[5] studies the spectrum of SSB-FM signals for specific classes of modulating signals. He also considers the recovery of the information signal from bandlimited versions of the SSB-FM signals. Barnard[3] genaralized Bedrosian's definition of SSB-FM signals in the space of tempered distributions (generalized functions) and gave an algorithm for demodulating bandlimited versions of single-sideband angle-modulated signals.

In the present paper we present general necessary and sufficient conditions for the hybrid modulation signal x(t) in eqn. (1) to be SSB (upper-sideband) in a generalized function context. In this summary we will present the main result in the form of a Theorem and one example. We will delete here the definitions and lemmas from generalized function theory (theory of distributions) which are necessary for the rigorous proof of our proposition, with the excemption of a few words about tempered distributions. Let S be the space of all "rapidly decreasing" test function[6], i.e. S is the space of all infinitely differentiable functions which, together with their derivatives tend to zero faster than any power of $1/|t|$, as $|t|\to\infty$.

Thus:

$$\xi \in S \to |t^m \xi^{(n)}(t)| \leqq C_{mn} \quad \text{or} \quad \lim_{|t|\to\infty} |t^m \xi^{(n)}(t)| = 0 \quad \text{for } m,n=0,1,2,\ldots \qquad (6)$$

The set of all generalized functions $f(\xi)$ defined over the space S of "rapidly decreasing" functions is denoted by S', S' being the space conjugate to S. Generalized functions in S' are called tempered distributions. Definitions of Hilbert and Fourier transforms of tempered distributions are given by Barnard[3]. Let g be a tempered distribution, i.e. $g \in S'$, we denote its Fourier transform by $\mathfrak{F}.g$. We define the space of bounded bandlimited tempered distributions, denoted by S_o'; as follows:

$$S_o' = \{g | g \in S',\ g \text{ real},\ |g|<\infty,\ |\hat{g}|<\infty,\ \mathfrak{F}.g = 0 \quad \text{for } |f|>f_o\} \qquad (7)$$

Where f is the frequency variable and f_o a given constant. We can now state our result:

Theorem

Let x(t) be the hybrid modulation signal:

$$x(t) = a(t)\cos\left[2\pi f_c t+\varphi(t)\right] \qquad (8)$$

Where $\varphi(t) \in S_o$, i.e. $\varphi(t)$ is a real bounded bandlimited tempered distribution, in other words*.

$$\text{Sup}[\mathfrak{F}\{\varphi(t)\}] \subseteq [-f_o, f_o] \quad \text{and} \quad |\varphi(t)|,\ |\hat{\varphi}(t)|<\infty$$

Where f_o is an arbitrarily large prespecified frequency and $\varphi(t)$

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*Sup|g| denotes the support of the distribution g.

is a real tempered distribution.

Then the necessary and sufficient condition for the signal $x(t)$ to be an SSB-FM signal (i.e. $\mathcal{F}[x(t)] = 0$ for $|f|<f_c$) is that,

$$a(t)\exp[\hat{\phi}(t)] \triangleq \pi(t) \qquad (9)$$

be an entire function of (the complex variable) t of order $\rho < 1$.

Or equivalently:

The spectrum (Fourier transform) of the hybrid modulation signal $x(t)$ in (8) is zero for $f \in (-f_c, f_c)$, i.e.

$$\mathcal{F}[x(t)] = 0 \qquad \text{for} \quad |f|<f_c$$

and hence $x(t)$ is an upper-sideband angle modulated signal, if and only if:

$$\pi(t) \triangleq a(t)\exp[\hat{\phi}(t)] = \sum_{n=0}^{\infty} c_n t^n \qquad (10)$$

$$\text{with} \qquad \lim \sqrt[n]{|c_n| n!} = 0 \qquad (11)$$

In the above $\hat{\phi}(t)$ is the Hilbert transform of $\varphi(t)$.

Remarks: 1. Special cases for $\pi(t)$ are all polynomials of t, since a polynomial is an entire function with order $\rho = 0$.

2. The condition $\text{Sup}[F\{\varphi(t)\}] \in [-f_o, f_o]$ where f_o is finite seems very restrictive and could possibly be relaxed as one can show with examples, including the one we present herein. However we need it in the proof of the theorem. Needless to say that since f_o could be taken arbitrarily large this assumption does not practically restrict the applicability of the theorem.

3. Entire functions $\pi(t)$ of order $\rho < 1$ have a Fourier transform which is a distribution with support the origin, i.e. if

$$\pi(t) = \sum_{n=0}^{\infty} c_n t^n$$

with $\rho = \lim_{n\to\infty}(n\log n/\log|1/c_n|) < 1$ then the Fourier transform of $\pi(t)$ is well defined in the generalized function context and we have:

$$\mathcal{F}[\pi(t)] = \sum_{n=0}^{\infty} (-j)^n c_n \delta^{(n)}(f) \qquad (12)$$

Where $\delta^{(n)}(f)$ is the n^{th} derivative of the delta function.

<u>Example</u>. Verify that signal:

$$x(t) = \frac{1+(t/T)}{\sqrt{1+(t/T)^2}} \cos[2\pi f_c t + \tan^{-1} \frac{t}{T}]$$

is a single-sideband signal and find its Fourier transform.

We have:

$$a(t) = (1+t/T)/[1+(t/T)^2]^{1/2} \quad \text{and}$$

$$\varphi(t) = \tan^{-1}(t/T)$$

In the paper we find:

$$\hat{\Phi}(t) = \frac{1}{2}\log[1+(t/T)^2] \qquad (13)$$

Thus we get:

$$\pi(t) \triangleq a(t)\, e^{\hat{\Phi}(t)} = 1 + \frac{t}{T} \qquad (14)$$

Here $\pi(t)$ is a polynomial. The Fourier transform of $x(t)$ is found in the paper to be:

$$X(f) = (1-j)\pi\, e^{-2\pi T(f-f_c)}\, u(f-f_c) + \frac{j}{2}\delta(f-f_c) \qquad \text{for} \quad f > 0$$

and $X(f) = X^*(-f)$ for $f < 0$, where $u(.)$ is the unit-step function.

We have therefore:

$$X(f) = 0 \qquad \text{for} \quad |f|<f_c$$

and thus $x(t)$ is an upper-sideband signal.

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