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On the Existence of Nash Strategies and Solutions to Coupled Riccati Equations in Linear–Quadratic Games[1]

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Abstract. The existence of linear Nash strategies for the linear-quadratic game is considered. The solvability of the coupled Riccati matrix equations and the stability of the closed-loop matrix are investigated by using Brower's fixed-point theorem. The conditions derived state that the linear closed-loop Nash strategies exist, if the open loop matrix A has a sufficient degree of stability which is determined in terms of the norms of the weighting matrices. When A is not necessarily stable, sufficient conditions for existence are given in terms of the solutions of auxiliary problems using the same procedure.

Key Words. Nonzero sum linear–quadratic games, Nash strategies, coupled Riccati equations, Brower's fixed-point theorem.

1. Introduction

The present work deals with the existence of linear, closed-loop Nash solutions to the continuous, time-invariant, nonzero-sum, linear–quadratic differential game, over an infinite period of time.

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The Nash solution for the linear–quadratic game has been studied in several papers (Refs. 1–10); yet, little is known compared to what is known about the corresponding classical control problem (Refs. 11–12). The closed-loop Nash strategies are not necessarily linear (Ref. 6); and, even if restriction to linear strategies is made, little is known concerning their existence, properties, interpretation in terms of solutions to the coupled algebraic Riccati system, and the stability of the closed-loop system. For the linear–quadratic game over a finite period of time $[0, T]$, there are certain existence results for closed-loop Nash strategies, assuming that T is sufficiently small and/or that the strategies lie in compact subsets of the admissible strategy spaces (Refs. 3, 7, 8). In Refs. 2, 5, the boundedness of the solutions of certain Riccati-type differential equations is assumed in order to guarantee the existence of Nash strategies. Finally, Ref. 15 deals with the static N-person Nash game, under compactness and convexity assumptions for the strategy spaces and concavity assumptions for the criteria.

For the infinite-time case considered here, there is no existence result available known to us. Although our results do not solve the problem completely, they are applicable to a subclass of problems. They are stated in terms of conditions on the norms of the matrices involved, and they do not depend on controllability or observability assumptions. They can be viewed as conditions for solution of certain coupled algebraic Riccati-type matrix equations.

The structure of this paper is as follows. In Section 2, we describe the system, the type of the Nash solution sought, and formulate the problem. These questions are pursued in Sections 3 and 4. The conditions derived in Section 3 state that linear, closed-loop Nash strategies exist, if the open-loop matrix A has a sufficient degree of stability, which is determined in terms of the norms of the weighting matrices. Section 4 contains some extensions of the conditions derived in Section 3 which do not require stability of the open-loop matrix. Our conclusions are given in Section 5.

2. Problem Statement

Consider the dynamic system described by

$$\dot{x} = Ax + B_1 u_1 + B_2 u_2, \qquad x(0) = x_0, \qquad t \in [0, +\infty), \tag{1}$$

and two functionals

$$\begin{aligned} J_1(u_1, u_2) &= \int_0^{+\infty} (x'Q_1 x + u_1' R_{11} u_1 + u_2' R_{12} u_2)\, dt, \\ J_2(u_1, u_2) &= \int_0^{+\infty} (x'Q_2 x + u_2' R_{22} u_2 + u_1' R_{21} u_1)\, dt, \end{aligned} \tag{2}$$

where x, u_1, u_2 are functions of time taking values in R^n, R^{m_1}, R^{m_2}, respectively, and A, B_1, B_2, $Q_i = Q_i'$, $R_{ij} = R_{ij}'$, $R_{ii} > 0$, $i, j = 1, 2$ are real, constant matrices of appropriate dimensions.

The problem is to find u_1^*, u_2^* as linear functions of x, i.e.,

$$u_i^* = -L_i^* x,$$

with L_i^* a real, constant matrix, such that $J_i(u_1^*, u_2^*)$ is finite (see Appendix B), $i = 1, 2$, and

$$\begin{aligned} J_1^* &= J_1(u_1^*, u_2^*) \le J_1(u_1, u_2^*) \qquad \text{for every } u_1 = -L_1 x, \\ J_2^* &= J_2(u_1^*, u_2^*) \le J_2(u_1^*, u_2) \qquad \text{for every } u_2 = -L_2 x. \end{aligned} \tag{3}$$

The conditions (3) are the Nash equilibrium conditions. It is known (see Ref. 1) that a necessary condition for the existence of such controls u_1^*, u_2^* is that there exist constant, real, symmetric matrices K_1, K_2 satisfying

$$\begin{aligned} 0 &= K_1 A + A' K_1 + Q_1 - K_1 B_1 R_{11}^{-1} B_1' K_1 - K_1 B_2 R_{22}^{-1} B_2' K_2 \\ &\quad - K_2 B_2 R_{22}^{-1} B_2' K_1 + K_2 B_2 R_{22}^{-1} R_{12} R_{22}^{-1} B_2' K_2, \\ 0 &= K_2 A + A' K_2 + Q_2 - K_2 B_2 R_{22}^{-1} B_2' K_2 - K_2 B_1 R_{11}^{-1} B_1' K_1 \\ &\quad - K_1 B_1 R_{11}^{-1} B_1' K_2 + K_1 B_1 R_{11}^{-1} R_{21} R_{11}^{-1} B_1' K_1. \end{aligned} \tag{4}$$

It can be proved that, if such K_i's exist and the closed-loop matrix

$$\tilde{A} = A - B_1 R_{11}^{-1} B_1' K_1 - B_2 R_{22}^{-1} B_2' K_2 \tag{5}$$

has Re $[\lambda(\tilde{A})] < 0$, i.e., $\tilde{A}$ is asymptotically stable (a.s.), and[5]

$$Q_i + K_j' B_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} B_j' K_j \ge 0, \qquad i \ne j, \qquad i,j = 1, 2, \tag{6}$$

then the strategies

$$u_i = -L_i^* x = -R_{ii}^{-1} B_i' K_i x, \qquad i = 1, 2, \tag{7}$$

satisfy (3) and J_1^*, J_2^* are finite. In relation to this, see Proposition 1 in Ref. 4. The proof of Proposition 1 in Ref. 4 does not hold under the assumptions stated there; see Appendix A.

In the next section, we will deal with the solution of (4) and try to find conditions under which solutions exist and yield a.s. $\tilde{A}$.

[5] For (6) to hold, it suffices for example that $Q_i \ge 0$, $R_{ij} \ge 0$.

3. Conditions for Existence of Solutions

We start by introducing the following notation:

$$K=\begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}, \quad F=\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \quad Q=\begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}, \quad J=\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix},$$

$$S_1=B_1R_{11}^{-1}B_1', \qquad S_2=B_2R_{22}^{-1}B_2',$$

$$S_{01}=B_2R_{22}^{-1}R_{12}R_{22}^{-1}B_2', \qquad S_{02}=B_1R_{11}^{-1}R_{21}R_{11}^{-1}B_1', \tag{8}$$

$$S=\begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}, \qquad S_0=\begin{bmatrix} S_{01} & 0 \\ 0 & S_{02} \end{bmatrix},$$

where I denotes the $n\times n$ unit matrix. Using this notation, (4) assumes the form

$$0=\mathscr{R}(K)\triangleq F'K+KF+Q-KSK-KJSKJ-JKSJK+JKJS_0JKJ. \tag{9}$$

Consider the space X of $2n\times 2n$ real, symmetric, constant matrices of the form

$$Y=\begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix},$$

where M and N are $n\times n$ matrices. X is a linear subspace of the space of $2n\times 2n$ real matrices. All norms of the matrices to be considered here are the sup norms,

$$\|A\|=\sup\{\|Ax\|:\|x\|=1\},$$

and the norms of the vectors are the square-root Euclidean norms. It is easy to see that, for $Y\in X$,

$$\|Y\|=\max(\|M\|, \|N\|).$$

We denote by I_0 the $2n\times 2n$ unit matrix; and, for $R\geq 0$, we set

$$B_R=\{Y\in X:\|Y\|\leq R\},$$

i.e., B_R is the compact ball of radius R centered at the zero element of X. We define the function Φ from X into X by

$$\Phi(K)=\mathscr{R}(K)+K. \tag{10}$$

Clearly, if $K\in X$, then $\Phi(K)\in X$, and Φ is continuous. The following lemma is proved by using Brower's fixed-point theorem (see Ref. 13, p. 161).

Lemma 3.1. If, for some $R\geq 0$,

$$(3\|S\|+\|S_0\|)R^2+(\|I_0+2F\|-1)R+\|Q\|\leq 0 \tag{11}$$

holds, then there exists $K\in X$, with $\|K\|\leq R$, which satisfies $\mathscr{R}(K)=0$.

Proof. For λ a fixed real number, we have

$$\Phi(K)=K(F+\lambda I_0)+(F'+(1-\lambda)I_0)K+Q$$
$$-KSK-KJSKJ-JKSJK+JKJS_0JKJ,$$

from which, for $K\in B_R$, using the obvious fact that $\|J\|=1$, we get

$$\|\Phi(K)\|\leq R(\|\lambda I_0+F\|+\|(1-\lambda)I_0+F\|)+\|Q\|+R^2(3\|S\|+\|S_0\|).$$

Since

$$\|I_0+2F\|\leq\|\lambda I_0+F\|+\|(1-\lambda)I_0+F\|,$$

with equality for $\lambda=\frac{1}{2}$, we set $\lambda=\frac{1}{2}$ (best λ). The result now follows by direct application of Brower's theorem. □

Let us introduce the transformation

$$K=\alpha\bar{K}, \tag{12}$$

where $\alpha\neq 0$ is a constant and $\bar{K}\in X$. Substituting $K=\alpha\bar{K}$ in (9), we obtain

$$\begin{aligned}0&=\mathcal{R}(K)=\mathcal{R}(\alpha\bar{K})\triangleq\mathcal{R}_\alpha(\bar{K})\\&=(\alpha F)'\bar{K}+\bar{K}(\alpha F)+Q-\bar{K}(\alpha^2S)\bar{K}-\bar{K}J(\alpha^2S)\bar{K}J\\&\quad-J\bar{K}(\alpha^2S)J\bar{K}+J\bar{K}J(\alpha^2S_0)J\bar{K}J.\end{aligned} \tag{13}$$

Applying Lemma 3.1 to

$$\Phi_\alpha(\bar{K})\triangleq\mathcal{R}_\alpha(\bar{K})+\bar{K},$$

we obtain that, if for some $R\geq 0$ it holds that

$$(3\|S\|+\|S_0\|)\alpha^2R^2+(\|I+2\alpha A\|-1)R+\|Q\|\leq 0, \tag{14}$$

then there exists $\bar{K}\in X$, $\|\bar{K}\|\leq R$, which satisfies

$$\mathcal{R}_\alpha(\bar{K})=0.$$

But then

$$K=\alpha\bar{K}$$

satisfies

$$\mathcal{R}(K)=\mathcal{R}_\alpha(\bar{K})=0 \qquad \text{and} \qquad \|K\|\leq|\alpha|R.$$

We thus have proved the following lemma.

Lemma 3.2. If for some $\alpha\neq 0$, $R\geq 0$, (14) holds, then there exists $K\in X$, $\|K\|\leq|\alpha|R$, which satisfies

$$\mathcal{R}(K)=0.$$

The scaling introduced in (12) helps to improve (11) and get (14), because in proving Lemma 3.2 we applied Lemma 3.1 to a whole class of Φ_α's which are nonlinear (quadratic in $\bar{K}$) and asked that at least one of them have a fixed point via Brower's theorem. As it turned out, if one of them (say, $\Phi_{\bar{\alpha}}$) has a fixed point, then all of them have, since

$$\mathcal{R}_\alpha(S) = \mathcal{R}(\alpha S) = \mathcal{R}_\beta((\alpha/\beta)S),$$

although (14) may not hold for $\beta \neq \bar{a}$.

Set

$$a = 3\|S\| + \|S_0\|, \qquad b = \|I + 2\alpha A\|, \qquad q = \|Q\|, \qquad \epsilon = \sqrt{(qa)}. \tag{15}$$

Then, (14) assumes the form

$$a\alpha^2 R^2 + (b-1)R + q \leq 0. \tag{16}$$

If $a = 0$, then

$$B_1 = 0 \qquad \text{and} \qquad B_2 = 0.$$

and the game is meaningless as such. Therefore, assume $a \neq 0$. Inequality (14) is satisfied for some $R \geq 0$ iff

(i) $1 \geq b + 2|\alpha|\epsilon$,

or

(ii) $q = 0$ and $1 < b$.

In Case (ii), $R = 0$ is the only solution to (16), and thus Lemma 3.2 guarantees only the solution $K_1 = 0$, $K_2 = 0$. Consequently, we will concentrate on Case (i), i.e., when

$$1 \geq \|I + 2\alpha A\| + 2|\alpha|\epsilon. \tag{17}$$

If (17) holds, then (16) is satisfied for all R: $R_1 \leq R \leq R_2$, where

$$R_{1,2} = \frac{1 - b \pm \sqrt{[(b-1)^2 - 4\alpha^2\epsilon^2]}}{2\alpha^2 a} \geq 0. \tag{18}$$

In this case, Lemma 3.2 guarantees the existence of solutions K_1, K_2 with

$$\|K_1\|, \|K_2\| \leq |\alpha| R.$$

We have the following theorem.

Theorem 3.1. Let $a > 0$. If, for some $\alpha \neq 0$, (17) is satisfied, then for every $R: R_1 \leq R \leq R_2$, where R_1, R_2 are as in (18), there exist K_1, K_2 satisfying (4); such that

$$\|K_i\| \leq |\alpha| R \leq |\alpha| R_2 \leq 2\|A\|/(3\|S\| + \|S_0\|) = M. \tag{19}$$

Proof. The proof has already been given, except for the right-hand side of (19). Since

$$1 = \|I + 2\alpha A - 2\alpha A\| \leq \|I + 2\alpha A\| + 2|\alpha|\|A\|,$$

we have

$$1 - b \leq 2|\alpha|\|A\|,$$

and so

$$|\alpha| R_2 = |\alpha| \frac{1 - b + \sqrt{[(1-b)^2 - 4\alpha^2 \epsilon]}}{2\alpha^2 a} \leq \frac{(1-b) + \sqrt{[(1-b)^2]}}{2|\alpha| a} \leq 2\|A\|/a = M.$$

□

Notice in passing that M is independent of the magnitude of α. Before giving the next theorem which provides us with necessary and sufficient conditions for the existence of an $\alpha \neq 0$ satisfying (17), we state the following lemma, the proof of which is given in Appendix C.

Lemma 3.3. Let Γ be a real $n \times n$ matrix, γ and ρ real numbers $\gamma \neq 0$, and $\lambda(\Gamma) = \sigma + jw$ be any eigenvalue of Γ. Then, the following results hold.

(i) If

$$\|I + \gamma \Gamma\| \leq \rho, \tag{20}$$

then

$$(\sigma + 1/\gamma)^2 + w^2 \leq (\rho/\gamma)^2. \tag{21}$$

(ii) If $\gamma > 0$ and $\rho = 1$, then $\sigma < 0$ or $\sigma = w = 0$ for every $\lambda(\Gamma)$.
(iii) If $\gamma < 0$ and $\rho = 1$, then $\sigma > 0$ or $\sigma = w = 0$ for every $\lambda(\Gamma)$.
(iv) Let $\Gamma = T \Lambda T^{-1}$, where Λ is the Jordan canonical form of Γ. We set

$$\rho' = \|T\| \cdot \|T^{-1}\|, \qquad \rho' \geq 1.$$

If

$$\|I + \gamma \Lambda\| \leq \rho/\rho', \tag{22}$$

then (20) holds.

(v) If Λ is diagonal and

$$(\sigma + 1/\gamma)^2 + w^2 \leq (\rho/\rho'\gamma)^2$$

holds, then (20) holds. In particular, if Γ is symmetric, then (20) is equivalent to (21).[6]

Theorem 3.2. Let $\lambda(A) = \sigma + jw$ be an eigenvalue of A.

(i) If $\epsilon = 0$ and (17) is satisfied for some $\alpha \neq 0$, then for every $\lambda(A)$ it holds that

$$\sigma = \operatorname{Re} \lambda(A) < 0 \text{ or } \lambda(A) = 0 \text{ if } \alpha > 0, \tag{23-1}$$

$$\sigma = \operatorname{Re} \lambda(A) > 0 \text{ or } \lambda(A) = 0 \text{ if } \alpha < 0, \tag{23-2}$$

$$(\sigma + 1/2\alpha)^2 + w^2 \leq (1/2\alpha)^2 \tag{23-3}$$

$$\|K_1\|, \|K_2\| \leq (1 - \|I + 2\alpha A\|)/|\alpha| a. \tag{24}$$

(ii) If $\epsilon \neq 0$ and $A = -\epsilon I$, then any $0 < \alpha \leq 1/2\epsilon$ satisfies (17) and

$$\|K_1\|, \|K_2\| \leq \alpha R_2 = \epsilon / a. \tag{25}$$

(iii) If $\epsilon \neq 0$, $A \neq -\epsilon I$, and (17) is satisfied for some $\alpha \neq 0$, then for any $\lambda(A)$ it holds that

$$\begin{aligned} &\sigma < -\epsilon \text{ or } \lambda(A) = -\epsilon \text{ if } \alpha > 0, \\ &\sigma > +\epsilon \text{ or } \lambda(A) = \epsilon \text{ if } \alpha > 0, \end{aligned} \tag{26}$$

and $\|K_1\|$, $\|K_2\|$ satisfy (19). Moreover, $|\alpha| < 1/2\epsilon$.

(iv) If $A \neq -\epsilon I (A \neq \epsilon I)$ is diagonalizable, $A = T \Lambda T^{-1}$, where Λ is the Jordan canonical form of A, $\rho' = \|T\| \cdot \|T^{-1}\|$, and for some $\gamma > 0 (\gamma < 0)$ it holds that

$$(\sigma + \epsilon + 1/\gamma)^2 + w^2 \leq 1/\rho'\gamma^2 \qquad ((\sigma - \epsilon + 1/\gamma)^2 + w^2 \leq 1/\rho'\gamma^2), \tag{27}$$

then

$$\alpha = \gamma / 2(1 + \epsilon\gamma) \qquad (\alpha = \gamma / 2(1 - \epsilon\gamma))$$

satisfies (17).

[6] The assumption that Λ is diagonal in (v) is essential. As a counterexample, let

$$\Gamma = \Lambda = \begin{bmatrix} -\frac{1}{2} & 1 \\ 0 & -\frac{1}{2} \end{bmatrix}, \qquad T = T^{-1} = 1, \qquad \rho = 1,$$

in which case (21) is satisfied for all $\gamma: 0 < \gamma \leq 4$; but, for $x = (1/\sqrt{2})(1, 1)'$, one has $\|x\| = 1$,

$$\|(I + \gamma\Gamma)x\| = \sqrt{(1 + \gamma^2/4)} > 1.$$

Proof. (i) Expressions (23) follow from Lemma 3.3 (i), (ii), (iii), and (14) follows from (19).

(ii) The first part is trivial. Expression (25) follows as (24).

(iii) Let $b = 1/|2\alpha|$. Then, (17) yields

$$b-\epsilon \geq \|bI + A\| \qquad \text{if } \alpha > 0,$$

$$b-\epsilon \geq \|bI + A\| \qquad \text{if } \alpha < 0.$$

Necessarily,

$$b-\epsilon > 0.$$

Let

$$\gamma = 1/(b-\epsilon);$$

then,

$$1 \geq \|I + \gamma(A + \epsilon I)\| \qquad \text{if } \alpha > 0,$$

$$1 \geq \|I + \gamma(-A + \epsilon I)\| \qquad \text{if } \alpha < 0.$$

We set

$$\Gamma = \pm A + \epsilon I,$$

and we have

$$1 \geq \|I + \gamma\Gamma\|. \tag{28}$$

Expressions (26) follow now from Lemma 3.3(ii)–(iii).

(iv) We bring (17) to the form (28) and apply part (v) of Lemma 3.3. □

If A is symmetric, then the existence of γ satisfying (27) is equivalent to the existence of α satisfying (17). For (17) to hold, it suffices that

$$1 \geq \sqrt{\{\mathrm{tr}\,[(I+2\alpha A)'(I+2\alpha A)]\}} + 2|\alpha|\epsilon.$$

By using the fact

$$\|M\|^2 \leq \mathrm{tr}\,(M'M),$$

it follows that the existence of an $\alpha > 0$ satisfying (17) is guaranteed in the following two cases (assuming that A is not a scalar).

(i) $\mathrm{tr}\, A \leq -\epsilon$,

$\mathrm{tr}\, A'A < \epsilon^2$,

$\Delta = (n-1)\,\mathrm{tr}\, A'A + (\mathrm{tr}\, A)^2 + 2\epsilon\, \mathrm{tr}\, A + (2-n)\epsilon^2 \geq 0$,

$1/2\epsilon \geq [-(\epsilon + \mathrm{tr}\, A) - \sqrt{\Delta}]/[2(\epsilon^2 - \mathrm{tr}\, A'A)]$;

(ii) $\mathrm{tr}\, A'A > \epsilon^2$,

$\mathrm{tr}\, A \leq -\epsilon$.

We shall now consider the stability of the closed-loop matrix (5). Let

$$a' = 4\|S\| + \|S_0\|,$$

$$\epsilon' = \sqrt{qa'}, \qquad \epsilon'^2 = \epsilon^2 + \|S\| \cdot \|Q\|,$$

$$R_2 = \frac{1-b+\sqrt{[(1-b)^2 - 4\alpha^2\epsilon^2]}}{2\alpha^2 a}, \tag{29}$$

$$R_2' = \frac{1-b+\sqrt{[(1-b)^2 - 4\alpha^2\epsilon'^2]}}{2\alpha^2 a'},$$

where a, ϵ, q, b are as in (15).

Theorem 3.3. Let $\mathrm{Re}[\lambda(A)] < 0$.

(i) If, for some $\alpha > 0$, it holds that

$$1 \geq \|I + 2\alpha A\| + 2\alpha\epsilon, \tag{30-1}$$

$$1 > \|I + 2\alpha A\|, \tag{30-2}$$

$$\alpha^2\|S\|R_2^2 < \alpha^2\|S_0\|R_2^2 + \|Q\|, \tag{30-3}$$

then there exist K_1, K_2, $\|K_i\| \leq \alpha R_2$, $i = 1, 2$, solving (9) and $\tilde{A}$ given by Eq. (5) is a.s.

(ii) If, for some $\alpha > 0$, it holds that

$$1 \geq \|I + 2\alpha A\| + 2\alpha\epsilon', \tag{31-1}$$

$$1 > \|I + 2\alpha A\|, \tag{31-2}$$

$$\|Q\| \text{ or } \|S_0\| \neq 0, \tag{31-3}$$

then there exist K_1, K_2, $\|K_i\| \leq \alpha R_2'$, $i = 1, 2$, solving (9) and $\tilde{A}$ given by Eq. (5) is a.s.

Proof. (i) Relationship (30-1) makes Theorem 3.1 applicable. We have

$$\|I + 2\alpha\tilde{A}\| = \|I + 2\alpha(A - S_1\alpha\bar{K}_1 - S_2\alpha\bar{K}_2)\| \leq \|I + 2\alpha A\| + 4\alpha^2\|S\|\,\|\bar{K}\|.$$

Since, by Lemma 3.3, $\tilde{A}$ will be a.s. if

$$\|I + 2\alpha\tilde{A}\| < 1$$

for some $\alpha > 0$, it suffices that

$$\|I + 2\alpha A\| + 4\alpha^2\|S\|\,\|\bar{K}\| < 1.$$

Relationship (30-2) implies $R_2 > 0$. Using $K = \alpha\bar{K}$ and (19), we have that it suffices that

$$(\|I + 2\alpha A\| - 1)R_2 + 4\alpha^2\|S\|R_2^2 < 0;$$

and, since (14) holds for $R = R_2$, it suffices that

$$4\alpha^2\|S\|R_2^2 < (3\|S\| + \|S_0\|)\alpha^2 R_2^2 + \|Q\|,$$

which is equivalent to (31-3).

(ii) Relationship (31-1) implies that the inequality

$$a'\alpha^2 R^2 + (\|I + 2\alpha A\| - 1)R + q \leq 0$$

is satisfied for all $R : R_1' \leq R \leq R_2'$, where

$$R_1', 2 = \frac{1 - \|I + 2\alpha A\| \pm \sqrt{[(1 - \|I + 2\alpha A\|)^2 - 4\alpha^2\epsilon'^2]}}{2\alpha^2 a'},$$

$R_2' \geq R_1' \geq 0$. Relationship (31-2) implies that $R_2' > 0$. If the above inequality holds for some R and α, then (14) holds for the same R and α, since $a \leq a'$. Therefore, there exist K_1, K_2, $\|K_i\| \leq \alpha R_2'$, $i = 1, 2$, solving (9). Repeating the analysis of (i), we have that, for $\tilde{A}$ to be a.s., it suffices that

$$4\alpha^2\|S\|R_2'^2 < (4\|S\| + \|S_0\|)\alpha^2 R_2'^2 + \|Q\|,$$

which holds by (31-3). □

If equality is allowed in (30-3) or

$$\|Q\| = \|S_0\| = 0$$

in (31-3), then the conclusions of (i) and (ii) in the given theorem change and allow $\tilde{A}$ stable, i.e.,

$$\mathrm{Re}[\lambda(\tilde{A})] \leq 0.$$

The geometric interpretation of the conditions given in Theorems 3.2 and 3.3 is given in Figs. 1 and 2. Figure 1 corresponds to Theorem 3.2, parts (i), (ii), (iii), which say that a necessary condition for the existence of an $\alpha > 0$ satisfying (17) is that the eigenvalues of A lie in a disk centered at $-1/2\alpha$ with radius $r = 1/2\alpha - \epsilon$, for some $\alpha > 0$, which is equivalent to saying that all $\lambda(A)$'s lie in the open half-plane on the left of the line ϵ_1, $\sigma = -\epsilon$, or at $-\epsilon$. Figure 2 corresponds to Theorem 3.2, part (iv), where it is assumed that A is diagonalizable. It shows that, if the eigenvalues of A lie in a disk as in Fig. 2 with radius $r = (1/\sqrt{\rho'})(1/2\alpha - \epsilon)$ and centered at $-1/2\alpha$, then this α satisfies (17), and thus (9) has a solution. If A is symmetric, then $\rho' = 1$ and $\theta = 90°$. If $\alpha < 0$, then we have the mirror images with respect to the jw-axis of the circles, cones, and lines depicted in Figs. 1, 2.

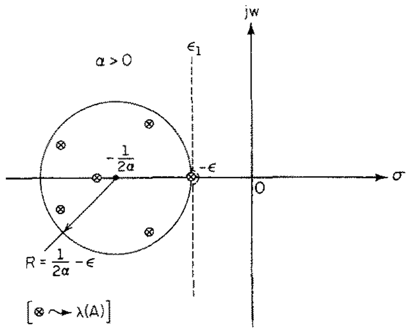

Fig. 1. Regions of eigenvalues of A in accordance with Theorem 3.2 (i)–(iii).

Employment of a different function Φ in (10) and application of Brower's theorem may in general provide different, perhaps better, existence results. Another suitable Φ can be defined as follows. If K solves (9) and A (and thus F) is a.s., then $\bar{K} = K/\alpha$, where $\alpha > 0$, solves equivalently (see Appendix D)

$$\bar{K} = \int_0^{+\infty} \exp(\alpha Ft)(Q - \bar{K}\alpha S\bar{K} - \bar{K}J\alpha^2 S\bar{K}J - J\bar{K}\alpha^2 SJ\bar{K} + J\bar{K}J\alpha^2 S_0 J\bar{K}J)\exp(\alpha F't)\, dt. \tag{32}$$

Let $\Phi_\alpha(\bar{K})$ denote the right-hand side of (32). Let also

$$A = T\Lambda T^{-1},$$

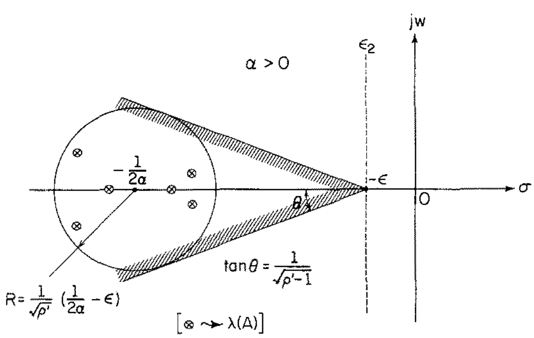

Fig. 2. Region of eigenvalues of A in accordance with Theorem 3.2 (iv).

Λ being the Jordan canonical form of A, with $m \times m$ the dimension of the largest Jordan block, $m \leq n$. Let also a, q, ϵ be as in (15), $\alpha > 0$, and

$$\begin{aligned} \rho &= \|T\| \, \|T^{-1}\|, \\ \bar{\lambda} &= \max \operatorname{Re}[\lambda(A)] < 0, \\ \pi(w) &= \sum_{i,j=0}^{m-1} [(i+j)!/i!j!](w/2)^{i+j+1} \qquad \text{for } w > 0, \\ \bar{\pi} &= \pi(1/(-\alpha\bar{\lambda})), \end{aligned} \tag{33}$$

and $\epsilon(m, \alpha) > 0$ be such that

$$\pi(1/\epsilon(m, \alpha)) = 1/2\alpha\epsilon.$$

Clearly, $\epsilon(m, \alpha)$ exists and is unique, for given α and m.

Theorem 3.4. Let A be a.s. and T, Λ, m, π, $\epsilon(m, \alpha)$ as above. If it holds that

$$\bar{\lambda} \leq -\epsilon(m, \alpha)(1/\alpha), \tag{34}$$

then there is $K \in X$ which satisfies (9), and

$$\|K\| \leq \alpha R_2 = \alpha(1/2a\alpha^2)\{1/\rho^2\bar{\pi} + \sqrt{(1/\rho^4\bar{\pi}^2 - 4qa\alpha^2)}\} \leq 2|\bar{\lambda}|/a\rho^2. \tag{35}$$

In addition, if A is diagonalizable (i.e., $m = 1$), then

$$\epsilon(1, \alpha) = \alpha\epsilon\rho^2$$

for every $\alpha > 0$.

Proof. Let $\bar{K} = \alpha K$, $\alpha > 0$, $\|\bar{K}\| \leq R$, $R \geq 0$. In order to use Brower's theorem we ask for $\|\Phi_\alpha(K)\| \leq R$ for some α and R. It suffices that

$$\begin{aligned} \|\Phi_\alpha(\bar{K})\| &= \Big\| \int_0^{+\infty} \exp(\alpha Ft)(Q - \bar{K}\alpha^2 S\bar{K} - \bar{K}J\alpha^2 S\bar{K}J - J\bar{K}\alpha^2 SJ\bar{K} \\ &\qquad + J\bar{K}J\alpha^2 S_0 J\bar{K}J) \exp(\alpha F't)\, dt \Big\| \\ &\leq [\|Q\| + (3\|S\| + \|S_0\|)\alpha^2 R^2] \int_0^{+\infty} \|\exp(\alpha Ft)\|^2 \, dt \leq R; \end{aligned}$$

or, by using (69) (see Appendix E),

$$a\alpha^2 R^2 - (1/\rho^2\bar{\pi})R + q \leq 0, \tag{36}$$

which holds because of (33)–(34). The rest is easy to prove. □

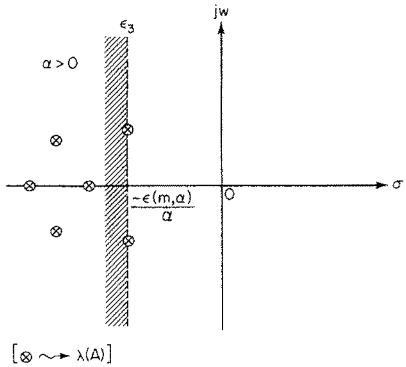

Fig. 3. Regions of eigenvalues of A in accordance with Theorem 3.4.

It is remarked that, if A is diagonalizable, then (34) gives $\lambda \leq -\epsilon\rho^2$. Introduction of $\alpha > 0$ induces no improvement of the result, which is in agreement with the fact that scaling cannot facilitate the existence of solutions of (9). In case A has $\mathrm{Re}[\lambda(A)] > 0$, we can have results similar to those of Theorem 3.4 by employing $\alpha < 0$.

The geometric interpretation of Theorem 3.4 is given in Fig. 3 and shows simply that, if the eigenvalues of A lie on the closed half-plane on the left of the line ϵ_2,

$$\sigma = -\epsilon(m, \alpha)/\alpha,$$

for some $\alpha > 0$, then there exists K which solves (9). If A is diagonalizable, then

$$-\epsilon(m, \alpha)/\alpha = -\epsilon\rho^2;$$

and, since $\rho \geq 1$, the line ϵ_2 is on the left of ϵ_1. In this case, a combination of Theorem 3.2(iv) and Theorem 3.4 gives an easily verified sufficient condition for solvability of (9).

Finally, Theorem 3.3 (ii) can be interpreted along the same lines as Theorem 3.2, using Figs. 1 and 2, where ϵ' is used instead of ϵ. So, if $\epsilon' \neq 0$, A is diagonalizable, and all $\lambda(A)$'s lie in a disk as in Fig. 2 with ϵ' in place of ϵ, then (9) has a solution and the closed-loop matrix $\tilde{A}$ is a.s. If $\epsilon' = 0$ and A is diagonalizable,

$$\|Q\| = 0, \qquad \epsilon = 0,$$

then if the eigenvalues of A lie in the *interior* of the disk in Fig. 2, the same conclusion holds. The version of Theorem 3.3 with $\alpha < 0$ and $\mathrm{Re}[\lambda(A)] > 0$ gives $\tilde{A}$ unstable, i.e., $\mathrm{Re}[\lambda(\tilde{A})] > 0$, and is thus of no interest to us.

We close this section with five remarks.

(i) The only assumption on the Q_i, R_{ij}'s used in developing the proofs in this section was that R_{11}^{-1}, R_{22}^{-1} exist. Neither $Q_i \geq 0$ or $Q_i \leq 0$ or $R_{ij} \geq 0$, nor any controllability, observability, or optimality conditions were used.

(ii) If

$$\bar{Q} = Q + KSK + JKJS_0JKJ \geq 0$$

(it suffices that R_{12} and $R_{21} \geq 0$) and $\tilde{A}$ is a.s., since

$$K\begin{bmatrix} \tilde{A} & 0 \\ 0 & \tilde{A} \end{bmatrix} + \begin{bmatrix} \tilde{A} & 0 \\ 0 & \tilde{A} \end{bmatrix}' K = -\bar{Q},$$

a standard result in Lyapunov theory yields $K \geq 0$. Since

$$J_i^* = x_0' K_i x_0$$

(see Ref. 4), we will have $J_i^* \geq 0$ for every x_0, as it should be expected in case Q_i, $R_{ij} \geq 0$, $i, j = 1, 2$.

(iii) Consider the single Riccati equation

$$KA + A'K + Q - KBR^{-1}B'K = 0, \tag{37}$$

where $R > 0$ and A is a.s., with Q not necessarily positive definite. Then, Brower's theorem provides results which can be used to verify easily whether the frequency condition of Lemma 5 in Ref. 11 holds. It is easy to prove (as in Theorems 3.1 and 3.2) the following results.

(a) If there is $\alpha > 0$ such that

$$1 \geq \|I + 2\alpha A\| + 2\alpha\sqrt{[\|Q\|\,\|BR^{-1}B'\|]}, \tag{38}$$

then (37) has a solution K. If in addition

$$\alpha^2\|S\|R_2 < \|Q\|, \qquad 1 > \|I + 2\alpha A\|, \tag{39}$$

where

$$R_2 = \frac{1 - \|I + 2\alpha A\| + \sqrt{[(1 - \|I + 2\alpha A\|)^2 - 4\|Q\|\,\|BR^{-1}B'\|]}}{2\|S\|}, \tag{40}$$

then $A - BR^{-1}B'K$ is a.s.

(b) If there is an $\alpha > 0$ such that

$$1 \geq \|I + 2\alpha A\| + 2\alpha\sqrt{2}\sqrt{[\|Q\|\,\|BR^{-1}B'\|]}, \tag{41}$$

then (37) has a solution K and $A - BR^{-1}B'K$ is a.s.

Let now $Q = -C'C \leq 0$, and assume (A, C) observable and (A, B) controllable. Using Lemma 5 of Ref. 11, we see that (38) and (39) or (41) imply that

$$I - B'(-jw - A')^{-1}C'C(jw - A)^{-1}B \geq 0 \qquad \text{for all real } w. \tag{42}$$

Also, since

$$K = \int_0^{+\infty} \exp(Ft)(Q - KSK)\exp(F't)\,dt \qquad \text{and} \qquad Q \leq 0,$$

it will be

$$K \leq 0.$$

(iv) In order to guarantee the fixed-point property of $\Phi(K)$, one could have employed contraction-mapping machinery. Then, Φ should map a closed set $D_0 \subset X$ into itself; in addition, Φ should be Lipschitzian with Lipschitz constant L, $0 \leq L < 1$, on D_0. But, since Φ is quadratic in K, D_0 should be bounded in order to guarantee that Φ is Lipschitzian there. However, this amounts to D_0 compact, and we could consider D_0 a ball B_R (w.l.o.g.). So, in order to use the contraction-mapping theorem, we should have made assumptions to guarantee that $L < 1$, in addition to those made to allow the use of Brower's theorem, and this would result in a weaker conclusion.

(v) The assumptions of Theorem 3.1 guarantee the existence of K_1, K_2 solving (4), which lie in B_{R_1}. Thus, if the solution of (4) is unique, it will be in B_{R_1}. If not, then there may be additional solutions K_1, K_2 in B_R, $R > R_1$, which are not in B_{R_1}, which solve (4).

4. Extensions

Let us now try to relax the assumption on A to be a.s. Two approaches will be considered. In both of them, we use the solution of an appropriately defined auxiliary problem in order to show existence of solutions to our main problem via Brower's theorem.

Consider first the optimal control problem

$$\begin{aligned} &\dot{x} = Ax + [B_1 \vdots B_2]\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \qquad x(0) = x_0, \qquad t \in [0, +\infty), \\ &\min \int_0^{\infty} \left(x'\tilde{Q}x + [u_1' \vdots u_2']\tilde{R}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right) dt, \end{aligned} \tag{43}$$

where

$$\tilde{Q} = \tfrac{1}{2}(Q_1 + Q_2), \qquad \tilde{R} = \begin{bmatrix} \tfrac{1}{2}(R_{11} + R_{21}) & 0 \\ 0 & \tfrac{1}{2}(R_{22} + R_{12}) \end{bmatrix} = \begin{bmatrix} \tilde{R}_1 & 0 \\ 0 & \tilde{R}_2 \end{bmatrix} \tag{44}$$

with $\tilde{R}_1$, $\tilde{R}_2 > 0$.[7] Under certain assumptions (controllability–observability

[7] For this to hold, it suffices that $R_{ii} > 0$, $R_{ij} \geq 0$, $i \neq j$, $i, j = 1, 2$.

and $\tilde{Q} \geq 0$, or see Theorem 2, page 167 in Ref. 14, or Remark (iii) in Section 3), there exists $\tilde{K}$ satisfying

$$0 = \tilde{K}A + A'\tilde{K} + \tilde{Q} - \tilde{K}(\tilde{S}_1 + \tilde{S}_2)\tilde{K}, \tag{45}$$

with

$$\tilde{S}_1 = B_1\tilde{R}_1^{-1}B_1', \qquad \tilde{S}_2 = B_2\tilde{R}_2^{-1}B_2',$$

such that

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = -\begin{bmatrix} \tilde{R}_1^{-1}B_1'\tilde{K}x \\ \tilde{R}_2^{-1}B_2'\tilde{K}x \end{bmatrix} \tag{46}$$

solves (43), and such that

$$\tilde{\tilde{A}} = A - (\tilde{S}_1 + \tilde{S}_2)\tilde{K}; \tag{47}$$

i.e., the closed-loop matrix for (43) is a.s. Let

$$K_1 = \tilde{K} + \Delta_1, \qquad K_2 = \tilde{K} + \Delta_2 \tag{48}$$

be substituted in (4); and, by using (45), we obtain

$$\begin{aligned} 0 = {} & \Delta_1\tilde{\tilde{A}} + \tilde{\tilde{A}}'\Delta_1 + \Delta_1(\tilde{S}_1 + \tilde{S}_2 - S_1 - S_2)\tilde{K} + \tilde{K}(\tilde{S}_1 + \tilde{S}_2 - S_1 - S_2)\Delta_1 \\ & + \Delta_2(S_{01} - S_2)\tilde{K} + \tilde{K}(S_{01} - S_2)\Delta_2 - \Delta_1S_1\Delta_1 - \Delta_1S_2\Delta_2 - \Delta_2S_2\Delta_1 \\ & + \Delta_2S_{01}\Delta_2 + \tfrac{1}{2}(Q_1 - Q_2) + \tilde{K}(\tilde{S}_1 + \tilde{S}_2 + S_{01} - S_1 - S_2 - S_2)\tilde{K}, \\ 0 = {} & \Delta_2\tilde{\tilde{A}} + \tilde{\tilde{A}}'\Delta_2 + \Delta_2(\tilde{S}_1 + \tilde{S}_2 - S_1 - S_2)\tilde{K} + \tilde{K}(\tilde{S}_1 + \tilde{S}_2 - S_1 - S_2)\Delta_2 \\ & + \Delta_1(S_{02} - S_1)\tilde{K} + \tilde{K}(S_{02} - S_1)\Delta_1 - \Delta_2S_2\Delta_2 - \Delta_2S_1\Delta_1 - \Delta_1S_1\Delta_2 \\ & + \Delta_1S_{02}\Delta_1 + \tfrac{1}{2}(Q_2 - Q_1) + \tilde{K}(\tilde{S}_1 + \tilde{S}_2 + S_{02} - S_1 - S_2 - S_1)\tilde{K}, \end{aligned} \tag{49}$$

where S_i, S_{0i} are given in (8).

Let a be as in (15) and

$$\begin{aligned} \Delta &= \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}, \\ \mu &= \|\tilde{K}(\tilde{S}_1 + \tilde{S}_2 - S_1 - S_2)\| + \mu_1, \\ \mu_1 &= \max\{\|\tilde{K}(S_{01} - S_2)\|, \|\tilde{K}(S_{02} - S_1)\|\}, \\ \tilde{q} &= \max\{\|\tfrac{1}{2}(Q_1 - Q_2) + \tilde{K}(\tilde{S}_1 + \tilde{S}_2 + S_{01} - S_1 - S_2 - S_2)\tilde{K}\|, \\ & \quad \cdot \|\tfrac{1}{2}(Q_2 - Q_1) + \tilde{K}(\tilde{S}_1 + \tilde{S}_2 + S_{02} - S_1 - S_2 - S_1)\tilde{K}\|\}, \\ \tilde{b} &= \|I + 2\alpha\tilde{\tilde{A}}\| + 2\alpha\mu, \\ \tilde{R}_2 &= \frac{1 - \tilde{b} + \sqrt{[(1 - \tilde{b})^2 - 4\alpha^2a^2\tilde{q}]}}{2\alpha^2a}. \end{aligned} \tag{50}$$

The proof of the following theorem is similar to the proofs of Theorems 3.1 and 3.3.

Theorem 4.1. Let $\tilde{\tilde{A}}$, μ, $\tilde{q}$ be as in (47), (50), $\alpha \neq 0$. Then, the following results hold.

(i) If, for some $\alpha > 0$,

$$1 \geq \|I + 2\alpha\tilde{\tilde{A}}\| + 2\alpha[\mu + \sqrt{(a\tilde{q})}], \tag{51}$$

then there are Δ_1, Δ_2 such that

$$K_1 = \tilde{K} + \Delta_1, \qquad K_2 = \tilde{K} + \Delta_2$$

solve (4) and

$$\|\Delta_i\| \leq \alpha\tilde{R}_2.$$

(ii) If, in addition to (51),

$$1 > \tilde{b}, \qquad \alpha^2\|S\|\tilde{R}_2^2 < \alpha^2\|S_0\|\tilde{R}_2^2 + 2\alpha\mu_1\tilde{R}_2 + \tilde{q}, \tag{52}$$

then the closed-loop matrix $\tilde{A}$ [as in (5)] is a.s.

(iii) If, for some $\alpha > 0$,

$$\begin{gathered} 1 \geq \|I + 2\alpha\tilde{\tilde{A}}\| + 2\alpha\{\mu + \sqrt{[(a + \|S\|)\tilde{q}]}\}, \\ 1 > \tilde{b}, \\ q \text{ or } \|S_0\| \text{ or } \mu_1 \neq 0, \end{gathered} \tag{53}$$

then both (i) and (ii) above hold.

Proof. Equation (49) can be written as

$$\begin{aligned} 0 = {} & \Delta\tilde{F} + \tilde{F}'\Delta + \psi(\Delta) \\ & + \begin{bmatrix} \tfrac{1}{2}(Q_1 - Q_2) + \tilde{K}(\tilde{S}_1 + \tilde{S}_2 + S_{01} - S_1 - S_2 - S_2)\tilde{K} & 0 \\ 0 & \tfrac{1}{2}(Q_2 - Q_1) + \tilde{K}(\tilde{S}_1 + \tilde{S}_2 + S_{02} - S_1 - S_2 - S_1)\tilde{K} \end{bmatrix}, \end{aligned}$$

where

$$F = \begin{bmatrix} \tilde{\tilde{A}} & 0 \\ 0 & \tilde{\tilde{A}} \end{bmatrix}.$$

Using similar methods as in the proof of Lemma 3.2, we conclude that it suffices that

$$a\alpha^2 R + [\|I + 2\alpha\tilde{\tilde{A}}\| - 1 + 2\mu\alpha]R + \tilde{q} \leq 0 \tag{54}$$

in order for (49) to have a solution Δ_1, Δ_2, where

$$\|\Delta_1\|, \|\Delta_2\| \leq \alpha R.$$

The rest follows as in the proofs of Theorems 3.1 and 3.3. □

The usefulness of the approach presented is clear in case the game is used to describe a situation where two independent controllers desire to achieve the same objective using slightly different information (Q_i) or control effort (R_{ij}).

Consider now the two independent control problems

$$\begin{gathered} \text{minimize} \int_0^\infty (x'Q_1x + u_1'R_{11}u_1)\,dt, \\ \dot{x} = Ax + B_1u_1, \qquad x(0) = x_0, \qquad t \in [0, +\infty). \end{gathered} \tag{55}$$

and

$$\begin{gathered} \text{minimize} \int_0^\infty (x'Q_2x + u_2'R_{22}u_2)\,dt, \\ \dot{x} = Ax + B_2u_2, \qquad x(0) = x_0, \qquad t \in [0, +\infty). \end{gathered} \tag{56}$$

Under proper assumptions, the two Riccati equations

$$\begin{aligned} 0 &= A'\bar{K}_1 + \bar{K}_1A + Q_1 - \bar{K}_1S_1\bar{K}_1, \\ 0 &= A'\bar{K}_2 + \bar{K}_2A + Q_2 - \bar{K}_2S_2\bar{K}_2 \end{aligned} \tag{57}$$

have solutions $\bar{K}_1$, $\bar{K}_2$,

$$u_1 = -R_{11}^{-1}B_1'\bar{K}_1x, \qquad u_2 = -R_{22}^{-1}B_2'\bar{K}_2x$$

solve (55) and (56), and

$$\bar{A}_1 = A - S_1\bar{K}_1, \qquad \bar{A}_2 = A - S_2\bar{K}_2 \tag{58}$$

are a.s. Let

$$\begin{aligned} &\bar{F} = \begin{bmatrix} \bar{A}_1 & 0 \\ 0 & \bar{A}_2 \end{bmatrix} = F - S\bar{K}, \qquad \bar{K} = \begin{bmatrix} \bar{K}_1 & 0 \\ 0 & \bar{K}_2 \end{bmatrix}, \\ &K_1 = \bar{K}_1 + \Delta_1, \qquad K_2 = \bar{K}_2 + \Delta_2, \\ &\Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}. \end{aligned} \tag{59}$$

Then, (4) can be written as

$$
\begin{aligned}
0 = \bar{F}'\Delta + \Delta\bar{F} - \Delta S\Delta - \Delta JS\Delta J - J\Delta SJ\Delta + J\Delta JS_0J\Delta J - \bar{K}JS\bar{K}J \\
- J\bar{K}SJ\bar{K} + J\bar{K}JS_0J\bar{K}J - \bar{K}JS\Delta J - \Delta JS\bar{K}J - J\bar{K}SJ\Delta \\
- J\Delta SJ\bar{K} + J\Delta JS_0J\bar{K}J + J\bar{K}JS_0J\Delta J.
\end{aligned} \tag{60}
$$

Let also

$$
\begin{aligned}
\bar{\mu} &= \|\bar{K}S\| + \|\bar{K}JS_0\| + \|\bar{K}JS\|, \\
\bar{q} &= \|\bar{K}JS\bar{K}J + J\bar{K}SJ\bar{K} - J\bar{K}JS_0J\bar{K}J\|, \\
\bar{b} &= \|I + 2\alpha\bar{F}\| + 2\alpha\bar{\mu}, \\
\bar{R}_2 &= \frac{1 - \bar{b} + \sqrt{[(1-\bar{b})^2 - 4\alpha^2 a\bar{q}]}}{2\alpha^2 a}.
\end{aligned} \tag{61}
$$

Theorem 4.2. Let $\bar{F}$, $\bar{\mu}$, $\bar{q}$ be as in (59), (61), $\alpha \neq 0$.

(i) If, for some $\alpha > 0$, it holds that

$$1 \geq \|I + 2\alpha\bar{F}\| + 2\alpha[\bar{\mu} + \sqrt{(a\bar{q})}], \tag{62}$$

then there exists Δ_1, Δ_2 such that

$$\bar{K}_1 + \Delta_1, \qquad \bar{K}_2 + \Delta_2$$

solve (4) and

$$\|\Delta_i\| \leq \alpha\bar{R}_2.$$

(ii) If, in addition to (62),

$$1 > \bar{b},\ \alpha^2\|S\|\bar{R}_2^2 < \alpha^2\|S_0\|\bar{R}_2^2 + 2\alpha(\|\bar{K}JS_0\| + \|\bar{K}JS\|)\bar{R}_2 + \bar{q}, \tag{63}$$

then the closed-loop matrix $\tilde{A}$, given by (5), is a.s.

(iii) If, for some $\alpha > 0$,

$$
\begin{aligned}
1 \geq \|I + 2\alpha\bar{F}\| + 2\alpha\{\bar{\mu} + \sqrt{[a(\bar{q} + \|S\|)]}\}, \\
1 > b, \\
\bar{q} \text{ or } \|S_0\| \text{ or } \|\bar{K}JS_0\| + \|\bar{K}JS\| \neq 0,
\end{aligned} \tag{64}
$$

then both the conclusions (i) and (ii) hold.

Proof. Working as in Theorem 4.1, it suffices that

$$a\alpha^2R^2 + [\|I + 2\alpha\bar{F}\| - 1 + 2\alpha\bar{\mu}]R + \bar{q} \leq 0 \tag{65}$$

in order for (60) to have a solution Δ, where

$$\|\Delta\| \leq \alpha R,$$

and so on. $\square$

The usefulness of this approach lies in the fact that the results which pertain to the case where the system is controlled separately by the decision-makers can be used to check the existence of the solution when the two decision-makers control it jointly and use Nash strategies.

Theorems 3.2, 3.3, 3.4 and the interpretations in Figs. 1, 2, 3 hold also for the two approaches presented, with the appropriate modifications. For example, for Theorems 3.2, 3.4, Figs. 1, 2, and the first approach, one should use $\tilde{\tilde{A}}$, $\mu + \sqrt{(a\tilde{q})}$, $\mu + \sqrt{[(a + \|S\|)\tilde{q}]}$ instead of A, ϵ, ϵ', respectively.

Finally, note that the existence results in all cases developed previously are dependent on the parameter ϵ (or ϵ'). Since ϵ is a function of the weighting matrices and since rescaling the criteria will affect the weighting matrices, it is of interest to point out how this scalling affects the existence results. Nothing changes in the game if we have

$$J_i' = r_i J_i,$$

instead of J_i, $r_i > 0$, $i = 1, 2$. So considering $r_i Q_i$, $r_i R_{ij}$ instead of Q_i, R_{ij} we have

$$\epsilon^2 = \max(r_1\|Q_1\|, r_2\|Q_2\|)[3\max(\|S_1\|/r_1, \|S_2\|/r_2) + \max(\|S_{01}\|/r_1, \|S_{02}\|/r_2)]$$

or

$$\epsilon^2 = \epsilon^2(r) = \max(\|Q_1\|/r, \|Q_2\|) \cdot [3\max(r\|S_1\|, \|S_2\|) + \max(r\|S_{01}\|, \|S_{02}\|)],$$

where $r = r_2/r_1$. Carrying out the minimization of $\epsilon^2(r)$ with respect to r, we find the minimum ϵ^*

$$\epsilon^* = \sqrt{\{3\max[\|Q_1\| \cdot \|S_1\|, \|Q_2\| \cdot \|S_2\|] + \max[\|Q_1\| \cdot \|S_{01}\|, \|Q_2\|\,\|S_{02}\|]\}}. \tag{66}$$

The value of r^* at which $\epsilon(r)$ becomes minimum is given by the following. Let

$$r_\alpha = \min(\|S_2\|/\|S_1\|, \|S_{02}\|/\|S_{01}\|), \qquad r_\beta = \max(\|S_2\|/\|S_1\|, \|S_{02}\|/\|S_{01}\|),$$
$$\bar{r} = \|Q_1\|/\|Q_2\|.$$

Ir $\bar{r} \leq r_\alpha \leq r_\beta$, then r^* is any point in $[\bar{r}, r_\alpha]$.
If $r_\alpha \leq \bar{r} \leq r_\beta$, then $r^* = \bar{r}$.
If $r_\alpha \leq r_\beta \leq \bar{r}$, then r^* is any point in $[r_\beta, \bar{r}]$.

For ϵ' as in (29), the same analysis holds, and the optimum ϵ'^* is given by a relation exactly the same as (66), but with 4 multiplying the first term instead of 3. We can consider in all of our conditions that $\epsilon(\epsilon')$ is given by (66). Notice also that a similar procedure will give the minimum values of $\mu+\sqrt{(aq)}$, see (50), (61). It is interesting to notice that, if $\bar{r}\leq r_\alpha$, then all $r:\bar{r}\leq r\leq r_\alpha$ give the same ϵ^*. Actually, as (3) indicates, the existence of solutions for the game should not depend on multiplying J_1 or J_2 by a positive constant. Our conditions have at least preserved this property for an interval $[\bar{r}, r_\alpha]$ or $[r_\beta, \bar{r}]$.

5. Conclusions

This paper provides partial results concerning the existence of linear Nash strategies. The applicability of fixed-point theorems (Brower's theorem) was demonstrated, and some existing results were interpreted in a new manner [Remark (iv) in Section 3]. The generalization of our results to the N-player case is obvious. It should be pointed out that, for many of the conditions presented, no assumptions of controllability, observability, or semidefiniteness were made. Therefore, we have singled out a region of parameter space (A, B_i, Q_i, R_{ij}) where the existence of solutions does not depend on controllability and observability. This region is necessarily contained in the region where A is asymptotically stable, or is the neighborhood of a parameter point for which a solution of an auxiliary control problem exists. Outside this region, the existence of solutions will depend in general on controllability and observability properties, but presently conditions under which existence can be guaranteed are not known.

6. Appendix A

In Ref. 4, Proposition 1 states that given (1), (2), where $R_{11}, R_{22}>0$, then if K_1, K_2 satisfying (4) exist and $\tilde{A}$, given by (5), is a.s., then the strategies (7) satisfy (3). This is not true, as the counterexample

$$\dot{x}=x+u+v, \qquad -J_2=J_1=\int_0^{+\infty}(x^2+u^2-2v^2)\,dt$$

demonstrates. This example is used in Ref. 9 to show that, in the zero-sum case, the linear solution for the game over a finite period of time $[0, T]$ does not, as $T\to+\infty$, tend to the linear solution of the infinite-time case. But, if one makes additional assumptions, then the conclusion holds.

The correct form of Proposition 1 in Ref. 4 is the following.

Proposition 6.1. Given the system (1) and the two functionals (2), where Q_i, R_{ij} are real, symmetric matrices and R_{11}, $R_{22}>0$, assume that there exist real, symmetric matrices K_1, K_2 satisfying (4) and (5) and either (i) or (ii) hold:

(i) $Q_i + K_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} K_j \geq 0, \quad i, j = 1, 2, \quad i \neq j.$

(ii) The two control problems

$$\min \int_0^{+\infty} (x'[Q_i + K_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} K_j]x + u_i' R_{ii} u_i)\, dt, \qquad i \neq j, \qquad i, j = 1, 2,$$

$$\dot{x} = (A - B_j R_{jj}^{-1} B_j' K_j)x + B_i u_i,$$

satisfy the conditions of Theorem 2, p. 167 of Ref. 14. Then, the strategies (7) satisfy (3) and J_1^*, J_2^* are finite.

7. Appendix B

The case where at least one of J_1^*, J_2^* is $\pm\infty$ can also be examined. For example, if we are interested in a linear Nash equilibrium where $J_1^* = \pm\infty$ and J_2^* is finite, then this amounts to seeking

$$u_i = -L_i^* x, \qquad i = 1, 2,$$

where (i), (ii), (iii) below hold.

(i) The control problem

$$\bar{J}_2 = \int_0^{+\infty} (x'(Q_2 + L_1^{*\prime} R_{21} L_1^*)x + u_2 R_{22} u_2)\, dt,$$

$$\dot{x} = (A - B_1 L_1^*)x + B_2 u_2$$

has $J_2^* = \min \bar{J}_2$ finite. For example, assume controllability and $Q_2 + L_1^{*\prime} R_{21} L_1^* \geq 0$.

(ii) The problem

$$\bar{J}_1 = \int_0^{+\infty} (x'(Q_1 + L_2^* R_{12} L_2^*)x + u_1' R_{11} u_1)\, dt,$$

$$\dot{x} = (A - B_2 L_2^*)x + B_1 u_1$$

has $\bar{J}_1 = +\infty$ for every $u_1 = -L_1 x$, which means roughly that some uncontrollable mode λ of the pair $(A - B_2 L_2^*, B_1)$, which does not lie in the null space of $Q_1 + L_2^* R_{12} L_2^*$, has $\mathrm{Re}[\lambda] \geq 0$.

(iii) The following hold:

$$L_2^* = R_{22}^{-1} B_2' K_2,$$

$$K_2(A - B_1 L_1^*) + (A - B_1 L_1^*)' K_2 + (Q_2 + L_1^{*\prime} R_{21} L_1^*) - K_2 B_2 R_{22}^{-1} B_2' K_2 = 0.$$

Similarly, one can form conditions for the cases

$$J_1^* = J_2^* = \pm\infty, \qquad J_1^* = -\infty, \qquad J_2^* \text{ finite.}$$

8. Appendix C: Proof of Lemma 3.3

(i) Let v be an eigenvector of Γ corresponding to $\sigma + jw = \lambda(\Gamma)$ and $\|v\| = 1$, w.l.o.g. Then,

$$\rho \geq \|I + \gamma\Gamma\| \geq \|(I + \gamma\Gamma)v\| = |1 + \gamma(\sigma + jw)|,$$

and (21) follows.

(ii) This follows trivally from (21) by noticing that (21) corresponds to a disk with center at $-1/\gamma$ and radius $1/|\gamma|$, which, in case $\gamma > 0$ and $\rho \leq 1$, lies in the left half-plane of the (σ, jw)-plane.

(iii) See (ii) above.

(iv) The proof is trivial.

(v) This follows by using (iii). If Γ is symmetric, then

$$T' = T^{-1} \qquad \text{and } \|T'\| = \|T\| = \sqrt{[\lambda \max(T'T)]} = 1,$$

and thus

$$\rho' = 1. \qquad \square$$

9. Appendix D

Consider the matrix differential equation

$$\dot{K}_1 = K_1A + A'K_1 + Q_1 - K_1S_1K_1 - K_1S_2K_2 - K_2S_2K_1 + K_2S_{01}K_2, \quad (67)$$

$$\dot{K}_2 = K_2A + A'K_2 + Q_2 - K_2S_2K_2 - K_2S_1K_1 - K_1S_1K_2 + K_1S_{02}K_1, \quad (68)$$

where K_1, K_2 are time-varying, $t \geq 0$, and

$$K_1(0) = \Gamma_1, \qquad K_2(0) = \Gamma_2$$

are the initial conditions. Then, it follows that, for $t \geq 0$ sufficiently small, it holds that

$$K_1(t) = \exp(At)\Gamma_1 \exp(A't) + \int_0^t \exp(A\sigma)$$

$$\times [Q_1 - K_1S_1K_1 - K_1S_2K_2 - K_2S_2K_1 + K_2S_{01}K_2] \exp(A'\sigma)\, d\sigma,$$

and a similar result holds for $K_2(t)$. The constant matrices Γ_1, Γ_2 solve (67), (68) iff

$$\Gamma_1 = \exp(At)\Gamma_1 \exp(A't) + \int_0^t \exp(A\sigma)$$
$$\times [Q_1 - \Gamma_1 S_1 \Gamma_1 - \Gamma_1 S_2 \Gamma_2 - \Gamma_2 S_2 \Gamma_1 + \Gamma_2 S_{01} \Gamma_2] \exp(A'\sigma)\, d\sigma,$$

and a similar result holds for Γ_2. Because A is a.s. and Γ_1, Γ_2 are constant, the integral

$$I_\infty = \int_0^{+\infty} \exp(A\sigma)[Q_1 - \Gamma_1 S_1 \Gamma_1 - \Gamma_1 S_2 \Gamma_2 - \Gamma_2 S_2 \Gamma_1 + \Gamma_2 S_{01} \Gamma_2] \exp(A'\sigma)\, d\sigma$$

exists. Also, with $w = \sigma - t$,

$$\int_t^{+\infty} \exp(A\sigma)[Q_1 - \Gamma_1 S_1 \Gamma_1 - \Gamma_1 S_2 \Gamma_2 - \Gamma_2 S_2 \Gamma_1 + \Gamma_2 S_{01} \Gamma_2] \exp(A'\sigma)\, d\sigma$$
$$= \int_0^{+\infty} \exp[A(w+t)][Q_1 - \Gamma_1 S_1 \Gamma_1 - \Gamma_1 S_2 \Gamma_2 - \Gamma_2 S_2 \Gamma_1$$
$$+ \Gamma_2 S_{01} \Gamma_2] \exp[A'(w+t)]\, dw$$
$$= \exp(At) I_\infty \exp(A't),$$

and thus

$$I_\infty = \int_0^t + \int_t^{+\infty} = \int_0^t + \exp(At) I_\infty \exp(A't),$$

from which we conclude that

$$I_\infty = \Gamma_1.$$

A similar result holds for Γ_2. Introducing scaling,

$$\Gamma_1 = \alpha \bar{\Gamma}_2, \qquad \alpha > 0,$$

and setting

$$K_i = \Gamma_i,$$

we have (32).

10. Appendix E

It is easy to see that

$$\|\exp(Ft)\| \le \|T\|\, \|T^{-1}\|\, \|\exp(\Lambda t)\|.$$

Let $t > 0$ and

$$\Lambda = \begin{bmatrix} J_1 & & & \\ & J_2 & & 0 \\ & & \ddots & \\ & 0 & & J_k \end{bmatrix},$$

where the J_i's are the Jordan blocks of dimension $m_1, \ldots, m_k$, with

$$m_1 + \cdots + m_k = n.$$

Let

$$\exp(J_i t) = \exp(\lambda_i t)\Delta_i = \exp(\lambda_i t) \begin{bmatrix} 1 & t/1! & t^2/2! & \ldots & t^{m_i-1}/(m_i-1)! \\ & 1 & \ddots & & \vdots \\ & & & \ddots & \\ 0 & & & & 1 \end{bmatrix}.$$

Then,

$$\|\exp(\Lambda t)\| = \max_{i=1,\ldots,k} \|\exp(\lambda_i t)\Delta_i\| \leq \exp(\bar{\lambda} t) \max_{i=1,\ldots,k} \|\Delta_i\|.$$

We have

$$\Delta_i = I + \left[\begin{array}{c|c} 0 & t\begin{bmatrix} 1 & t/2! & \cdots & t^{i-2}/(i-1)! \\ & 1 & \ddots & \vdots \\ & & \ddots & \vdots \\ 0 & & & 1 \end{bmatrix} \\ \hline 0 & 0 \end{array}\right] = I + \left[\begin{array}{c|c} 0 & t\bar{\Delta}_i \\ \hline 0 & 0 \end{array}\right].$$

So, if

$$x = (x_1, \ldots, x_i)' \in R^i, \qquad \tilde{x} = (x_2, \ldots, x_i)',$$

then

$$\begin{aligned} \|\Delta_i\| &= \sup_{\|x\|=1} \|\Delta_i x\| \leq \sup_{\|x\|=1} \{\|x\| + t\|\bar{\Delta}_i \tilde{x}\|\} \\ &= 1 + t \sup_{x_2^2 + \cdots + x_i^2 = 1} \|\bar{\Delta}_i \tilde{x}\| = 1 + t \sup_{\|\tilde{x}\| \leq 1} \|\bar{\Delta}_i \tilde{x}\| = 1 + t\|\bar{\Delta}_i\|. \end{aligned}$$

For $\bar{\Delta}_i$, we have

$$\bar{\Delta}_i = \begin{bmatrix} 1 & t/2! & t^2/2 \cdot 3 & \ldots & t^{i-2}/2 \ldots (i-1) \\ & \ddots & & & \vdots \\ & & & \ddots & \\ 0 & & & & 1 \end{bmatrix}$$

$$= I + \left[\begin{array}{c|c} 0 & (t/2)\begin{bmatrix} 1 & t/3 & \dots & t^{i-3}/3 \dots (i-1) \\ & & & \\ 0 & & & 1 \end{bmatrix} \\ \hline 0 & 0 \end{array}\right]$$

$$= I + \left[\begin{array}{c|c} 0 & (t/2)\bar{\bar{\Delta}}_i \\ \hline 0 & 0 \end{array}\right].$$

We have again

$$\|\bar{\Delta}_i\| \leq 1 + (t/2)\|\bar{\bar{\Delta}}_i\|,$$

and thus

$$\|\Delta_i\| \leq 1 + t[1 + (t/2)\|\bar{\bar{\Delta}}_i\|].$$

Continuing similarly, we end up with

$$\begin{aligned} \|\Delta_i\| &\leq 1 + t(1 + t/2(1 + t/3(\dots((1 + t/i - 2)(1 + t/i - 1))\dots))) \\ &= 1 + t/1! + t^2/2! + \cdots + t^{i-1}/(i-1)!. \end{aligned}$$

Therefore, if

$$m = \max(m_1, \dots, m_k),$$

then

$$\begin{aligned} \|\exp(Ft)\| &\leq \|T\| \, \|T^{-1}\| \, \|\exp(\Lambda t)\| \\ &\leq \|T\| \, \|T^{-1}\| \exp(\lambda t)\left(1 + \frac{t}{1!} + \cdots + \frac{t^{m-1}}{(m-1)!}\right) = \rho \sum_{j=0}^{m-1} \left(\frac{t^j}{j!}\right). \end{aligned}$$

Direct calculation (recall that $\lambda < 0$) gives

$$\begin{aligned} \int_0^{+\infty} \|\exp(Ft)\|^2 \, dt &\leq \rho^2 \int_0^{+\infty} \exp(2\bar{\lambda} t)\left[\sum_{j=0}^{m-1} \left(\frac{t^j}{j!}\right)\right]^2 dt \\ &= \rho^2 \int_0^{+\infty} \sum_{i,j=0}^{m-1} \exp(2\bar{\lambda} t) \frac{t^{i+j}}{i!\,j!} \, dt \\ &= \rho^2 \sum_{i,j=0}^{m-1} \frac{(i+j)!}{i!\,j!} \left(\frac{1}{-2\bar{\lambda}}\right)^{i+j+1} = \rho^2 \pi(-1/\bar{\lambda}). \end{aligned} \tag{69}$$

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