Nonclassical Control Problems and Stackelberg Games

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Abstract—**A nonclassical control problem, where the control depends on state and time, and its partial derivative with respect to the state appears in the state equation and in the cost function is analyzed. Stackelberg dynamic games which lead to such nonclassical control problems are considered and studied.**

Introduction

HIERARCHICAL and large scale systems have received considerable attention during the last few years; firstly because of their importance in engineering, economics, and other areas, and secondly because of the increased capability of computer facilities [13], [14]. An important characteristic of many large scale systems is the presence of many decision makers with different and usually conflicting goals. The existence of many decision makers who interact through the system and have different goals may be an inherent property of the system under consideration (e.g., a market situation), or may be simply the result of modeling the system as such (e.g., a large system decomposed to subsystems for calculation purposes). Differential games are useful in modeling and studying dynamic systems where more than one decision maker is involved. Most of the questions posed in the area of the classical control problem may be considered in a game situation, but their resolution is generally more difficult. In addition, many questions can be posed in a game framework, which are meaningless or trivial in a classical control problem framework. The superior conceptual wealth of game over control problems, which makes them potentially much more applicable, counterbalances the additional difficulties encountered in their solution.

A particular class of games are the so-called Stackelberg differential games [1]–[8]. Stackelberg games provide a natural formalism for describing systems which operate on many different levels with a corresponding hierarchy of decisions. The mathematical definition of a general two-level Stackelberg game is as follows. Let U, V be two sets and J_1, J_2 two real-valued functions

$$J_i : U \times V \to R, \qquad i = 1,2. \tag{1}$$

We consider the set valued mapping T

$$T: U \to V, \quad u \overset{T}{\mapsto} Tu \subseteq V \tag{2}$$

defined by

$$Tu = \{ v | v = \arg\inf [J_2(u,\bar{v});\ \bar{v} \in V] \}. \tag{3}$$

Clearly $Tu = \varnothing$ if the inf in definition (3) is not achieved. We also consider the minimization problem

$$\begin{aligned} &\inf J_1(u,v) \\ &\text{subject to: } u \in U, v \in Tu, \end{aligned} \tag{4}$$

where we use the usual convention $J_1(u,v) = +\infty$ if $v \in Tu = \varnothing$.

Definition: A pair $(u^*, v^*) \in U \times V$ is called a Stackelberg equilibrium pair if (u^*, v^*) solves (4).

The sets U and V are called the leader's and follower's strategy spaces, respectively. The game situation described by the mathematical formulation above is as follows. The follower tries to minimize his cost function J_2, for a given choice of $u \in U$ by the leader. The leader knowing the follower's rationale, wishes to announce a u^* such that the follower's reaction v^* to this given u^* will result to the minimum possible $J_1(u^*, v^*)$. The general N-level Stackelberg game is defined analogously. Stackelberg differential games were first introduced and studied in the engineering literature in [2] and further studied in [3]–[8]. They are mathematically formalized as follows:

$$\begin{aligned} &\dot{x}(t) = f(x(t), \bar{u}(t), \bar{v}(t), t), \qquad x(t_0) = x_0 \\ &J_i(u,v) = g_i(x(t_f), t_f) + \int_{t_0}^{t_f} L_i(x(t), \bar{u}(t), \bar{v}(t), t)\, dt, \\ &\qquad\qquad\qquad i = 1,2 \end{aligned} \tag{5}$$

where f, g_i, L_i are appropriately defined functions. Also, $u \in U$, $v \in V$, where U, V are appropriately defined function spaces and $\bar{u}(t)$, $\bar{v}(t)$ are the values of u and v, respectively, at time t, i.e., $\bar{u}(t) = u|_t$, $\bar{v}(t) = v|_t$. The type of strategy spaces U and V which were considered and

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treated successfully in the previous literature were the spaces of piecewise continuous functions of time. In this case, the problem of deriving necessary conditions for the Stackelberg differential game with fixed time interval and initial condition x_0, falls within the area of classical control. Thus, variational techniques can be used in a straightforward manner. The case where the strategy spaces are spaces of functions whose values at instant t depend on the current state $x(t)$ and time t, i.e., $\bar{u}(t)=u|_t=u(x(t),t)$, $\bar{v}(t)=v|_t=v(x(t),t)$, was not treated. This case results in a nonclassical control problem because $\partial u/\partial x$ appears in the follower's necessary conditions. Since the follower's necessary conditions are seen as state differential equations by the leader, the presence of $\partial u/\partial x$ in them makes the leader face a nonclassical control problem.

In the present paper, the nonclassical control problem arising from the consideration of the above strategy spaces is embedded in a more general class of nonclassical control problems; see (20)–(22). The characteristics of this general class of problems are the following: 1) each of the components u^i, of the control m-vector u, depends on the current time t and on a given function of the current state and time, i.e., $u^i|_t=u^i(h^i(x(t),t),t)$; 2) the state equation and the cost functional depend on the first-order partial derivative of u with respect to the state x. The vector valued functions h^i may represent outputs or measurements available to the ith "subcontroller," in a decentralized control seting. The only restriction to be imposed on h^i is to be twice continuously differentiable with respect to x. This allows for a quite large class of h^i's which can model output feedback or open-loop control laws. It can also model mixed cases of open-loop and output feedback control laws where during only certain intervals of time an output is available. The appearance of the partial derivative of u with respect to x prohibits the restriction of the admissible controls to those which are functions of time only. It will become clear that the extension of our results to the case where higher order partial derivatives of u with respect to x, up to order N, appear is straightforward. This case is of interest in hierarchical systems since it arises, for example, in an N-level Stackelberg game where the players use control values dependent on the current state and time. Although the bulk of the analysis provided in this paper concerns continuous-time problems, the corresponding discrete-time results can be derived in a very similar manner.

The structure of the present paper is as follows: In Section I a two-level Stackelberg differential game is introduced for a fixed time interval $[t_0,t_f]$ and initial condition $x(t_0)=x_0$. The leader's and follower's strategies are functions of the current state and time. This game leads to the consideration of a nonclassical control problem which is studied in Section II. In Section III we use the results of Section II to study further the game of Section I, and in particular we work out a linear quadratic Stackelberg game. In Section IV the relation of the Stackelberg game, introduced in Section I, to the principle of optimality is investigated. Finally we have a Conclusions section and two Appendices.

Notation and Abbreviations

R^n: n-dimensional real Euclidean space with the Euclidean metric;

$\|\ \|$: denotes the Euclidean norm for vectors and the sup norm for matrices;

$'$: denotes transposition for vectors and matrices.

For a function f: $R^n \to R^m$ we say that $f \in C^k$ if f has continuous mixed partial derivatives of order k. For f: $R^n \to R$, ∇f is considered an $n \times 1$ column vector and f_{xx} denotes the Hessian of f. For f: $R^n \to R^m$, ∇f is considered an $n \times m$ matrix (Jacobian). For f: $R^n \times R^k \to R^m$, where $x \in R^n$, $y \in R^k$, $f(x,y) \in R^m$, we denote by $\partial f/\partial x$ or f_x or $\nabla_x f$ the Jacobian matrix of the partial derivatives of f with respect to x and is considered as $n \times m$ matrix.

w.r. to: with respect to;

w.l.o.g.: without loss of generality;

n.b.d.: neighborhood.

I. A Stackelberg Game

In this section we introduce a two-level Stackelberg game and show how it leads us to the consideration of a nonclassical control problem. This nonclassical control problem falls into the general class to be considered in Section II.

Let

$$U=\{u \mid u\colon R^n \times [t_0,t_f] \to R^{m_1},\ u(x,t) \in R^{m_1} \text{ for } x \in R^n \text{ and } t \in [t_0,t_f],\ u_x(x,t) \text{ exists and } u(x,t),\ u_x(x,t) \text{ are continuous in } x \text{ and piecewise continuous in } t\} \tag{6}$$

$$V=\{v \mid v\colon [t_0,t_f] \to R^{m_2},\ v \text{ is piecewise continuous in } t\}. \tag{7}$$

Consider the dynamic system

$$\dot{x}(t)=f(x(t),\bar{u}(t),\bar{v}(t),t), \quad x(t_0)=x_0, \quad t \in [t_0,t_f] \tag{8}$$

and the functionals

$$J_1(u,v)=g(x(t_f))+\int_{t_0}^{t_f} L(x(t),\bar{u}(t),\bar{v}(t),t)\,dt \tag{9}$$

$$J_2(u,v)=h(x(t_f))+\int_{t_0}^{t_f} M(x(t),\bar{u}(t),\bar{v}(t),t) \tag{10}$$

where $u \in U$, $v \in V$, x is the state of the system, assumed to be a continuous function of t and piecewise in C^1 w.r. to t, x: $[t_0,t_f] \to R^n$, and the functions f: $R^n \times R^{m_1} \times R^{m_2} \times [t_0,t_f] \to R^n$, g,h: $R^n \to R$, L,M: $R^n \times R^{m_1} \times R^{m_2} \times [t_0,t_f] \to R$, are in C^1 w.r. to the x, u, v arguments and continuous in t. The u and v are called strategies and are chosen from U and V which are called the strategy spaces, by the two players, the leader and the follower, respectively. With the given definitions, for each choice of u and v, the behavior of the dynamic system is unambiguously determined, assuming of course, that for the selected pair (u,v) the solution of the differential equation (8) exists over $[t_0,t_f]$.

Let us assume that a Stackelberg equilibrium pair $(u^*,v^*)\in U\times V$ exists. For fixed $u\in U$, Tu is determined by the minimization problem

$$\begin{gathered}\text{minimize } J_2(u,v)\\ \text{subject to: } v\in V\\ \dot{x}=f(x,u(x,t),v,t),\quad x(t_0)=x_0,\ t\in[t_0,t_f]\end{gathered}\tag{11}$$

and thus, applying the minimum principle we conclude that for $v\in V$ to be in Tu, there must exist a function p: $[t_0,t_f]\to R^n$ such that

$$\dot{x}=f(x,u,v,t)\tag{12a}$$

$$M_v+f_v'p=0\tag{12b}$$

$$-\dot{p}=M_x+u_xM_u+(f_x+u_xf_u)p\tag{12c}$$

$$x(t_0)=x_0,\quad p(t_f)=\frac{\partial h(x(t_f))}{\partial x}.\tag{12d}$$

We further assume that U is properly topologized. Conditions (12) define a set valued mapping T': $U\to V$. By using the nature of the defined U and V and the fact that (12) are necessary but not sufficient conditions it is easily proven that

i) $Tu\subseteq T'u$,

ii) $J_2(u,v')\geqslant J_2(u,v)\quad \forall: v'\in T'u,\ v\in Tu$,

iii) $T'u^*\cap Tu^*\supseteq\{v^*\}\neq\varnothing$.

Notice that $J_2(u,v)$ takes one value for given u and *any* $v\in Tu$, while $J_2(u,v')$, $v'\in T'u$ does not necessarily do so. We assume now the following.

Assumption A:

$$J_1(u,v')\geqslant J_1(u,v)\qquad \text{for } v'\in T'u,\ v\in Tu,\ u\in U_N^*\tag{13}$$

where U_N^* is a n.b.d. of u^* in U.

For Assumption A to hold it suffices for example: $T=T'$ on U_N^*.[1] We conclude that if Assumption A holds, then u^* is a local minimum of the problem

$$\begin{gathered}\text{minimize } J_1(u,v)\\ \text{subject to: } u\in U,\ v\in T'u\end{gathered}$$

or equivalently

$$\begin{gathered}\text{minimize } J_1(u,v)\\ \text{subject to: } u\in U,\ v\in V\end{gathered}\tag{14}$$

$$\dot{x}=f(x,u,v,t)\tag{15a}$$

$$-\dot{p}=M_x+u_xM_u+(f_x+u_xf_u)p\tag{15b}$$

$$M_v+f_v'p=0\tag{15c}$$

$$x(t_0)=x_0,\quad p(t_f)=\frac{\partial h(x(t_f))}{\partial x}.\tag{15d}$$

The problem (14) is a nonclassical control problem of the type to be considered in the next section, since the partial derivative of the control u w.r. to x appears in the constraints (15) which play the role of the system differential equations and state control constraints, with new state $(x',p')'$. Notice that the leader uses only $x(t)$ and t in evaluating $u(x(t),t)$ and not the whole state $(x',p')'$; i.e., the value of u at time t is composed in a partial feedback form with respect to the state $(x',p')'$; (recall the output feedback in contrast to the state feedback control laws). If one were concerned with a Stackelberg game composed of N ($\geqslant 2$) hierarchical decision levels [7], [8], then the leader would face a nonclassical control problem where the $(N-1)$th partial of u with respect to x would appear.

We will assume that the state-control constraint (15c) can be solved for v over the whole domain of interest to give

$$v=S(x,p,u,t)\tag{16}$$

where S is continuous and in C^1 w.r. to x and p. This assumption holds in many cases, as for example in the linear quadratic case to be considered in the next section. In any case, direct handling of the constraint (15c) by appending it, or assumption of its solvability in v, does not seem to be the core of the matter from a game point of view. However, the following remark is pertinent here. Assume that we allow $v\in\bar{V}$, where

$$\bar{V}=\{v\,|\,v: R^n\times[t_0,t_f]\to R^{m_2},\ v(x,t)\ \text{piecewise continuous in } t \text{ and Lipschitzian in } x \text{ where } x\in R^n \text{ and } t\in[t_0,t_f]\}\tag{17}$$

instead of $v\in V$. The assumption of solvability of (15c) will again give

$$v(x,t)=S(x,p,u,t).\tag{18}$$

Since $v(x,t)$ will be substituted in the rest of (15) with $S(x,p,u,t)$ from (18), the leader will be faced with exactly the same problem as after substituting $v(t)$ with S from (16). Therefore, no additional difficulty arises if one allows $\bar{V}$ instead of V and assumes solvability of (15c).

Substituting v from (16) to (15) we obtain

$$\underset{u\in U}{\text{minimize}}\ J(u)=g(x(t_f))+\int_{t_0}^{t_f}\tilde{L}(x,p,u,t)\,dt$$

subject to:

$$\begin{bmatrix}\dot{x}\\ \dot{p}\end{bmatrix}=\begin{bmatrix}F_1(x,p,u,t)\\ F_{21}(x,p,u,t)+u_xF_{22}(x,p,u,t)\end{bmatrix}\tag{19}$$

$$x(t_0)=x_0,\quad p(t_f)=\frac{\partial h\ (x(t_f))}{\partial x}$$

where $\tilde{L}$, F_1, F_{21}, F_{22} stand for the resulting composite functions. Problem (19) is a nonclassical control problem like the one treated in Section II where $(x',p')'$ is the state of the system.

Besides the procedure described above which leads to the consideration of the problem (19), there are other cases in which such problems arise. For example, in a

[1]See Appendix A.

control problem where the state x is available, stochastic disturbances are present, and the time interval $[t_0,t_f]$ is very large, synthesis of the control law as a function of x and t is preferable over a synthesis not using x (open loop). In addition, u_x might be penalized in the cost function or be subjected to bounds of the form $|u_x(x(t),t)| \leq K$, $t\in[t_0,t_f]$, where $K\geq 0$ is a constant.

II. A Nonclassical Problem

Consider the dynamic system described by

$$\begin{aligned}\dot{x}(t)=f(x(t),u^1(h^1(x(t),t),t),\\ u^2(h^2(x(t),t),t),\cdots,u^m(h^m(x(t),t),t),\\ u_x^1(h^1(x(t),t),t),\cdots,u_x^m(h^m(x(t),t),t),t) \quad (20)\\ x(t_0)=x_0, \qquad t\in[t_0,t_f]\end{aligned}$$

and the functional

$$\begin{aligned}J(u)=g(x(t_f))+\int_{t_0}^{t_f}L(x(t),\\ u^1(h^1(x(t),t),t),\cdots,u^m(h^m(x(t),t),t),\\ u_x^1(h^1(x(t),t),t),\cdots,u_x^m(h^m(x(t),t),t),t)\,dt\end{aligned} \quad (21)$$

where the functions f: $R^{n+m+mn+1}\to R^n$, L: $R^{n+m+mn+1}\to R$, h^i: $R^{n+1}\to R^{q_i}$, $i=1,\cdots,m$, g: $R^n\to R$ are continuous in all arguments and in C^1 with respect to the x, u, u_x^i. The functions h^i: $R^{n+1}\to R^{q_i}$ are continuous, and in C^2 w.r. to x.[2] The solution $x(t)$ of (20) is assumed to be continuous and piecewise in C^1 w.r. to t. The time interval $[t_0,t_f]$ is considered fixed w.l.o.g. (see [10, p. 27]). We want to find a function u where

$$u=\begin{bmatrix}u^1\\ \cdot\\ \cdot\\ \cdot\\ u^m\end{bmatrix}$$

$$u^i: R^{q_i}\times[t_0,t_f]\to R, \qquad i=1,\cdots,m.$$

$u_x^i(h^i(x,t),t)$ exists and $u^i(h^i(x,t),t)$, $u_x^i(h^i(x,t),t)$ are continuous in x and piecewise continuous in t, for $x\in R^n$, $t\in[t_0,t_f]$, $i=1,\cdots,m$ so as to minimize $J(u)$. We denote by $\overline{U}$ the set of all such u's. Therefore, the problem under investigation is

$$\begin{aligned}&\text{minimize } J(u)\\ &\text{subject to } u\in\overline{U} \text{ and (20).}\end{aligned} \quad (22)$$

This problem is posed for a fixed time interval $[t_0,t_f]$ and a fixed initial condition $x(t_0)=x_0$. Therefore, the solution u^*, if it exists, will in general be a function of t_0, t_f, x_0, in addition to being a function of $h(x,t)$, t, but we do not show this dependence on t_0, t_f, x_0 explicitly.

We use the notation

$$f_u=\begin{bmatrix}\dfrac{\partial f}{\partial u^1}\\ \cdot\\ \cdot\\ \cdot\\ \dfrac{\partial f}{\partial u^m}\end{bmatrix}, \quad m\times n \text{ matrix,}$$

$$L_u=\begin{bmatrix}\dfrac{\partial L}{\partial u^1}\\ \cdot\\ \cdot\\ \cdot\\ \dfrac{\partial L}{\partial u^m}\end{bmatrix}, \quad m\times 1 \text{ vector}$$

$$f_i=\frac{\partial f}{\partial(u_x^i)}, \quad n\times n \text{ matrix,} \quad i=1,\cdots,m$$

$$L_i=\frac{\partial L}{\partial(u_x^i)}, \quad n\times n \text{ vector,} \quad i=1,\cdots,m \quad (23)$$

$$u_j^i=\frac{\partial u^i(y^i,t)}{\partial y_j^i}, \quad y^i=(y_1^i,\cdots,y_j^i,\cdots,y_{q_i}^i)'\in R^{q_i}$$

$$h^i=(h_1^i,\cdots,h_j^i,\cdots,h_{q_i}^i)', \quad i=1,\cdots,m,\ j=1,\cdots,q_i$$

$$u_{y^i}^i=(u_1^i,\cdots,u_{q_i}^i), \quad u_{y^iy^i}^i=q_i\times q_i \text{ Hessian}$$

$$u_x^i=\frac{\partial h^i(x,t)}{\partial x}\cdot\frac{\partial u^i(y^i,t)}{\partial y^i}\bigg|_{y^i=h^i(x,t)}, \quad n\times 1 \text{ vector,} \quad i=1,\cdots,m$$

$$u_x=[u_x^1 \,\vdots\, \cdots \,\vdots\, u_x^m], \quad n\times m \text{ matrix.}$$

It should be pointed out that the arguments used in classical control theory for showing that for the fixed initial point case, it is irrelevant for the optimal trajectory and cost whether the control value at time t is composed by using $x(t)$ and t or only t,[3] do not apply here in general. If $u|_t=u(t)$, $t\in[t_0,t_f]$, then $u_x=0$ and this changes the structure of problem (22). Consideration of variations of u_x is also needed and this was where the previous researchers stopped; see [4]. This problem is successfully treated here. We provide two different ways of doing that, the first of which is based on an extension (Lemma 2.1) of the so-called "fundamental lemma" in the Calculus of Variations (see [12]).

[2] The restriction $h\in C^2$ w.r. to x is somewhat strong. For example, the case $h(x,t)=x$ if $t_0<t<t_1$, $h(x,t)=0$ if $t_1<t<t_f$, i.e., the state is available only during a part of the $[t_0,t_f]$ is not included. Nonetheless, it can be approximated arbitrarily close by a C^2 function, like any function which is only piecewise C^2. Consequently, from an engineering point of view, $h\in C^2$ w.r. to x is not a serious restriction.

[3] This holds if 1) the set of the admissible closed-loop control laws contains the set of the admissible open-loop control laws and 2) if u^* is an optimal closed-loop control law generating an optimal trajectory $x^*(t)$, then $v^*(t)\triangleq u^*(x^*(t),t)$ is an admissible open-loop control law.

The following theorem provides necessary conditions for a function $u\in\bar{U}$ to be a solution to the problem (22) in a local sense; (we assume that $\bar{U}$ is properly topologized). It is assumed in this theorem that the optimum u^* has strong differentiability properties, an assumption which will be relaxed later, in Theorem 2.1. The proof of this theorem is based on the following lemma.

Lemma 2.1: Let M: $[t_0,t_f]\to R^m$, N_i: $[t_0,t_f]\to R^n$, $i=1,\cdots,m$, y: $[t_0,t_f]\to R^n$, be continuous functions, such that

$$\int_{t_0}^{t_f} M'(t)\varphi(y(t),t)\,dt+\sum_{i=1}^{m}\int_{t_0}^{t_f} N_i'(t)\varphi_y^i(y(t),t)\,dt=0$$

for every continuous function φ: $R^n\times[t_0,t_f]\to R^m$, where $\varphi=(\varphi^1,\cdots,\varphi^m)'$, and φ is in C^1 w.r. to y. Then $M,N_1,\cdots,N_m$ are identically zero on $[t_0,t_f]$.

Proof of Lemma 2.1: The choice $\varphi_i=(0,\cdots,0,\varphi^i,0,\cdots,0)'$, φ^i: $[t_0,t_f]\to R$, φ^i continuous in t, $i=1,\cdots,m$, yields $M\equiv 0$ on $[t_0,t_f]$. Since $M\equiv 0$, the choice $\bar{\varphi}_i=(0,\cdots,y'\psi,0,\cdots,0)'$, $\varphi^i=y'\psi$, where $\psi=(\psi_1,\cdots,\psi_n)'$, ψ: $[t_0,t_f]\to R^n$, ψ continuous in t, results in $\int_{t_0}^{t_f}N_i'(t)\psi(t)dt=0$, for every such ψ, and thus $N_i\equiv 0$ on $[t_0,t_f]$ is proven in the same way as $M\equiv 0$ was proven. □

The conclusion of the above lemma holds even if the restriction $\varphi^i(x,t)=y_1^{k_{1i}}\cdots y_n^{k_{ni}}\cdot t^{\lambda_i}$ is imposed, where $k_{1i},\cdots,k_{ni},\lambda_i$ are nonnegative integers, since the polynomials are dense in the space of measurable functions on $[t_0,t_f]$.

Theorem 2.1': Let $u^*\in\bar{U}$ be a solution of (22) which gives rise to a trajectory $\Gamma_1=\{(x^*(t),t)|t\in[t_0,t_f]\}$, such that $u_{y^i}^i$ are in C^1 w.r. to x in a n.b.d. of $\{(h^i(x^*(t),t),t),\ t\in[t_0,t_f]\}$. Then there exists a function p: $[t_0,t_f]\to R^n$ such that

$$-\dot{p}(t)=L_x+f_xp+\sum_{i=1}^{m}\sum_{j=1}^{q_i}u_j^i\nabla_{xx}h_j^i(L_i+f_ip) \tag{24}$$

$$L_u+f_up=0 \tag{25}$$

$$\nabla_xh^{i'}(L_i+f_ip)=0,\qquad i=1,\cdots,m \tag{26}$$

$$p(t_f)=\frac{\partial g(x(t_f))}{\partial x} \tag{27}$$

hold for $t\in[t_0,t_f]$, where all the partial derivatives are evaluated at

$$x^*(t),u^{i*}(h^i(x^*(t),t),t),u_x^{i*}(h^i(x^*(t),t),t),t.$$

The proof of this theorem by using variational techniques and Lemma 2.1 is simple but lengthy. For the sake of completeness, we give it in Appendix B.

We now give a different derivation of the results of Theorem 2.1', under weaker assumptions, which provides an interpretation for them and at the same time an extension of the region of their validity. Let

$$\bar{U}_k=\{\bar{u}|\bar{u}\colon [t_0,t_f]\to R^k,\ \bar{u}\text{ piecewise continuous}\}. \tag{28}$$

Consider the problem

$$\text{minimize } J(\bar{u},\bar{u}_1,\cdots,\bar{u}_m)=g(x(t_f))+\int_{t_0}^{t_f}L(x,\bar{u},\nabla_xh^1(x,t)\bar{u}_1,\cdots,\nabla_xh^m(x,t)\bar{u}_m,t)\,dt$$

$$\text{subject to } \dot{x}=f(x,\bar{u},\nabla_xh^1(x,t)\bar{u}_1,\cdots,\nabla_xh^m(x,t)\bar{u}_m,t),\qquad x(t_0)=x_0,\ t\in[t_0,t_f]$$

$$\bar{u}\in\bar{U}_m,\ \bar{u}_i\in\bar{U}_{q_i},\ i=1,\cdots,m. \tag{29}$$

Clearly, if J_1^*, J_2^* are the infima of (22) and (29), respectively, it will be $J_1^*\le J_2^*$. Also, if $\bar{u}=(\bar{u}^1,\cdots,\bar{u}^m)'$, $\bar{u}_1,\cdots,\bar{u}_m$ solve (29) and give rise to $x(t)$, then a $u=(u^1,\cdots,u^m)'\in\bar{U}$ with

$$\begin{bmatrix} u^1(h^1(x(t),t),t)\\ \vdots\\ u^m(h^m(x(t),t),t)\end{bmatrix}=\bar{u}(t), \tag{30}$$

$$u_x^i(h^i(x(t),t),t)=\nabla_xh^i(x(t),t)\bar{u}_i(t)\qquad i=1,\cdots,m$$

results in $J_2(u)=J(\bar{u},\bar{u}_1,\cdots,\bar{u}_m)$ and gives rise to the same $x(t)$. However, such $u\in\bar{U}$ does exist. For example, we set

$$u^i(h^i(x,t),t)=a_i'(t)h^i(x,t)+b_i(t) \tag{31}$$

where

$$a_i(t)=\bar{u}_i(t) \tag{32}$$

$$b_i(t)=\bar{u}^i(t)-a_i'(t)h^i(x(t),t),\qquad i=1,\cdots,m. \tag{33}$$

This u satisfies (30). Thus, problems (29) and (22) are actually equivalent, in the sense that for each given (x_0,t_0) they have the same optimal trajectories and costs and their optimal controls are related by (30).

The conditions of Theorem 2.1' are now directly verified to be the necessary conditions for problem (29), where one should use $\bar{u}$ and $\bar{u}_i$ in place of u and $u_{y^i}^i$, respectively. More importantly, the conditions of Theorem 2.1' hold if one considers simply $u^*\in\bar{U}$, without assuming that $u_{y^i}^{i*}$ is in C^1 w.r. to x in a n.b.d. of $\{(h^i(x^*(t),t),t),\ t\in[t_0,t_f]\}$. This weakens the strong differentiability property of u^* assumed in Theorem 2.1'. The relative independence of u, $u_{y^i}^i$, was exploited in proving Theorem 2.1', when the special form of the perturbation $\varphi(y,t)$, $y'\psi(t)$ (see proof of Lemma 2.1), sufficed to conclude (25) and (26). This independence of u and $u_{y^i}^i$ was taken *a priori* into consideration when problem (29) was formulated. Clearly, even if higher order partial derivatives of u w.r. to x appear in f and L, or if u, $u_{y^i}^i$ are restricted to take values within certain closed sets, the equivalence of the corresponding problems (22) and (29) holds again (with appropriate modifications of the definitions of $\bar{U}$, $\bar{U}_k$, f, and L). We formalize the discussion above in the following theorem.

Theorem 2.1: Let $u^* \in \bar{U}$ be a solution to the problem

$$\text{minimize } J(u) = g(x(t_f)) + \int_{t_0}^{t_f} L(x, u, u_x^i, \cdots, u_x^m, t)\, dt \quad (34)$$

$$\text{subject to: } \dot{x} = f(x, u, u_x^i, \cdots, u_x^m, t),$$

$$x(t_0) = x_0,\ t \in [t_0, t_f],\ u \in U,$$

$$\left(u^1(h^1(x(t),t),t), \cdots, u^m(h^m(x(t),t),\ u_{y^1}^1(h^1(x(t),t),t)', \cdots, u_{y^m}^m(h^m(x(t),t),t)'\right) \in V_0 \quad (35)$$

where $V_0 \subseteq R^{m+nm}$ is closed. Then there exists

$$p: [t_0, t_f] \to R^n \text{ such that}$$

$$-\dot{p} = L_x + f_x p + \sum_{i=1}^{m} \sum_{j=1}^{q_i} u_j^i \nabla_{xx} h_j^i (L_i + f_i p) \quad (36)$$

$$\begin{aligned} &L(x^*(t), u^{1*}(h^1(x^*(t),t),t), \cdots, \\ &\quad u^{m*}(h^m(x^*(t),t),t), u_x^{1*}(h^1(x^*(t),t),t), \cdots, \\ &\quad u_x^{m*}(h^m(x^*(t),t),t),t) \\ &\quad + f'(x^*(t), u^{1*}(h^1(x^*(t),t),t), \cdots, \\ &\quad u^{m*}(h^m(x^*(t),t),t), u_x^{1*}(h^1(x^*(t),t),t), \cdots, \\ &\quad u_x^{m*}(h^m(x^*(t),t),t),t) \cdot p(t) \\ &< L(x^*(t), q_0^1, \cdots, q_0^m, \nabla_x h^1(x^*(t),t) q_1, \cdots, \\ &\quad \nabla_x h^m(x^*(t),t) q_m, t) \\ &\quad + f'(x^*(t), q_0^1, \cdots, q_0^m, \nabla_x h^1(x^*(t),t) q_1, \cdots, \\ &\quad \nabla_x h^m(x^*(t),t) q_m, t) \cdot p(t) \end{aligned}$$

$$\forall (q_0^1, \cdots, q_0^m, q_1', \cdots, q_m')' \in V_0. \quad (37)$$

$$p(t_f) = \frac{\partial g(x^*(t_f))}{\partial x} \quad (38)$$

for $t \in [t_0, t_f]$. □

It is remarkable that the established equivalence of the problems (22) and (29) refers to the optimal trajectories, costs, and control values. It does not refer to any other properties, such as sensitivity, for example. It is thus possible that different realizations of $u^i(h^i(x,t),t)$ other than (31) may enjoy sensitivity or other advantages. The following proposition provides information for tackling such problems.

Proposition 2.1:

i) If u and v are elements of U, both satisfying (30), so does $\lambda u + (1-\lambda)u$, $\forall \lambda \in R$.

ii) Let $m = 1$, $h^1(x,t) = x_1$, and $\bar{x}_1$, $\bar{u}$, $\bar{u}_1$ be scalar valued functions of $t, t \in [t_0, t_f]$. Then the function

$$u(x,t) = e^{x_1(x_1 - \bar{x}_1(t))} \bar{u}(t) + [\bar{u}_1(t) - \bar{x}_1(t)\bar{u}(t)] \cdot [x_1 - \bar{x}_1(t)]$$

satisfies $u(\bar{x}(t),t) = \bar{u}(t)$, $u_x(\bar{x}(t),t) = \bar{u}_1(t)$.

iii) Let $\bar{x}$, $\bar{u}$, $\bar{u}_1$ be as in ii). Assume that the scalar valued functions $u(x,t)$, $v(x,t)$ satisfy $u(\bar{x}(t),t) = v(\bar{x}(t),t) = \bar{u}(t)$ and $u_x(\bar{x}(t),t) = v_x(\bar{x}(t),t) = \bar{u}_1(t)$. Then so do the functions $2uv/(u+v)$, $\sqrt{uv}$, $\sqrt{(u^2+v^2)/2}$, assuming that u and v are properly behaved. □

The proof of this proposition is a matter of straightforward verification. The assumption in parts ii) and iii) for scalar valued quantities actually induces no loss of conceptual generality since it can be abandoned at the expense of increased complexity of the corresponding expressions, of course.

The nonuniqueness of the solution u to problem (22) is obvious in the light of (30) and Proposition 2.1. Nonetheless, this nonuniqueness is a nonuniqueness in the representation of u^i as a function of h^i and t, while $u|_t$, $u_{y^i}^i|_t$ are the same for all these representations. The nonuniqueness of $u|_t$, $u_{y^i}^i|_t$, if any, can be characterized in terms of the possible nonuniqueness of the $a_i(t)$, $b_i(t)$ [see (31)], where one, w.l.o.g, restricts u^i to affine in h^i controls.

One very basic difference between problems (22) and (29) is the following. It is clear that the principle of optimality holds for both of these problems, in the sense that the last piece of each optimal trajectory is optimal. The existence of a closed-loop control law $(\bar{u}(x,t), \bar{u}_1(x,t), \cdots, \bar{u}_m(x,t)$ which results in an optimal solution to problem (29) for every initial point (x_0, t_0) in a subset of R^{n+1} is guaranteed under certain assumptions; see [11]. A corresponding statement does not hold for problem (22), i.e., in general, there do not exist functions u^i of $h^i(x,t)$ and t such that $u = (u^1, \cdots, u^m)'$ is an optimal solution to problem (22) for every initial point (x_0, t_0) in a subset of R^{n+1}. This can be easily seen to hold by the following argument. Let such u exist. Then

$$\left(u^1(h^1(x,t),t), \cdots, u^m(h^m(x,t),t),\ \cdot u_{y^1}^1(h^1(x,t),t)', \cdots, u_{y^m}^m(h^m(x,t),t)'\right)'$$

is a closed-loop control law for problem (29). This implies that there must exist a solution $(\bar{u}, \bar{u}_1, \cdots, \bar{u}_m)$ with $\bar{u} = (\bar{u}^1, \cdots, \bar{u}^m)$ of the partial differential equation of dynamic programming associated with problem (29) which satisfies $\bar{u}^i(x,t) = u^i(h^i(x,t),t)$ and $(\partial \bar{u}^i(x,t))/\partial x = \nabla_x h^i(x,t) \cdot \bar{u}_i(x,t)$, $i = 1, \cdots, m$, which is not in general true.[4] This difference between problems (22) and (29) emphasizes the fact that their equivalence holds in a restricted fashion, i.e., for each initial point considered independently and not in a global fashion, as a closed-loop control law treats the initial points.

Two final remarks before entering the next section are pertinent here. First, that the established equivalence of

[4]Imposing the condition $(\partial \bar{u}^i)/\partial x = \nabla_x h^i \cdot \bar{u}_i$, where $\bar{u}^i(x,t)$, $\bar{u}_i(x,t)$ are given in terms of $(\partial V(x,t))/\partial x$, where $V(x,t)$ is the value function for the control problem (29), results in a condition that V must satisfy in addition to being a solution of the dynamic programming partial differential equation. This procedure can be used to single out a class of control problems (22) where a closed-loop control law exists.

the problems (22) and (29) reduces all questions of existence, uniqueness, controllability, and of sufficiency conditions for problem (22) to the corresponding ones for (29). Any problem of the form (22) where terminal constraints and control constraints are present can be solved and necessary and sufficient conditions can be written down in as much as this can be done for the problem (29) with the corresponding constraints considered in addition. Second, Theorem 2.1 still holds if instead of the initial condition $x(t_0)=x_0$, it is given: $x^\alpha(t_0)=x_0^\alpha$ and $x^\beta(t_f)=x_f^\beta$, where $x=(x^{\alpha'},x^{\beta'})'$. In this case, (38) is modified to

$$p^\alpha(t_f)=\frac{\partial g(x^\alpha(t_f))}{\partial(x^\alpha)} \quad \text{and} \quad p^\beta(t_0)=\frac{\partial h(x^\beta(t_0))}{\partial(x^\beta)} \tag{39}$$

where the more general cost functional

$$J=g(x^\alpha(t_f))+h(x^\beta(t_0))+\int_{t_0}^{t_f}L(x,u,t)\,dt \tag{40}$$

is considered (see [10]).

III. Solution of the Stackelberg Game

In this section we analyze the Stackelberg game of Section I by using the results of Section II. In particular, we work out a linear quadratic Stackelberg game, where the leader is penalized for u_x^i as well.

Let us consider the Stackelberg game of Section I. In this case, the h^i's for the leader (u), are

$$h^i((x,p),t)=\begin{bmatrix} I_{n\times n} & 0_{n\times n}\end{bmatrix}\begin{bmatrix} x \\ p\end{bmatrix}=x, \qquad i=1,\cdots,m_1$$

and the h^i's for the follower (v) are identically zero. Different h^i's may be used to model different information structures in terms of $x(t)$, and t available to the leader and follower at time t. Thus, Theorem 2.1 is applicable and can be used for writing down the leader's necessary conditions. From the results of the previous section, we conclude that the solution for the leader's u, if it exists, is not unique. It is interesting to notice that (31) implies that the leader has nothing to lose if he commits himself to an affine in x, time-varying strategy. With such a commitment, the leader does not deteriorate this cost, does not alter the optimal trajectory, and also the follower's optimal cost is not affected. More noteworthy is that the affine choice for the leader can be made even if f, L, M are nonlinear and u, u_x^i are constrained to take values in given closed sets. In addition, v may be constrained to take values in a given closed set in which case (15c) should be substituted by an appropriate inequality. In accordance with the discussion in the previous section, we have that in general there does not exist a strategy $u(x,t)$ which is optimal for every initial point (x_0,t_0) in a subset of R^{n+1}.

Let $\lambda=(\lambda_1',\lambda_2')'$ denote the adjoint variable for problem (19) with λ_1, λ_2 corresponding to x and p, respectively. Then, condition (37) results in

$$[M_u(x,u,S(x,p,u,t),t) + f_u(x,u,S(x,p,u,t),t)p]\lambda_2'=0 \qquad \forall t\in[t_0,t_f], \tag{41}$$

which will generally make the leader's problem singular [9]. This is to be expected because the leader exerts his influence through the time functions resulting from u and u_x, which are actually quite independent, and u_x is not penalized or subjected to any constraint in the initial formulation (8)–(9). In other words, the leader is more powerful than what a first inspection of the original problem indicates. One way to restrict the leader's strength or to avoid the singular problem could be the inclusion of u_x^i in L, i.e., $L=L(x,u,u_x^1,\cdots,u_x^m,t)$, which would model a self-disciplined leader, or to impose *a priori* bounds on u_x, for example, $\|u_x^i\|\leqslant K$, $\forall t\in[t_0,t_f]$, which could be interpreted as a constitutional restriction on a real life leader.

It could be suggested to the follower to penalize u_x^i in his criterion while u_x^i is not penalized in the leader's criterion. This would lead to the appearance of u_{xx}^i in (19) (assuming u_{xx}^i exists). Thus, in addition to (41), a similar condition due to u_{xx}^i appears which reinforces the singular character of the problem. If the leader now restricts himself to affine strategies in x, then $u_{xx}^i=0$ and the resulting optimum is as before. Actually, the leader can restrict himself to a quadratic strategy in x (without affecting his global optimum cost and trajectory) having thus three influences on the system, namely u, u_x, u_{xx}^i, from which only u is penalized in the leader's criterion. Therefore, the leader will do better. For the follower, it is not obvious if he will do better or not.

Let us work out a linear quadratic Stackelberg game. The leader is penalized for u_x^i as well, by including it in L. We consider the dynamic system

$$\dot{x}=Ax+B_1u+B_2v, \qquad x(t_0)=x_0, \quad t\in[t_0,t_f] \tag{42}$$

and the cost functionals

$$J_1(u,v)=\tfrac{1}{2}\left[x_f'K_{1f}x_f+\int_{t_0}^{t_f}\left(x'Q_1x+u'R_{11}u + v'R_{12}v+\sum_{i=1}^{m_1}u_x^{i'}R_iu_x^i\right)dt\right] \tag{43}$$

$$J_2(u,v)=\tfrac{1}{2}\left[x_f'K_{2f}x_f + \int_{t_0}^{t_f}(x'Q_2x+u'R_{21}u+v'R_{22}v)\,dt\right] \tag{44}$$

where the matrices A, B_i, Q_i, R_{ij}, R_i are continuous functions of time and Q_i, R_{ij}, $R_i>0$, $R_{11}>0$ are symmetric. $R_{22}>0$ is nonsingular $\forall t\in[t_0,t_f]$, which guarantees (16). The follower's necessary conditions are [recall (15)]

$$v = -R_{22}^{-1}B_2'p \tag{45}$$
$$\dot{x} = Ax + B_1u - B_2R_{22}^{-1}B'p \tag{46}$$
$$\dot{p} = -Q_2x - u_xR_{21}u - A'p - u_xB_1'p \tag{47}$$
$$x(t_0) = x_0, \quad p(t_f) = K_{2f}x_f. \tag{48}$$

Therefore, the leader's problem is [recall (14), (19)][5]

$$\text{minimize } J(u) = \tfrac{1}{2}\left[x_f'K_{1f}x_f + \int_{t_0}^{t_f}\left(x'Q_1x + u'R_{11}u + p'B_2R_{22}^{-1}R_{12}R_{22}^{-1}B_2'p + \sum_{i=1}^{m_1} u_x^{i'}R_iu_x^i\right)dt\right] \tag{49}$$

subject to:

$$\dot{x} = Ax - B_2R_{22}^{-1}B_2'p + B_1u \tag{50}$$
$$\dot{p} = -Q_2x - A'p - u_xB_1'p - u_xR_{21}u \tag{51}$$
$$x(t_0) = x_0, \quad p(t_f) = K_{2f}x_f. \tag{52}$$

The necessary conditions for the leader in accordance with Theorem 2.1 are (50), (51), (52) and

$$R_{11}u + B_1'\lambda_1 - R_{21}'u_x'\lambda_2 = 0 \tag{53}$$
$$\left[\,R_1u_x^1 \;\vdots\; \cdots \;\vdots\; R_{m_1}u_x^{m_1}\,\right] + \lambda_2(R_{21}u + B_1'p)' = 0 \tag{54}$$
$$\dot{\lambda}_1 = -Q_1x - A'\lambda_1 + Q_2'\lambda_2 \tag{55}$$
$$\dot{\lambda}_2 = -B_2R_{22}^{-1}R_{12}R_{22}^{-1}B_2'p + B_2R_{22}^{-1}B_2'\lambda_1 + A'\lambda_2 + B_1u_x\lambda_2 \tag{56}$$
$$\lambda_1(t_f) = K_{1f}x_f, \quad \lambda_2(t_0) = 0. \tag{57}$$

For simplification we assume further that

$$R_i = \gamma_i I, \quad \gamma_i = \gamma > 0, \qquad i = 1,\cdots,m_1$$
$$R_{11} = I, \quad R_{22} = I \tag{58}$$

and (53), (54) are easily solved for u and u_x to yield

$$u = -\left[I + \frac{\|\lambda_2\|^2}{\gamma}R_{21}'R_{21}\right]^{-1}\left[B_1'\lambda_1 + \frac{\|\lambda_2\|^2}{\gamma}R_{21}'B_1'p\right] \tag{59}$$

$$u_x = -\frac{1}{\gamma}\lambda_2[\,p'B_1 + u'R_{21}'\,] \tag{60}$$

which can be substituted into (50), (51), (55), (56) to yield a nonlinear system of differential equations, with unknown $x, p, \lambda_1, \lambda_2$ and boundary conditions (52) and (57). If $\gamma \to +\infty$, then (59) and (60) yield $u_x \to 0$ and $u \to -B_1'\lambda_1$, and thus the solution tends to the open-loop solution, i.e., $u = u(t)$, $v = v(t)$, as the resulting form of (50), (51), (55), (56) indicates $\gamma \to +\infty$ [2], [3].

[5]We assume that Assumption A holds. See also Appendix A.

IV. Relation to the Principle of Optimality

It has been shown in [4] through a counterexample that the principle of optimality does not hold for Stackelberg games. To make this statement more precise, let us assume that the Stackelberg problem of Section I has been solved in $[t_0, t_f]$ and x^* is the optimal trajectory. While the process is at $(x^*(\bar{t}), \bar{t})$, where $t_0 < \bar{t} < t_f$, we stop and solve the same Stackelberg game on $[\bar{t}, t_f]$ with initial condition $x(\bar{t}) = x^*(\bar{t})$. Let $\bar{x}^*$ be the optimal trajectory for the second problem. Then $\bar{x}^*$ does not have to coincide with the restriction of x^* on $[\bar{t}, t_f]$. The explanation is the following. The leader is faced with the control problem (19) which has boundary conditions $x(t_0) = x_0$ and $p(t_f) = (\partial h(x(t_f)))/\partial x$, given at t_0 and t_f. Let (x^*, p^*) be the optimal trajectory of this problem. If the leader is asked to solve the same control problem on $[\bar{t}, t_f]$ with boundary conditions $x(\bar{t}) = x^*(\bar{t})$ and $p(t_f) = (\partial h(x(t_f)))/\partial x$, there is no necessity for $p(\bar{t}) = p^*(\bar{t})$! Even more, if λ_1, λ_2 are the adjoint variables of the leader's control problem on $[t_0, t_f]$ and $\bar{\lambda}_1, \bar{\lambda}_2$ are the adjoint variables of the leader's control problem on $[\bar{t}, t_f]$, corresponding to x and p, respectively, it will be $\lambda_1(t_f) = (\partial g(x^*(t_f)))/\partial x$, $\lambda_2(t_0) = 0$, $\bar{\lambda}_1(t_f) = (\partial g(\bar{x}^*(t_f)))/\partial x$, $\bar{\lambda}_2(\bar{t}) = 0$. If the principle of optimality were holding, it should be $\lambda_2(\bar{t}) = \bar{\lambda}_2(\bar{t}) = 0$, which is not true. Actually, $\lambda_2(\bar{t}) = 0$, $\forall \bar{t} \in [t_0, t_f]$ is a necessary condition for the principle of optimality to hold. The condition $\lambda_2(\bar{t}) = 0$, $\forall \bar{t} \in [t_0, t_f]$ can be used for deriving more explicit conditions in terms of the data of the problem for the principle of optimality to hold.

Let us consider the linear quadratic game of Section III. As it was shown in the previous paragraph, $\lambda_2(t) = 0$, $\forall t \in [t_0, t_f]$ is a necessary condition for the principle of optimality to hold. With $\lambda_2 \equiv 0$, (56) yields

$$-B_2R_{22}^{-1}R_{12}R_{22}^{-1}B_2'p + B_2R_{22}^{-1}B_2'\lambda_1 = 0$$

from which, by assuming rank $B_2 = m_2$, we obtain equivalently

$$-R_{12}R_{22}^{-1}B_2'p + B_2'\lambda_1 = 0.$$

Also, (54) yields

$$u_x^i = 0, \qquad i = 1,\cdots,m_1. \tag{61}$$

We conclude that under the assumption rank $B_2 = m_2$, (50)–(57) simplify to give

$$\dot{x} = Ax + B_1u + B_2v \tag{62}$$
$$\dot{\lambda}_1 = -Q_1x - A'\lambda_1 \tag{63}$$
$$R_{11}u + B_1'\lambda_1 = 0, \quad R_{12}v + B_2'\lambda_1 = 0 \tag{64}$$
$$x(t_0) = x_0, \quad \lambda_1(t_f) = K_{1f}x_f \tag{65}$$
$$\dot{p} = -Q_2x - A'p \tag{66}$$
$$v = -R_{22}^{-1}B_2'p \tag{67}$$
$$p(t_f) = K_{2f}x_f. \tag{68}$$

Equations (62)–(65) show that the leader's problem can be considered as a team problem under the "constraint" (61), with optimal solution, say (u^*, v^*) and (66)–(68) show that the same v^* must be the follower's optimal reaction to the leader's choice u^*. Actually, (61) is not at all a constraint, since with $\lambda_2 \equiv 0$, (51), (where u_x^i appears) is not really considered by the leader. So, the leader operating under (50) and wanting to minimize (43) may as well choose $u_x^i = 0$, since he is penalized for u_x^i, while u_x^i does not appear in (50).

The same analysis and conclusions carry over to the more general game of Section I [see (6)–(10) and (16)] since the condition $\lambda_2 \equiv 0$ on $[t_0, t_f]$ comes from the demand that the transversality conditions hold $\forall t \in [t_0, t_f]$ and is not affected by the fact that in (9) u_x^i is not penalized. Notice that if the leader's cost functional (9) is substituted by

$$J_1(u,v) = g(x(t_f)) + \int_{t_0}^{t_f} \left\{ L(x,u,v,t) + \sum_{i=1}^{m_1} u_x^{i\prime} R_i u_x^i \right\} dt$$

$$R_i > 0, \quad i = 1, \cdots, m_1, \qquad (69)$$

then (61) holds again.

The idea behind the condition $\lambda_2 \equiv 0$ on $[t_0, t_f]$ is that the leader is not really constrained by the follower's adjoint equation and therefore the leader's problem, being independent of the follower's problem, becomes a team control problem.

In conclusion, a necessary condition for the principle of optimality to hold for the Stackelberg games of Section I (and III), is that the leader's problem is actually a team control problem. But for a control problem with fixed initial conditions, the principle of optimality does hold. We thus have the "if and only if" statement: the principle of optimality holds for the problems of Section I (and III) (see (6)–(10), (16) and (42)–(44), respectively) if and only if the leader's problem is a team control problem for both the leader and follower.

V. Conclusions

In the present paper, a nonclassical control problem motivated by Stackelberg games was introduced and analyzed. Problems of this type arise in the study of hierarchical systems and take into account several information patterns that might be available to the controllers. Two different approaches were presented. The first uses variational techniques, while the second reduces the nonclassical problem to a classical one. The nonexistence of closed-loop control laws for this problem was shown. The nonuniqueness of the solution of this problem was considered and explained. The results obtained for this nonclassical control problem were used to study a Stackelberg differential game where the players have current state information only $((x(t),t))$. Necessary conditions that the optimal strategies must satisfy were derived. The inapplicability of dynamic programming to Stackelberg dynamic games was explained and discussed. The singular character of the leader's problem was proven and the nonuniqueness of his strategies was proven and characterized. In particular, it was shown that commitment of the leader to an affine time-varying strategy does not induce any change to the optimal costs and trajectory. A linear quadratic Stackelberg game was also worked out as a specific application.

We end by outlining certain generalizations of the work presented here. We consider, first, the discrete-time versions. Consider the dynamic system

$$x_{k+1} = f(x_k, u^1(h^1(x_k,k),k), \cdots, u^m(h^m(x_k,k),k),$$
$$u_x^1(h^1(x_k,k),k), \cdots, u_x^m(h^m(x_k,k),k),k)$$
$$x_0 \text{ given} \qquad k = 0, \cdots, N-1$$

and the cost

$$J(u) = g(x_N) + \sum_{k=0}^{N-1} L(x_k u^1(h^1(x_k,k),k), \cdots,$$
$$u^m(h^m(x_k,k),k),$$
$$u_x^1(h^1(x_k,k)k), \cdots, u_x^m(h^m(x_k,k),k).$$

The proof of the corresponding Theorem 2.1 is straightforward. An immediate consequence is that the restriction

$$u^i(h^i(x_k,k),k) = A_k^i h^i(x_k,k) + B_k^i, \qquad i = 1, \cdots, m$$

where A_k^i, B_k^i are matrices, does not induce any loss of generality as far as the optimal cost and trajectory are concerned, [compare to (31)]. Clearly Proposition 2.1 carries over, too.

A discrete-time version of the Stackelberg game of Section I can be defined (see [8]), and analyzed similarly to Section III. Several information patterns can be exploited by employing different h^i's (see [8]). The restriction of the leader to affine strategies can also be imposed in the discrete case. The linear quadratic discrete analog of problem (42)–(44) can also be worked out in a similar way.

The case where higher order partial derivatives of u w.r. to x appear in (20) and (21) can be treated, and all the analysis of Section II carries over. One should assume higher order differentiability of the functions involved. Lemma 2.1 can easily be extended to the case where higher order of partials of φ w.r. to y appear, making the proof of the corresponding Theory 2.1′ possible. We can also restrict u^i to a polynomial form in terms of the h^i's. The analog of Theorem 2.1 can be easily stated and proven and Proposition 2.1 also carries over.

Finally, an N-level Stackelberg game where on each i-level $(i = 1, \cdots, N)$ n_i followers operate $(u_1^i, \cdots, u_{n_i}^i)$, play Nash (or Pareto) among them, and $u_j^i|_t = u_j^i(h_j^i(x,t),t)$ $j = 1, \cdots, n_i$, $i = 1, \cdots, N$, with given h_j^i and fixed x_0, t_0, t_f can be easily treated by using the analysis for the nonclassical control problem supplied here.

Appendix A

In this Appendix we give certain conditions under which Assumption A (Section I) holds.

Lemma A.1: Let U_l be a subset of U [see (6)], defined as

$$U_l=\{u\in U \mid u(x,t)=C(t)x+D(t), \text{ where the } m_1\times n \text{ matrix } C(t) \text{ and the } m_1\times 1 \text{ vector } D(t) \text{ are piecewise continuous functions of time over } [t_0,t_f]\}. \tag{A-1}$$

Then it holds:

$$\begin{aligned}&\inf[J_1(u,v);\ u\in U_l,\ v\in Tu]\\ &\quad\geq \inf[J_1(u,v);\ u\in U,\ v\in Tu]\\ &\quad\geq \inf[J_1(u,v);\ u\in U,\ v\in T'u]\\ &\quad= \inf[J_1(u,v);\ u\in U_l,\ v\in T'u].\end{aligned} \tag{A-2}$$

Proof: The inequalities follow from the facts $U_l\subseteq U$, $Tu\subseteq T'u$ $\forall u\in U$. The last equality is obvious in the light of (31) and the proof of Theorem 2.1. □

An immediate conclusion of Lemma A.1 is that if

$$\inf[J_1(u,v);\ u\in U_l,\ v\in Tu]=\inf[J_1(u,v);\ u\in U_l,\ v\in T'u] \tag{A-3}$$

holds, then Assumption A holds (with $U_N^*=U$). For (A-3) to hold, it suffices that the first-order necessary conditions for the follower's problem are also sufficient, for each fixed $u\in U_l$. More specifically, for fixed $C(t)$, $D(t)$ as in definition (A-1), we consider the problem

$$\begin{aligned}&\text{minimize } h(x(t_f))+\int_{t_0}^{t_f}M(x,C(t)x+D(t),v,t)\,dt\\ &\text{subject to: } v\in V\\ &\dot{x}=f(x,C(t)x+D(t),v,t),\quad x(t_0)=x_0,\ t\in[t_0,t_f]\end{aligned} \tag{A-4}$$

and seek conditions under which the first-order necessary conditions for an optimal v^* for problem (A-4) [see (15b)–(15d)] are also sufficient. Such conditions can be found in [15, ch. 5-2]. We formalize this discussion in the following proposition.

Proposition A.1: If for each $u\in U_l$, the first-order necessary conditions (15b)–(15d) for problem (A-4) are also sufficient, then Assumption A holds.

The discussion in the present Appendix generalizes clearly to the case where each u^i depends on $h^i(x,t)$ instead of x and to the case where different U_l's are considered; see for example Proposition 2.1ii).

As an example where Proposition A.1 can be applied, we consider the linear quadratic game of Section III. Then, [15, Theorem 5, p. 341] in conjunction with Proposition A.1 yields that if $Q_2\geq 0$, $R_{22}>0$, $R_{21}\geq 0$, $K_{2f}\geq 0$, then Assumption A holds.

Appendix B

Proof of Theorem 2.1′: Let $g\equiv 0$ w.l.o.g. (see [10]). Consider a function $\varphi\in U$, $\varphi=(\varphi^1,\cdots,\varphi^m)$ which has the same continuity and differentiability properties as u^*. Such a φ will be called admissible. Using the known theorems on the dependence of solutions of differential equations on parameters, we conclude that for $\epsilon\in R$, ϵ sufficiently small, $u^*+\epsilon\varphi$ gives rise to a trajectory $\{(x(\epsilon,t),t)\mid t\in[t_0,t_f]\}$, $x(0,t)=x^*(t)$, and that $x(\epsilon,t)$ is in C^1 w.r.t. to ϵ. Direct calculation yields

$$\frac{d}{dt}\left(\frac{\partial x(\epsilon,t)}{\partial\epsilon}\right)=\left[f_x+(u_x+\epsilon\varphi_x)f_u+\sum_{i=1}^m(u^i_{xx}+\epsilon\varphi^i_{xx})f_i\right]'\cdot\frac{\partial x(\epsilon,t)}{\partial\epsilon}+f_u'\varphi+\sum_{i=1}^m f_i'\nabla_x h^i\varphi^i_{y^i},\qquad \left.\frac{\partial x(\epsilon,t)}{\partial\epsilon}\right|_{t=t_0}=0. \tag{B-1}$$

We set

$$z(t)=\left.\frac{\partial x(\epsilon,t)}{\partial\epsilon}\right|_{\epsilon=0} \tag{B-2}$$

$$A(t)=f_x+u_xf_u+\sum_{i=1}^m\left[\nabla_x h^i u^i_{y^iy^i}\nabla_x h^{i'}+\sum_{j=1}^{q_i}u^i_j\nabla_{xx}h^i_jf_i\right]p \tag{B-3}$$

$$B_1(t)=f_u' \tag{B-4}$$

$$B_2^i(t)=f_i'\nabla_x h^i,\qquad i=1,\cdots,m \tag{B-5}$$

where A, B_1, B_2^i are evaluated at t, x^*, u^*, u_x^* and, thus, for $\epsilon=0$, (B-1) can be written as

$$\dot{z}=Az+B_1\varphi+\sum_{i=1}^m B_2^i\varphi^i_{y^i},\qquad z(t_0)=0. \tag{B-6}$$

For fixed φ we consider

$$\bar{J}(\epsilon)=J(u+\epsilon\varphi).$$

Since $\bar{J}(\epsilon)$ is in C^1 w.r.t. to ϵ and u^* is a local optimum, it must hold that

$$\left.\frac{d\bar{J}(\epsilon)}{d\epsilon}\right|_{\epsilon=0}=0.$$

Direct calculation yields

$$\frac{d\bar{J}(\epsilon)}{d\epsilon}=\int_{t_0}^{t_f}\left\{\left[L_x+(u_x+\epsilon\varphi_x)L_u+\sum_{i=1}^m(u^i_{xx}+\epsilon\varphi^i_{xx})L_i\right]'\frac{\partial x(\epsilon,t)}{\partial\epsilon}+L_u'\varphi+\sum_{i=1}^m L_i'\nabla_x h^i\varphi^i_{y^i}\right\}dt. \tag{B-7}$$

Setting

$$\Gamma(t)=L_x+u_xL_u+\sum_{i=1}^{m}\left[\nabla_x h^i u^i_{y_iy_i}\nabla_x h^{i'}+\sum_{j=1}^{q_i}u^i_j\nabla_{xx}h^i_j\right]L_i \quad \text{(B-8)}$$

$$\Delta_1(t)=L_u' \quad \text{(B-9)}$$

$$\Delta_2^i(t)=L_i'\nabla_x h^i, \qquad i=1,\cdots,m \quad \text{(B-10)}$$

with Γ, Δ_1, Δ_2^i evaluated at x^*, u^*, u_x^*, we conclude from (B-7)–(B-10) that

$$\int_{t_0}^{t_f}\left[\Gamma z+\Delta_1\varphi+\sum_{i=1}^{m}\Delta_2^i\varphi^i_{y^i}\right]dt=0. \quad \text{(B-11)}$$

Therefore, (B-11) must hold for every admissible φ. Let $\Phi(t,\tau)$ be the transition matrix of $A(t)$. Let also $\bar{\varphi}(t)$ denote the vector $(\varphi^1(h^1(x^*(t),t),t),\cdots,\varphi^m(h^m(x^*(t),t),t))'$ and $\bar{\varphi}^i(t)$ the vector $(\partial\varphi^i(h^i(x^*(t),t),t))/\partial x$. Then from (B-6) we obtain

$$z(t)=\int_{t_0}^{t}\Phi(t,\tau)\left[B_1(\tau)\bar{\varphi}(\tau)+\sum_{i=1}^{m}B_2^i(\tau)\bar{\varphi}^i(\tau)\right]d\tau \qquad t\in[t_0,t_f] \quad \text{(B-12)}$$

and substituting in (B-11) we obtain

$$\int_{t_0}^{t_f}\left\{\Gamma(t)\int_{t_0}^{t}\Phi(t,\tau)\left[B_1(\tau)\bar{\varphi}(\tau)+\sum_{i=1}^{m}B_2^i(\tau)\bar{\varphi}^i(\tau)\right]d\tau +\Delta_1(t)\bar{\varphi}(t)+\sum_{i=1}^{m}\Delta_2^i(t)\bar{\varphi}^i(t)\right\}dt=0. \quad \text{(B-13)}$$

Let $X_{[a,b]}$ denote the indicator function of $[a,b]\subseteq[t_0,t_f]$. We can interchange the order of integration in (B-13) since the integrated quantities are bounded on $[t_0,t_f]\times[t_0,t_f]$ (Fubini's theorem). Using the fact $X(c)_{[t_0,b]}=X(b)_{[c,t_f]}$ we have successively

$$\int_{t_0}^{t_f}\int_{t_0}^{t_f}\left[\Gamma(t)\Phi(t,\tau)B_1(\tau)\bar{\varphi}(\tau)+\Gamma(t)\Phi(t,\tau)\sum_{i=1}^{m}B_2^i(\tau)\bar{\varphi}^i(\tau)\right]\cdot X(t)_{[\tau,t_f]}\,d\tau\,dt$$
$$=\int_{t_0}^{t_f}\left[\int_{\tau}^{t_f}\Gamma(t)\Phi(t,\tau)\,dt\right]B_1(\tau)\bar{\varphi}(\tau)\,d\tau+\sum_{i=1}^{m}\int_{t_0}^{t_f}\left[\int_{\tau}^{t_f}\Gamma(t)\Phi(t,\tau)\,dt\right]B_2^i(\tau)\bar{\varphi}^i(\tau)\,d\tau. \quad \text{(B-14)}$$

By introducing

$$p'(\tau)=\int_{\tau}^{t_f}\Gamma(t)\Phi(t,\tau)\,d\tau, \quad \text{(B-15)}$$

(B-13) can be written as

$$\int_{t_0}^{t_f}[p'(\tau)B_1(\tau)+\Delta_1(\tau)]\bar{\varphi}(\tau)+\sum_{i=1}^{m}\int_{t_0}^{t_f}[p'(\tau)B_2^i(\tau)+\Delta_2^i(\tau)]\cdot\bar{\varphi}^i(\tau)\,d\tau=0. \quad \text{(B-16)}$$

Applying Lemma 2.1 to (B-16), we obtain

$$p'(\tau)B_1(\tau)+\Delta_1(\tau)\equiv 0, \qquad \text{on } [t_0,t_f] \quad \text{(B-17)}$$

$$p'(\tau)B_2^i(\tau)+\Delta_2^i(\tau)\equiv 0, \qquad \text{on } [t_0,t_f]. \quad \text{(B-18)}$$

Using (B-4), (B-5) and (B-9), (B-10) in (B-17), (B-18), we have equivalently (25) and (26). Differentiation of (B-15) and use of (B-3) and (B-8) give the equivalent to (B-15)

$$-\dot{p}=L_x+f_xp+\sum_{i=1}^{m}\sum_{j=1}^{q_i}u^i_j\nabla_{xx}h^i_j(L_i+f_ip)$$

$$p(t_f)=0.$$

The assumption $g\equiv 0$, is removed in the known way, resulting in (27). □

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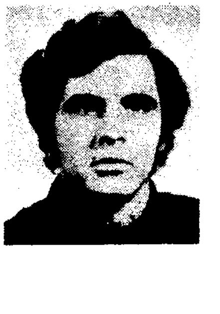

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