Technical Notes and Correspondence

On the Rank Minimization Problem Over a Positive Semidefinite Linear Matrix Inequality

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***Abstract*— We consider the problem of minimizing the rank of a positive semidefinite matrix, subject to the constraint that an affine transformation of it is also positive semidefinite. Our method for solving this problem employs ideas from the ordered linear complementarity theory and the notion of the least element in a vector lattice. This problem is of importance in many contexts, for example in feedback synthesis problems; such an example is also provided.**

***Index Terms*—Feedback synthesis, least element theory, linear matrix inequalities, rank minimization problem.**

I. Introduction

The analogies between the cone of positive semidefinite matrices and the positive orthant in the Euclidean space have been the focus of many interesting investigations in matrix theory over the years. Recently, these analogies have been quite useful in devising efficient algorithms for the eigenvalue optimization problems, and more generally, for the semidefinite programming (SDP) and the linear matrix inequality (LMI) problems [1], [3]. These analogies can in fact be made more explicit by associating to a positive semidefinite matrix its vector of eigenvalues, arranged in a nondecreasing order. Through this association, many properties and questions about a positive semidefinite matrix can be "translated," almost *mechanically*, in terms of the attributes of the corresponding nonnegative vector of eigenvalues. For example, the rank of a matrix can be viewed in terms of the cardinality of the support set of the vector of eigenvalues (counting multiplicities), the latter being the set of indexes for which the vector has a nonzero component.

In this paper, we explore the possibility of using the analogy between the rank of a positive semidefinite matrix and the cardinality of the support set of the associated vector of eigenvalues to solve an important problem which has found many applications in system and control theory. The problem is that of minimizing the rank of a matrix, subject to the constraint that the matrix and an affine transformation of it are positive semidefinite. This problem will be referred to as the MIN-RANK problem and is stated as follows:

$$\min \mathbf{rank}\, X \tag{1}$$

$$\text{subject to: } Q + M(X) \succeq 0 \tag{2}$$

$$X \succeq 0. \tag{3}$$

In (1)–(3), M is a symmetry preserving linear map on the space of symmetric matrices, Q is a symmetric matrix (of appropriate dimensions), and the ordering "$\succeq$" is to be interpreted in the sense of Löwner, i.e., $A \succeq B$ if and only if $A - B$ is positive semidefinite; similarly, $A \succ B$ indicates that $A - B$ is positive-definite.

The MIN-RANK problem has various applications in control and system theory. For example, the bilinear matrix inequality problem (BMI) can be shown to be closely related to the MIN-RANK problem [8], [11]. The BMI, on the other hand, has been shown by Safonov *et al.* [10] to be a unifying formulation for a wide array of control synthesis problems, including the fixed-order H^∞ control, μ/k_m-synthesis, decentralized control, robust gain-scheduling, and simultaneous stabilization. Similarly in [5], El Ghaoui and Gahinet have shown that the important problems of static output feedback stabilization, dynamic reduced-order output-feedback stabilization, reduced-order H^∞ synthesis, and μ-synthesis with constant scaling can be formulated as a rank minimization under an LMI constraint, clearly an instance of the MIN-RANK problem.

Coming back to the MIN-RANK problem and using our "dictionary," the associated problem in the Euclidean space would be the problem of minimizing the cardinality of the support set of a vector, subject to the constraint that the vector and an affine transformation of it have nonnegative components, i.e.,

$$\min |\mathbf{support}\ x| \tag{4}$$

$$\text{subject to: } q + Hx \geq 0 \tag{5}$$

$$x \geq 0 \tag{6}$$

where H is an $n \times n$ matrix, q is an $n \times 1$ vector, and $|\mathbf{support}\ x|$ denotes the cardinality of the support set of the vector x (counting multiplicities). Problem (4)–(6) shall be referred to as the MIN-SUPP problem. Let Λ denote the feasible set of the MIN-SUPP problem, i.e.,

$$\Lambda := \{x \geq 0 : q + Hx \geq 0\}. \tag{7}$$

One way of solving this problem is to start checking for the existence of a particular support configuration in Λ. For example, to see whether a vector with cardinality one exists in Λ, one can examine the positivity of a column of H. Similarly, to check whether a vector with a support cardinality k exists in Λ, the consistency of the following system of linear inequalities can be examined:

$$x_{i_1} h_{i_1} + \cdots + x_{i_k} h_{i_k} \geq -q \tag{8}$$

$$x_{i_j} > 0 \quad (j = 1, \cdots, k) \tag{9}$$

where h_l is the lth column of the matrix H and $(i_1, \cdots, i_k)$ is some k combination of the n indexes, corresponding to the n columns of H. Hence, checking for the existence of a solution with a support cardinality k amounts to solving at most $n!/(k!(n-k)!)$ systems of linear inequalities. Therefore, the MIN-SUPP problem can in principle be solved via 2^n linear programs. Evidently, this approach for solving the MIN-SUPP problem is not quite acceptable. However, it should be noted that checking for the existence of a solution with a *particular* support cardinality can be done efficiently. For example, the easiest case is to examine the existence of a vector in Λ with a support cardinality one, which amounts to simply checking for the existence of a positive column in H.

A special case of the MIN-SUPP problem which can be solved efficiently is the case where the matrix H in (5) is a Z matrix. A square matrix is a Z matrix when all of its off-diagonal elements are nonpositive. When H is a Z matrix, the set Λ (7) has an element

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whose every component is less than or equal to the corresponding component of every other element in Λ. This so-called *least* element of Λ has to have the minimum support in Λ, since if it does not, then it has a positive component which majorizes the corresponding zero component of some other vector in Λ that contradicts its least element property. To summarize, a set $T \subseteq R^n$ has a least element x when $x \in T$, and for all $u \in T$, $x \leq u$ (the inequality is interpreted componentwise). Moreover, if x is the least element of a subset of the positive orthant, it has the minimum support cardinality in that subset as well. Consequently, when H is a Z matrix, one can replace the task of minimizing the cardinality of the support set of the vector satisfying (5), (6) by finding the least element of Λ.

The notion of the least element is not restricted to polyhedral sets like $\Lambda \subseteq R^n$. In fact, to study sets with the least element property, one merely has to have a *Hilbert lattice*, i.e., a Hilbert space $\mathcal{H}$, a pointed convex cone which induces an ordering "$\geq$" on $\mathcal{H}$, and an infimum operation, "inf," with respect to the ordering $\geq$: given $x, y \in \mathcal{H},\ z := \inf\{x, y\}$ is such that $z \leq x,\ z \leq y$, and for all $w \leq x$ and $w \leq y,\ w \leq z$. This in effect means that $\inf\{x, y\}$ is the greatest lower bound of the set $\{x, y\}$ with respect to the ordering $\geq$. It can be seen easily that for vectors x and y in R^n, with the componentwise ordering, if we let $z_i = \min\{x_i, y_i\} (i = 1, \cdots, n)$, then $z = \inf\{x, y\}$. The least element theory in the case of vector inequalities relies on the important observation that when H is a Z matrix, the set Λ is closed with respect to the inf operation, i.e., if $x, y \in \Lambda$, then $\inf\{x, y\} \in \Lambda$. In this case the set Λ is called a *meet semilattice*, since the operation inf is exactly the operation of taking the meet of two vectors [4]. Having a meet semilattice structure for Λ, and noting that Λ is bounded from below (by the vector 0) and it is closed, one can actually find the least element of Λ efficiently via a *linear program*. Thereby, when H is a Z matrix in the MIN-SUPP problem (4)–(6), the minimum support element can be found by a linear program.

There are certain issues that arise pertaining to our comparison between the MIN-RANK and the MIN-SUPP problems. To what extent can the result regarding the MIN-SUPP problem with a Z matrix be generalized for the MIN-RANK problem? What is the analogue of the Z matrix, the least element, and the meet semilattice property for problems defined over the space of symmetric matrices? Can one solve certain classes of the MIN-RANK problem via an SDP (a linear program over the cone of positive semidefinite matrices)?

In this paper, we try to generalize certain aspects of the theory of Z matrices to address the problem pertaining to the minimum rank element of the set defined by (2), (3). The outline of the paper is as follows. In Section II, we provide some definitions and properties which allow us to motivate, and subsequently introduce, the generalization of the meet semilattice and the Z matrices (Section II-A). In Section II-B, we use these generalizations to show that a special class of MIN-RANK problems can be solved by a convex program. Finally, a control example is provided, and a few remarks then conclude the paper.

A few words on the notation are necessary. T' and $\lambda(T)$ denote the transpose and an eigenvalue of the matrix T, respectively. The space of $n \times n$ real matrices is denoted by $R^{n \times n}$, its symmetric subset by $SR^{n \times n}$, its positive semidefinite subset by $SR_+^{n \times n}$, and its identity matrix by I_n. Finally, the inner product of two square matrices A and B in $SR^{n \times n}$ is denoted by $A \bullet B$, which is equal to the trace of the product AB.

II. The MIN-RANK Problem

Consider again the MIN-RANK problem (1)–(3) with the Löwner ordering "$\succeq$." Two very useful properties of the Löwner ordering are as follows [6].

1) Given symmetric $n \times n$ matrices A and B, for any $n \times n$ matrix T
$$A \succeq B \Rightarrow T'AT \succeq T'BT.$$
2) If $A \succeq B$, then $\lambda_i(A) \geq \lambda_i(B)\ (i = 1, \cdots, n)$, where the eigenvalues λ_i's of both matrices A and B are arranged in the nondecreasing order.

One should note that the implication (2) does not hold in the reverse direction. For example, the matrices
$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$
have the same set of eigenvalues, but neither $A \succeq B$ nor $B \succeq A$.

Let $M : SR^{n \times n} \rightarrow SR^{n \times n}$, $Q \in SR^{n \times n}$, and define
$$\Gamma := \{X \succeq 0 : Q + M(X) \succeq 0\} \tag{10}$$
to be the feasible set of the MIN-RANK problem (1)–(3).

As mentioned at the end of the previous section, we now consider the possibility of using ideas from the least element theory and Z matrices to approach the problem of determining the minimal rank matrix of the set Γ (10). The main obstacle in this avenue is that the Löwner ordering cannot be used to introduce a lattice structure on the space of symmetric matrices. Given two symmetric matrices A and B, the inf operation that yields the matrix $C := \inf\{A, B\}$ *cannot* be defined such that $C \preceq A,\ C \preceq B$, and the implication
$$D \preceq A,\ D \preceq B \Rightarrow D \preceq C \tag{11}$$
holds in general. In particular, the matrix $\inf\{A, B\}$ and an arbitrary matrix D such that $D \preceq A$ and $D \preceq B$ *do not have to be even comparable.* Hence, any attempt to define a greatest lower bound (in the sense of Löwner) for a set of symmetric matrices which parallels the vector case (with componentwise ordering) runs into difficulty.

Fortunately, there is a remedy for this problem. Ando [2] realized that for a given pair of symmetric positive semidefinite matrices, although the set
$$\Delta(A, B) := \{X \in SR^{n \times n} : 0 \preceq X \preceq A,\ 0 \preceq X \preceq B\}$$
does not possess a maximal point, it has in a sense "many maximal elements."

The set of the maximal points of $\Delta(A, B)$, which shall be denoted by $\Delta_{\sup}(A, B)$, has the following property:
$$\begin{aligned} \forall D \in \Delta(A, B), \exists Z \in \Delta_{\sup}(A, B) :& \\ Z \in \Delta(A, B), D \preceq Z;& \\ \&\ \nexists W \in \Delta(A, B) : W \neq Z;\ W \succeq Z.& \end{aligned} \tag{12}$$
The matrix $Z \in \Delta_{\sup}(A, B)$ that satisfies (12) not only depends on the matrices A and B, but also on the specific matrix D.

In [2], a complete characterization of the maximal points of the set $\Delta(A, B)$, along with an algorithm for their computation, is provided. More explicitly, in [2] the set $\Delta_{\sup}(A, B)$ is parameterized by a subspace $\mathcal{N} \subset \operatorname{range}(A) \cap \operatorname{range}(B)$ and an n_2-by-n_1 matrix K such that $K^*K \prec I_{n_1}$, where n_1 (respectively, n_2) is the number of positive (respectively, negative) eigenvalues of the matrix $[\mathcal{N}]\mathcal{A} - [\mathcal{N}]\mathcal{B}$ with multiplicity counted; the notation $[\mathcal{N}]\mathcal{A}$ denotes the *short* of the matrix A to the subspace $\mathcal{N}$ [2]. Moreover, given a matrix $D \in \Delta(A, B)$, a matrix $Z \in \Delta_{\sup}(A, B)$ satisfying (12) is constructed as
$$Z = \frac{1}{2}\{[\mathcal{N}]A + [\mathcal{N}]B - L|L^{-1}([\mathcal{N}]B - [\mathcal{N}]A)L^{-1}|L\} \tag{13}$$
where $L := ([\mathcal{N}]A + [\mathcal{N}](B) - 2D)^{1/2},\ L^{-1}$ is the inverse of L *restricted* to the range of $[\mathcal{N}]A - [\mathcal{N}]B$ and $|A|$ denotes the positive

square root of the matrix A^2. For more details on this construction and, in particular, the reason for the existence of the restricted inverse of L, the reader is referred to [2, p. 5, lines 15–16; p. 10, lines 5–7].

Analogous to the case of the componentwise ordering for vectors, we define the following generalization of a (meet semi-) lattice.

Definition II.1: A set $\Gamma \subseteq SR_+^{n\times n}$ is called a (meet semi-) hyper-lattice if for all pairs X and Y in Γ there exists $Z \in \Delta(X, Y)$ such that $Z \in \Gamma$.

In the next section, we demonstrate that for an important class of linear maps M and *a negative semidefinite* matrix Q, the set Γ (10) is in fact a (meet semi-) hyper-lattice. In the spirit of the Z matrix theory, we then proceed to demonstrate that having a (meet semi-) hyper-lattice, the minimal rank element can in fact be found via *convex* optimization.

A note on the terminology is necessary before we start our main discussion. Following Alizadeh [1], a constraint optimization problem is called an SDP, if its variables are either (symmetric) matrices or scalar valued, the objective is a linear functional on the product space of the spaces of the variables, and the constraint set is defined by linear equalities or inequalities (either componentwise or Löwner ordering).

A. When is the Set Γ a (Meet Semi-) Hyper-Lattice?

In this section we show that when the linear map M in the definition of the set Γ (10) has a particular form, and the matrix Q is negative semidefinite, the resulting Γ (10) is a (meet semi-) hyper-lattice. For this purpose, we consider a generalization of the Z matrices.

Definition II.2: A symmetric preserving linear map $M: SR^{n\times n} \to SR^{n\times n}$ is of type $\mathcal{Z}$ if it can be represented as

$$M(X) = X - \sum_{i=1}^{k} M_i X M_i' \tag{14}$$

for some matrices $M_i \in R^{n\times n}$ $(1 \le i \le k)$ and integer $k \ge 1$. A control problem which can be formulated as a MIN-RANK problem with a type $\mathcal{Z}$ linear map is considered in Section III.

The main result of this section is now stated.

Lemma II.1: Let the linear map M in the definition of the set Γ (10) be of type $\mathcal{Z}$ and the matrix Q be negative semidefinite. Then the set Γ is a (meet semi-) hyper-lattice.

Proof: We would like to show that for two symmetric matrices A and B in Γ, there exists $Z \in \Delta(A, B)$ such that $Z \in \Gamma$.

We first note that the set $\Delta(A, B)$ is compact. It suffices to show that for some $Z \in \Delta(A, B)$

$$Z \succeq -Q + \sum_{i=1}^{k} M_i Z M_i'.$$

Since $Z \preceq A$ and $Z \preceq B$, one has

$$\sum_i M_i Z M_i' \preceq \sum_i M_i A M_i'$$

and

$$\sum_i M_i Z M_i' \preceq \sum_i M_i B M_i'.$$

As a result of the assumption $A, B \in \Gamma$, one concludes that

$$A \succeq -Q + \sum_i M_i A M_i' \succeq -Q + \sum_i M_i Z M_i' \succeq 0$$

and

$$B \succeq -Q + \sum_i M_i B M_i' \succeq -Q + \sum_i M_i Z M_i' \succeq 0$$

for all $Z \in \Delta(A, B)$ (recall that Q is assumed to be negative semidefinite). Hence for all $Z \in \Delta(A, B)$, $(-Q + \sum_i M_i Z M_i') \in \Delta(A, B)$.

In particular, for all $Z \in \Delta(A, B)$, there exists $Y \in \Delta_{\sup}(A, B)$ such that

$$Y \succeq -Q + \sum_i M_i Z M_i' \tag{15}$$

by the definition of the set $\Delta_{\sup}(A, B)$. Let $g: \Delta(A, B) \to \Delta(A, B)$ be the point-to-set map such that

$$g(Z) = \left\{ Y \in \Delta(A, B) : Y \succeq -Q + \sum_i M_i Z M_i' \right\}. \tag{16}$$

The map g is upper semicontinuous. To see this, let $\{Z_k\}_{k\ge 1}$ and $\{Y_k\}_{k\ge 1}$ be a sequence of matrices such that

$$Y_k \succeq -Q + \sum_i M_i Z_k M_i'$$

and let $Z_k \to Z^*$ and $Y_k \to Y^*$. Define

$$M(Z_k, Y_k) := Q + Y_k - \sum_i M_i Z_k M_i'.$$

The map M is linear on $SR^{n\times n} \times SR^{n\times n}$ and is therefore continuous. Since the cone of positive semidefinite matrices is closed

$$0 \preceq \lim_{k\to\infty} M(Z_k, Y_k) = M(Z^*, Y^*)$$

and therefore

$$Y^* \succeq -Q + \sum_i M_i Z^* M_i'$$

hence $Y^* \in g(Z^*)$.

Since g is upper semicontinuous on the convex set $\Delta(A, B)$, it has a fixed point via the Kakutani's Fixed Point theorem [7]. That is, there exists a matrix $\hat{Z} \in \Delta(A, B)$ such that $\hat{Z} \succeq -Q + \sum_i M_i \hat{Z} M_i'$. Hence, Γ is indeed a (meet semi-) hyper-lattice.

B. Finding a Minimal Rank Matrix in a (Meet Semi-) Hyper-Lattice

We now consider the problem of finding the minimal rank matrix of the set Γ defined by

$$\Gamma := \left\{ X \succeq 0,\ Q + X - \sum_i M_i X M_i' \succeq 0 \right\} \tag{17}$$

with $Q \preceq 0$. As we discussed in the previous section, the set Γ is a (meet semi-) hyper-lattice (Definition II.1). We shall assume subsequently that Γ is nonempty.

The following theorem provides us with an algorithm for finding a minimal rank matrix of the set Γ (17).

Theorem II.2: A minimal rank element of Γ can be found by a semidefinite program.

Proof: Consider the following semidefinite program:

$$\min I \bullet X \tag{18}$$

$$\text{subject to: } Q + X - \sum_i M_i X M_i' \succeq 0 \tag{19}$$

$$X \succeq 0. \tag{20}$$

Since Γ is assumed to be nonempty, let $A \in \Gamma$ (17) (such a matrix can be found be a semidefinite program itself). Now consider instead the problem

$$\min I \bullet X \tag{21}$$

$$\text{subject to: } Q + X - \sum_i M_i X M_i' \succeq 0 \tag{22}$$

$$X \succeq 0 \tag{23}$$

$$I \bullet X \le I \bullet A. \tag{24}$$

It should be clear that the optimum of both SDP's, (18)–(20) and (21)–(24), are the same. The latter SDP has an optimum since $\Gamma \cap \{X : I \bullet X \leq I \bullet A\}$ is a compact set, and $I \bullet X$ is a linear functional in X. Let $\tilde{X}$ be the optimal solution of (18)–(20). We now claim that $\tilde{X}$ is of minimal rank in Γ. To show this, let $Y \in \Gamma$ and $Z \in \Delta(\tilde{X}, Y)$, such that $Z \in \Gamma$ (this is possible since Γ (17) is a (meet semi-) hyper-lattice). By the optimality of $\tilde{X}$

$$\sum_i \lambda_i(\tilde{X}) \leq \sum_i \lambda_i(Z). \tag{25}$$

On the other hand, since $Z \in \Delta(\tilde{X}, Y)$, one has

$$\lambda_i(Z) \leq \lambda_i(\tilde{X}) \quad (i = 1, \cdots, n) \tag{26}$$

and

$$\lambda_i(Z) \leq \lambda_i(Y) \quad (i = 1, \cdots, n). \tag{27}$$

In view of (25), (26) implies that $\lambda_i(Z) = \lambda_i(\tilde{X})$ $(i = 1, \cdots, n)$. Thus by (27), for an arbitrary matrix $Y \in \Gamma$

$$\lambda_i(\tilde{X}) \leq \lambda_i(Y) \quad (i = 1, \cdots, n). \tag{28}$$

Suppose now that $\tilde{X}$ is not of minimal rank in Γ. Then there exists $\tilde{Y}$ such that $\lambda_i(\tilde{Y}) = 0$ and $\lambda_i(\tilde{X}) \neq 0$ for some index i. Since $\tilde{X} \succeq 0$, $\lambda_i(\tilde{X}) > 0$, which violates (28). Hence $\tilde{X}$ is of minimal rank in Γ. □

III. Fixed-Order Output Feedback Problem

Let Σ be a continuous-time linear time-invariant dynamical system

$$\Sigma: \quad \dot{x} = Ax + Bu \tag{29}$$

$$y = Cx \tag{30}$$

with matrix $A \in R^{n \times n}$ (and all other matrices of appropriate dimensions).

Suppose that it is desired to design a stabilizing controller of order k for Σ

$$\dot{z} = A_K z + B_K y \tag{31}$$

$$u = C_K z + D_K y \tag{32}$$

where $A_K \in R^{k \times k}$. We would like to check, for a given k, whether such a controller (of fixed order) exists.

In [5], El Ghaoui and Gahinet show that this important problem in control theory can be reduced to a MIN-RANK problem.

Theorem III.1 [5]: There exists a stabilizing dynamic output feedback law of order k for Σ if and only if there exist matrices R and S and scalar $\gamma > 0$ such that

$$AR + RA' \prec BB' \tag{33}$$

$$A'S + SA \prec C'C \tag{34}$$

and

$$\begin{pmatrix} \gamma R & I \\ I & \gamma S \end{pmatrix} \succeq 0 \tag{35}$$

$$\text{rank} \begin{pmatrix} \gamma R & I \\ I & \gamma S \end{pmatrix} \leq n + k. \tag{36}$$

Let

$$X = \begin{pmatrix} \gamma R & I \\ I & \gamma S \end{pmatrix}.$$

It can be shown that the above problem can be reduced to solving the following instance of the MIN-RANK problem [9]:

$$\min \text{ rank } X \tag{37}$$

$$\text{subject to: } \tilde{A} X \tilde{A}' - X \prec Q_\gamma \tag{38}$$

$$X \in \mathcal{L} \tag{39}$$

$$X \succeq 0 \tag{40}$$

for an appropriate choice of the matrices $\tilde{A}$ and (symmetric) Q_γ which is affine in γ; moreover, the set $\mathcal{L}$ is defined as

$$\mathcal{L} := \left\{ X \in SR^{2n \times 2n} : X = \begin{pmatrix} U & I \\ I & V \end{pmatrix};\ U, V \in SR^{n \times n} \right\}.$$

The subset $\mathcal{L}$ can, for example, be defined by a set of linear equalities of the form $\frac{1}{2} E_{ij} \bullet X = 1$, where E_{ij} is a matrix whose entries are all zero, except the ijth entry which is one (this fixes the ijth entry of the matrix X to one).

Let us rewrite the above problem for $\epsilon > 0$ as

$$\min_{X, \gamma} \text{ rank } X \tag{41}$$

$$\text{subject to: } (Q_\gamma - \epsilon I) + X - \tilde{A} X \tilde{A}' \succeq 0 \tag{42}$$

$$X \in \mathcal{L} \tag{43}$$

$$X \succeq 0. \tag{44}$$

We now realize that the above problem is exactly a MIN-RANK problem with a linear map of type $\mathcal{Z}$, except that the solution has to be found in the affine set $\mathcal{L}$. This additional constraint does not introduce a difficulty for the applicability of the approach described earlier, provided that the restriction of the set Γ (10) with the linear map

$$M(X) := X - \tilde{A} X \tilde{A}'$$

to $\mathcal{L}$ (if nonempty) is a (meet semi-) hyper-lattice (refer to [9] and [2, pp. 8–10]). Assuming that this is in fact the case, in order to solve this instance of the MIN-RANK problem arising from the fixed-order output feedback synthesis, one thus solves the following semidefinite program for an appropriate choice of $\epsilon > 0$:

$$\min_{X, \gamma} I \bullet X \tag{45}$$

$$\text{subject to: } (Q_\gamma - \epsilon I) + X - \tilde{A} X \tilde{A}' \succeq 0 \tag{46}$$

$$X \in \mathcal{L} \tag{47}$$

$$X \succeq 0 \tag{48}$$

$$\gamma > 0. \tag{49}$$

The constraint that $Q_\gamma - \epsilon I \preceq 0$ can be added as an additional constraint to the above semidefinite program (note that in the case where $\epsilon > 0$ has to be chosen very large, the rank of an optimal solution of the above SDP might only provide us with an upper bound on the minimal rank solution). The above approach consequently results in an efficient way that can be used to study the fixed-order output feedback synthesis problem for the continuous-time linear time-invariant systems.

IV. Concluding Remarks

In this paper, we have described an approach for solving the problem of minimizing the rank of a positive semidefinite matrix, subject to the constraint that an affine transformation of it is also positive semidefinite. In this direction, an approach analogous to finding the least element of a meet semilattice, with componentwise ordering for vectors, is developed. However, our analysis uses some additional ideas and concepts since the positive semidefinite ordering cannot be used to introduce a lattice structure on the space of symmetric matrices. The applicability of our results to the fixed-order output feedback synthesis problem is also provided; this application also reinforces the usefulness of exploiting the structure of the nonconvex optimization problems arising in control theory.

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Robust H_∞ Stabilization via Parameterized Lyapunov Bounds

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***Abstract*—The parameterized Lyapunov bounding technique of Haddad and Bernstein is extended to include an H_∞-disturbance attenuation constraint. The results presented in this paper provide a framework for designing fixed-order (i.e., full- and reduced-order) controllers that guarantee robust H_2 and H_∞ performance in the presence of structured constant real parameter variations in the state space model.**

***Index Terms*—H_2/H_∞ design, real parameter uncertainty, parameter-dependent Lyapunov functions.**

NOMENCLATURE

$\Re$, $\Re^{r\times s}$, $\Re^r$	Real numbers, $r\times s$ real matrices, $\Re^{r\times 1}$.
$(\,)^T$, $(\,)^{-1}$, tr$(\,)$, $\mathcal{E}$	Transpose, inverse, trace, expectation.
I_r, 0_r	$r\times r$ identity matrix, $r\times r$ zero matrix.
$\mathcal{S}^r$, $\mathcal{N}^r$, $\mathcal{P}^r$	$r\times r$ symmetric, nonnegative-definite, positive-definite matrices.
$Z_1 \le Z_2$, $Z_1 < Z_2$	$Z_2 - Z_1 \in \mathcal{N}^r$, $Z_2 - Z_1 \in \mathcal{P}^r$; $Z_1, Z_2 \in \mathcal{S}^r$.
$n, l, m, p, p_\infty, q, n_c, \tilde{n}$	Positive integers; $1 \le n_c \le n$; $\tilde{n} = n + n_c$.
$x, u, y, z, x_c, \tilde{x}$	n-, m-, l-, q-, n_c-, $\tilde{n}$-dimensional vectors.
$w(\cdot)$, $w_\infty(\cdot)$	p-, p_∞-dimensional white noise, L_2 signals.
A, B, C	$n\times n$, $n\times m$, $l\times n$ matrices.
A_c, B_c, C_c	$n_c\times n_c$, $n_c\times l$, $m\times n_c$ matrices.
$D_1, D_2, D_{1\infty}, D_{2\infty}$	$n\times p$, $l\times p$, $n\times p_\infty$, $l\times p_\infty$ matrices.
E_1, E_2	$q\times n$, $q\times m$ matrices.

I. INTRODUCTION

In a recent series of papers [9]–[12], a refined Lyapunov function technique was developed to overcome some of the current limitations of Lyapunov function theory for the problem of robust stability and performance in the presence of constant real parameter uncertainty. Since, as noted in [9]–[11], conventional Lyapunov bounding techniques guarantee stability with respect to time-varying parameter perturbations, a feedback controller designed for time-varying parameter variations will unnecessarily sacrifice performance when the uncertain real parameters are actually constant. To overcome some of the limitations of conventional Lyapunov bounding techniques, the authors in [9]–[11] developed a general framework for robust controller analysis and synthesis based on *parameter-dependent Lyapunov functions* that is both flexible in addressing a large class of uncertainty structures and restrictive in excluding uncertainties that are not physically meaningful. Specifically, in this framework, the Lyapunov function is allowed to be a function of the uncertain parameters, thus guaranteeing robust stability and performance via a family of Lyapunov functions. As demonstrated in [9]–[11], the form of the parameterized Lyapunov bounding function proves to be critical because the presence of uncertainty within the Lyapunov function curtails arbitrary time-variation of the uncertain parameters, thus yielding a more effective robust analysis and synthesis framework for constant real parameter uncertainty.

In this paper, we extend the results of [9]–[11] to guarantee robust H_2 and H_∞ performance in the presence of constant real-valued parameter uncertainty. Thus, the results presented herein provide a further refinement of the results in [13] which considered the design of H_∞ robust controllers in the presence of arbitrarily time-varying real-valued parameter variations.

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