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TECHNICAL NOTE

On the Probability of Existence of Pure Equilibria in Matrix Games[1]

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Abstract. In a recent paper (Ref. 1), Papavassilopoulos obtained results on the probability of the existence of pure equilibrium solutions in stochastic matrix games. We report a similar result, but where the payoffs are drawn from a finite set of numbers N. In the limiting case, as N tends to infinity, our result and that of Papavassilopoulos are identical. We also cite similar results obtained independently by others, some of which were already independently brought to the notice of Papavassilopoulos by Li Calzi as reported in Papavassilopoulos (Ref. 2). We cite a much earlier result obtained by Goldman (Ref. 3). We also cite our related work (Ref. 4), in which we derive the conditions for the existence of mixed strategy equilibria in two-person zero-sum games.

Key Words. Pure equilibrium, matrix games, stochastic games, separation of diagonals, mixed equilibrium.

1. Introduction

In January 1996, we presented a revised version (Ref. 4) of our earlier results (Ref. 5) on the probability of obtaining a pure strategy equilibrium in matrix games with random payoffs at the 2nd International Conference on Game Theory and Economic Applications, Bangalore, India. After the conference, in April 1996 R. B. Bapat brought to our attention a recent paper by Papavassilopoulos (Ref. 1), which contained somewhat similar results. In

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his paper, Papavassilopoulos (Ref. 1) had asked the readers to bring to his attention any related work. After going through his paper, we were tempted to write this note with two main objectives: first, to bring to the notice of Papavassilopoulos and the readers of JOTA some related published material; second, to highlight certain features of our own work not covered in Ref. 1 and other works.

When we sent the first draft of this note to Papavassilopoulos, he sent us a copy of his Technical Note (Ref. 2), which does refer to many of the earlier works that we cite, these having been brought to his notice by Li Calzi. Our note draws attention to a much earlier result by Goldman (Ref 3). Papavassilopoulos points out that Li Calzi also obtained similar results independently. If different persons independently rediscovered an old result, it perhaps points out how interesting and important these concepts and results are. We hope that researchers will examine these results again, taking into account much of the literature that has emerged in game theory since the results were first obtained by Goldman (Ref. 3) and Dresher (Ref. 6).

2. Probability of Pure Strategy Equilibria in Two-Person Zero-Sum Games

In this note, we will restrict the analysis to two-person zero-sum games as it is in this regard that our research is similar and/or different from Ref. 1. Papavassilopoulos assumes that the payoffs a_{ij} of an $m \times n$ matrix game are chosen independently and randomly from a uniform distribution, which means that the set of numbers N, from which the payoffs are drawn, is infinite. Papavassilopoulos derives the probability p of having a pure strategy pair solution in the $m \times n$ matrix game. He obtained the following result:

$$p = m!n!/(m+n-1)!. \tag{1}$$

This formula was also derived by Goldman (Ref. 3), not mentioned in Ref. 1. However, Papavassilopoulos makes certain remarks which were not mentioned explicitly by Goldman.

The most insightful result of Papavassilopoulos is that, as $mn \to \infty$, $p \to 0$, with $\min(m, n) > 1$. Papavassilopoulos extends this result to a situation where m, n, and a_{ij} are chosen randomly and concludes that the probability of obtaining a pure strategy equilibrium is zero.

We were, however, more interested in situations where m, n are finite and deterministic and N is also finite. We derived the probability of obtaining a pure strategy equilibrium for this case (Ref. 4). If m and n are finite and N tends to infinity, then Eq. (1) holds good. However, if N is also finite, then the probability of obtaining a pure strategy equilibrium can be derived using our formula, which is a weighted average of certain probabilities as given below:

$$p = \Sigma_s p_s q_s, \qquad s = 1, \ldots, mn, \tag{2}$$

where p_s is the probability of obtaining a pure strategy equilibrium in an $m \times n$ matrix game, given that there are s distinct payoffs and q_s is the probability of obtaining an $m \times n$ matrix game with s distinct payoffs from a set of N finite numbers.

Let us restrict the analysis to situations where all the payoffs are distinct, $s = mn$ (strictly ordinal games), as this is the common theme of both Refs. 1 and 4. Hence, we derive formulas for p_{mn} and q_{mn}. First, let us show the former,

$$p_{mn} = \Sigma_t p_{mn}^t, \qquad t = 1, \ldots, mn. \tag{3}$$

where p_{mn}^t is the probability of obtaining a pure strategy equilibrium when the value of the game v_t is the tth ordinal payoff.

It may be noted that, if the mn payoffs are put in an ascending order, then the lowest $m = 1$ ordinal payoffs and the largest $n - 1$ ordinal payoffs cannot be the value of the game in a pure strategy equilibrium. Hence,

$$p_{mn}^t = 0, \qquad t < m \text{ and } t > mn - (n-1).$$

But for $t = m, \ldots, mn - (n-1)$, the formula for p_{mn}^t is

$$\begin{aligned} p_{mn}^t = \{&mn(m-1)!(n-1)![(m-1)(n-1)]! \\ &\times [(t-1)C_{(m-1)}(nm-t)C_{(n-1)}]\}/(mn)!. \end{aligned} \tag{4}$$

Here, mn corresponds to the event that the value of the game v_t, can be in any of the mn cells; $(m-1)!$ denotes the number of ways of ordering of $m-1$ distinct payoffs larger than v_t which are in the same column as v_t; $(n-1)!$ denotes the number of ways of ordering of $n-1$ distinct payoffs smaller than v_t which are in the same row as v_t; $[(m-1)(n-1)]!$ denotes the number of ways of ordering of $(m-1)(n-1)$ payoffs excluding the row and column containing v_t; $(t-1)C_{(m-1)}(nm-t)C_{(n-1)}$ denotes the combinatorial when the value of the game is the tth ordinal payoff; $(mn)!$ denotes the total number of games possible from mn distinct payoffs.

Now, we give the expression for q_{mn} as follows:

$$q_{mn} = (mn)!\, NC_{mn}/N^{mn}. \tag{5}$$

Here, $(mn)!$ denotes the total number of games possible from mn distinct payoffs; NC_{mn} denotes the number of mn distinct payoffs possible from a set of N numbers; N^{mn} denotes the total number of $m \times n$ games possible from a set of N numbers. It follows that

$$p_{mn}q_{mn}=m!n![(m-1)(n-1)]!$$
$$\times\Sigma_t[(t-1)C_{(m-1)}(nm-t)C_{(n-1)}]NC_{mn}/N^{mn}, \qquad (6)$$

where $t=m,\ldots,mn-(n-1)$, because the lowest $m-1$ ordinal payoffs and the largest $n-1$ ordinal payoffs cannot be the value of the game in a pure strategy equilibrium.

We had shown in Ref. 4 that Eq. (6) is equivalent to (3) and (1) as $N\to\infty$. Though (1) is much simpler, our alternative formula can be used to derive the probability of obtaining a pure strategy equilibrium when the value of the game v_t is the tth ordinal payoff as in (4). Further, (6) can also be used when N is finite.

In Ref. 4, we till the ground further in the field of two-person zero-sum games and derive the necessary and sufficient conditions for the value of the $m\times n$ matrix game to be associated with a mixed strategy equilibrium. We introduce the concept of row arrays and column arrays, and then the concept of the separation of a row array from a column array. This last concept of separation of arrays is a generalization of the concept of dominant diagonals and separation of diagonals introduced by Von Neumann and Morgenstern in Ref. 7. We developed these conditions in the hope that we can later derive the probability of these conditions being satisfied when the payoffs are drawn randomly. However, Papavassilopoulos takes a different trajectory and derives the probability of obtaining pure strategy equilibrium for N-person nonzero-sum games.

3. Probability of Pure Strategy Equilibria in N-Person Nonzero-Sum Games

As this is our first attempt at working on game theory, it would not be apt on our part to comment on the latter part of Ref. 1. Nevertheless, as suggested by Papavassilopoulos in his note to the reader, we give reference to some related literature that came to our attention.

While discussing the two-player nonzero-sum game, Papavassilopoulos shows that the probability of obtaining a pure strategy equilibrium for a randomly chosen game is $1-e^{-1}$. The same result was also derived through a different method by Dresher (Ref. 6). This result, as derived by Powers (Ref. 8), also holds good when there are more than two players and the number of strategies are infinite. This latter result might be of greater interest to Papavassilopoulos, because of his conclusion that the probability of obtaining a pure strategy equilibrium is zero when the number of players are infinite even if strategies are finite.

Apart from the above-mentioned literature, we have come across references to some other papers, but could not get copies of those (Refs. 9–11). However, as we have not read these papers, we do not comment on their contents, but feel that these papers may also be of interest to researchers interested in the interesting topic discussed by Papavassilopoulos. We will also be grateful if Papavassilopoulos or any other reader would enlighten us regarding some related research, with specific reference to behavioristic interpretations of interest to social scientists.

We feel that the revival of interest in this topic is justified, as learning through repetition and through evolution, with a fixed set of strategies, will lead to more precise knowledge on the payoffs. Such knowledge might lead to elimination of inferior strategies, reduction in the dimension of the matrix of payoffs, etc., leading to an increase in the probability of obtaining pure strategy equilibria.

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