# Robust Variable Structure and Switching-$\Sigma$ Adaptive Control of Single-Arm Dynamics

Li-Wen Chen and G. P. Papavassilopoulos

*Abstract*— In this paper, the variable structure control (VSC) and switching-$\sigma$ adaptive laws are used to design a new robust controller for single-arm rigid manipulators in joint space. The controller is shown to be robust with respect to bounded disturbance. More particularly, a bound on the tracking error is determined and shown to be smaller than those resulting from the VSC law proposed in Slotine and Li as well as the robust adaptive control law derived in Reed and Ioannou. The simulations also show that the proposed control law has better tracking precision performance than the VSC law and switching-$\sigma$ adaptive control law. It can also suppress chattering and maintain good tracking precision even if actuator unmodeled dynamics are considered.

## I. Introduction

In the area of robotic control, the design of a manipulator's controller is a difficult problem owing to the nonlinearity, disturbance, and unknown parameters of the manipulator's dynamics. This leads to interesting and difficult robust variable structure control (VSC) [9]–[18] and adaptive control [19]–[27], [7]–[8] problems.

Slotine and Li [23]–[26] exploited the structure of manipulator dynamics, which is assumed to be disturbance free, to develop a globally convergent adaptive scheme for position control of a single-arm manipulator based on variable structure control (VSC) law. The analysis and simulations show that the unknown parameters can be precisely estimated if disturbances are not involved in the manipulator dynamics. Slotine and Li mentioned the possibility of applying the VSC for the control of single-arm dynamics with bounded disturbance, but there are several important aspects they did not consider. First, they did not apply the estimation law which can guarantee that the estimated parameter will not drift to infinity in the presence of bounded disturbances. Second, they did not simulate and discuss VSC law while considering disturbances and unmodeled dynamics. Reed and Ioannou [19], [27] developed two new robust adaptive controllers which are based on the switching-$\sigma$ modification control and computed torque method [1] for the control of a single-arm manipulator with rigid links. Both the VSC and switching-$\sigma$ adaptive control laws can guarantee that the tracking errors belong to some error set in the presence of bounded disturbance and time-varying parameters. In this paper we develop a new control law which is a combination of VSC law and switching-$\sigma$ adaptive law to enhance the control of single-arm dynamics with unknown parameters and bounded disturbances.

The paper is organized as follows. In Section II we analyze the new composite control law and compare it with the continuous VSC law and switching-$\sigma$ adaptive law. In Section III we simulate and compare these control laws in the presence of bounded disturbance and unknown parameters with and without unmodeled dynamics involved. Finally, in Section IV we summarize the results.

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## II. Combination of the Continuous VSC Law and Switching-$\Sigma$ Adaptive Law

The dynamic equation of a single-arm manipulator with $n$ links in joint space is [1]–[4]

$$\tau = D(q)\ddot{q} + H(q,\dot{q})\dot{q} + G(q) - d(t) \tag{2.1}$$

where $\tau \in R^n$ is the vector of joint torques supplied by the actuators; $D(q) \in R^{n\times n}$ is the arm mass (inertial) matrix which is symmetric and positive definite; $q, \dot{q}, \ddot{q} \in R^n$ are the vectors of joint displacement, velocity, and acceleration, respectively; $H(q,\dot{q}) \in R^{n\times n}$ is the matrix from centrifugal, Coriolis, and frictional forces; $G(q) \in R^n$ is the vector of gravitational force; and $d(t) \in R^n$ is a uniformly bounded disturbance. Our objective is to find a controller which uses the control law $\tau$ as a function of the state $q, \dot{q}$ and the estimated unknown parameter $\hat{P}$ in (2.3) which will make (2.1) to have $q \to q_d$ in the presence of disturbance $d(t)$ and unknown parameter $P$, where $q_d$ is the desired trajectory. The combination of the continuous VSC law and switching-$\sigma$ adaptive law for the control of a single-arm manipulator with bounded disturbance and unknown parameters is presented here for the first time. Our error bound is smaller than those of the VSC law proposed in Slotine and Li [26] and the robust adaptive control law derived in Reed and Ioannou [19].

*Lemma 1:* Consider the following robust adaptive control law for (2.1)

$$\tau = \hat{D}(q,\hat{P})\ddot{q}_r + \hat{H}(q,\dot{q},\hat{P})\dot{q}_r + \hat{G}(q,\hat{P}) - K_d S - d_0\,\mathrm{sat}(S/\phi) \tag{2.2}$$

$$\dot{\Phi}(t) = \dot{\hat{P}}(t) = -\Gamma^{-1}W^{T}(q,\dot{q},\dot{q}_r,\ddot{q}_r)S - \sigma\Gamma^{-1}\hat{P} \tag{2.3}$$

where

$$\sigma = \begin{cases} 0, & \|\hat{P}\| \le P_0 \\ \sigma_0\left(\frac{\|\hat{P}\|}{P_0} - 1\right), & P_0 < \|\hat{P}\| \le 2P_0 \\ \sigma_0, & 2P_0 < \|\hat{P}\| \end{cases} \tag{2.4}$$

$\sigma_0 > 0$ is a scalar, $P_0 > \|P\|$, and $\|x\|$ is $l_2$ norm for the vector $x$.

$$d_0 = \mathrm{diag}(d_{10}, d_{20}, \cdots, d_{n0}) \in R^{n\times n} \tag{2.5}$$

$$|d_i(t)| \le d_{i0}, \quad i = 1, 2, \cdots, n \tag{2.6}$$

$$d(t) = (d_1(t), d_2(t), \cdots, d_n(t))^T \in R^n \tag{2.7}$$

$d_1(t), d_2(t), \cdots, d_n(t)$ are disturbances, and $d_{10}(t), d_{20}(t), \cdots, d_{n0}(t)$ are upper bounds of disturbances.

$$\mathrm{sat}(S/\phi) = (\mathrm{sat}(S_1/\phi_1), \mathrm{sat}(S_2/\phi_2), \cdots, \mathrm{sat}(S_n/\phi_n))^T \in R^n \tag{2.8}$$

$$\phi = (\phi_1, \phi_2, \cdots, \phi_n)^T, \quad S = (S_1, S_2, \cdots, S_n)^T$$

$$\mathrm{sat}(S_i/\phi_i) = \begin{cases} S_i/\phi_i, & \text{if } |S_i/\phi_i| \le 1 \\ \mathrm{sgn}(S_i/\phi_i), & \text{otherwise} \end{cases} \tag{2.9}$$

$$\mathrm{sgn}(S_i/\phi_i) = \begin{cases} 1, & \text{if } S_i > \phi_i \\ -1, & \text{if } S_i < -\phi_i \end{cases}$$

$\phi_i$ is called the boundary layer for the corresponding variable $S_i$, $S$ is defined in (2.12), and the plot of $\mathrm{sat}(S_i/\phi_i)$ is shown in Fig. 1. The variables $q_r$, $\Phi(\hat{P})$, $W$, and $\hat{H}$, $\hat{D}$, $\hat{G}$ are defined in (2.14), (2.11), (2.22), and (2.23), respectively. $K_d$ and $\Gamma$ are arbitrary constant diagonal positive definite matrices chosen by the designer. Then we can guarantee $q$ is close to $q_d$ in a bound as in (2.42).

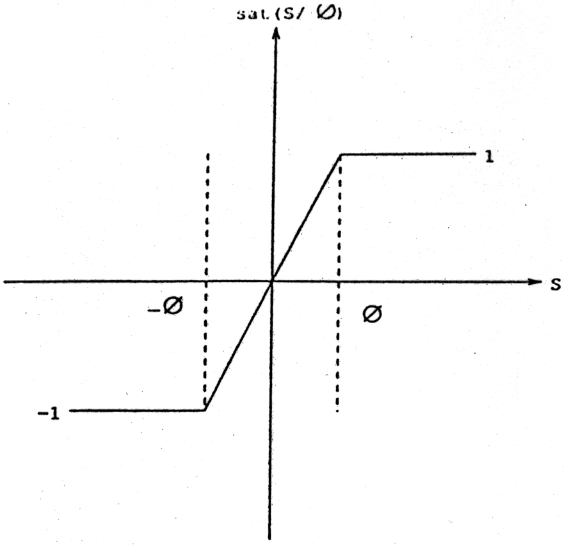


Fig. 1. Continuous VSC law.

*Proof:* In the following proof we drop the independent variables of all functions for simplicity. We choose the following function as a Lyapunov function candidate of the dynamic equation (2.1).

$$V(t, S, \Phi) = (1/2)S(t)^T D(q)S(t) + (1/2)\Phi(t)^T \Gamma \Phi(t) \quad (2.10)$$

where $D(q) \in R^{n\times n}$ is the arm inertia matrix which is symmetric and positive definite, $\Gamma \in R^{m1\times m1}$ is a diagonal positive definite constant matrix chosen by the designer, and $\Phi(t) \in R^{m1}$ is as

$$\Phi(t) = \hat{P}(t) - P \quad (2.11)$$

where $m1$ is the number of unknown parameters, $P$ is an unknown constant parameter vector, $\hat{P}(t)$ is the estimate of $P$, and $\Phi(t)$ is the estimate error of the parameter vector. We assume that the desired trajectory $q_d$ is twice differentiable, then we define $S(t) \in R^n$ as

$$S(t) = \dot{\overline{q}}(t) + \Lambda\overline{q}(t), \; \dot{S}(t) = \ddot{\overline{q}}(t) + \Lambda\dot{\overline{q}}(t) \quad (2.12)$$

where

$$\overline{q}(t) = q(t) - q_d(t), \; \dot{\overline{q}}(t) = \dot{q}(t) - \dot{q}_d(t), \; \ddot{\overline{q}}(t) = \ddot{q}(t) - \ddot{q}_d(t) \quad (2.13)$$

$q_d(t)$, $\dot{q}_d(t)$, and $\ddot{q}_d(t) \in R^n$ are the desired joint position, velocity, and acceleration of the single arm, $\overline{q}(t)$, $\dot{\overline{q}}(t)$, and $\ddot{\overline{q}}(t) \in R^n$ are the joint position error, velocity error and acceleration error of the single arm, and $\Lambda \in R^{n\times n}$ is a constant diagonal positive definite matrix chosen by the designer. We also define the reference variable $q_r(t) \in R^n$ as

$$q_r(t) = q_d(t) - \Lambda \int_0^t \overline{q}(t)\, dt. \quad (2.14)$$

Therefore

$$\dot{q}_r(t) = \dot{q}_d(t) - \Lambda\overline{q}(t), \; \ddot{q}_r(t) = \ddot{q}_d(t) - \Lambda\dot{\overline{q}}(t). \quad (2.15)$$

From (2.12), (2.13), and (2.15), we then obtain

$$S(t) = \dot{q}(t) - \dot{q}_r(t) = \dot{\overline{q}}_r(t). \quad (2.16)$$

From (2.12), (2.13), and (2.1), we then obtain

$$\begin{aligned} D\dot{S} &= D(\ddot{\overline{q}} + \Lambda\dot{\overline{q}}) \\ &= D(\ddot{q} - \ddot{q}_d + \Lambda\dot{\overline{q}}) \\ &= D\ddot{q} + D(-\ddot{q}_d + \Lambda\dot{\overline{q}}) \\ &= (\tau - H\dot{q} - G + d) + D(-\ddot{q}_d + \Lambda\dot{\overline{q}}) \\ &= \tau - H\dot{q} - G + d - D\ddot{q}_d + D\Lambda\dot{\overline{q}}. \end{aligned} \quad (2.17)$$

Differentiating the Lyapunov function candidate (2.10) and using (2.17), we have

$$\dot{V} = S^T(\tau - H\dot{q} - G + d - D\ddot{q}_d + D\Lambda\dot{\overline{q}}) + S^T((1/2)(\dot{D} - 2H) + H)S + \Phi^T\Gamma\dot{\Phi}. \quad (2.18)$$

Since $\dot{D} - 2H$ is skew-symmetric [23]–[26], it holds [5]

$$S^T(\dot{D} - 2H)S = 0. \quad (2.19)$$

Inserting (2.19) into (2.18), we get

$$\dot{V} = S^T(\tau - H(\dot{q} - S) - G + d - D\ddot{q}_d + D\Lambda\dot{\overline{q}}) + \Phi^T\Gamma\dot{\Phi}. \quad (2.20)$$

Inserting (2.15) and (2.16) into (2.20), we get

$$\dot{V} = S^T(\tau - H\dot{q}_r - G + d - D\ddot{q}_r) + \Phi^T\Gamma\dot{\Phi}. \quad (2.21)$$

Exploiting the structure of the manipulator dynamics, we can obtain the relations [23]–[26]

$$W(q, \dot{q}, \dot{q}_r, \ddot{q}_r)\Phi = \overline{H}(q, \dot{q}, \hat{p})\dot{q}_r + \overline{D}(q, \hat{P})\ddot{q}_r + \overline{G}(q, \hat{P}) \quad (2.22)$$

where

$$\overline{H} = \hat{H} - H, \overline{D} = \hat{D} - D, \overline{G} = \hat{G} - G \quad (2.23)$$

where

$\hat{H}, \hat{D}, \hat{G}$ are the estimates of $H, D, G$,

and

$\overline{H}, \overline{D}, \overline{G}$ are the errors.

Inserting (2.2) and (2.22) into (2.21), we then obtain

$$\dot{V} = S^T(W\Phi - K_d S) + S^T(d - d_0 \,\mathrm{sat}\,(S/\phi)) + \Phi^T\Gamma\dot{\Phi}. \quad (2.24)$$

Now, we analyze the derivative of Lyapunov function inside and outside the boundary layer $\phi_i$ as follows:

1) Outside the boundary layer $|S_i| > \phi_i, i = 1, 2, \cdots, n$. From (2.9) and (2.24), we get

$$\dot{V} = S^T(W\Phi - K_d S) + S^T(d - d_0 \,\mathrm{sgn}\,(S/\phi)) + \Phi^T\Gamma\dot{\Phi}. \quad (2.25)$$

From (2.6) and (2.9), we get

$$S_i d_i \le S_i d_{i0} \,\mathrm{sgn}\,(S_i/\phi_i), \quad \text{as } S_i/\phi_i > 1$$

$$S_i d_i \le S_i d_{i0} \,\mathrm{sgn}\,(S_i/\phi_i), \quad \text{as } S_i/\phi_i < 1. \quad (2.26)$$

From (2.5), (2.7), and the definition of $S/\phi$ in (2.8), we then obtain

$$S^T d - S^T d_0 \,\mathrm{sgn}\,(S/\phi) \le 0. \quad (2.27)$$

Inserting (2.27) and the switching-$\sigma$ adaptive law (2.3) into (2.25), we get

$$\dot{V} \le -S^T K_d S - (\hat{P} - P)^T \sigma \hat{P}. \quad (2.28)$$

From the definition of the inner product of two vectors, we know

$$\hat{P}^T P \le \|\hat{P}\| \, \|P\|. \quad (2.29)$$

From (2.29) and using the fact $\sigma \ge 0$ and $P_0 > \|P\|$, we obtain

$$\sigma\hat{P}^T(\hat{P} - P) \ge 0. \quad (2.30)$$

Since $\dot{V}$ in (2.28) is always negative owing to (2.30) and the positive definite matrix $k_d$, we know that the joint position error $\overline{q}$ will go to some finite value proportional to the boundary layer $\Phi$ if $\Lambda$ in (2.12) is a positive definite matrix.

2) Inside the boundary layer $|S_i| < \phi_i, i = 1, 2, \cdots, n$.
From (2.9) and (2.24), we get

$$\dot{V} = S^T(W\Phi - K_d S) + S^T(d - d_0(S/\phi)) + \Phi^T \Gamma \dot{\Phi}. \quad (2.31)$$

Inserting the switching-$\sigma$ adaptive law (2.3) into (2.31), we get

$$\dot{V} = -S^T K_d S + S^T(d - d_0(S/\phi)) - (\hat{P} - P)^T \sigma \hat{P}. \quad (2.32)$$

Define

$$d_0/\phi = \mathrm{diag}\,(d_{10}/\phi_1, d_{20}/\phi_2, \cdots, d_{n0}/\phi_n). \quad (2.33)$$

From (2.6) and (2.7), we get

$$S^T d \le |S|^T d_{00}, \quad \text{where } d_{00} = (d_{10}, d_{20}, \cdots, d_{n0})^T. \quad (2.34)$$

Inserting (2.33) and (2.34) into (2.32), we get

$$\dot{V} = -S^T(K_d + d_0/\phi)S + |S|^T d_{00} - (\hat{P} - P)^T \sigma \hat{P}. \quad (2.35)$$

Let

$$K_d + d_0/\phi = K_h K_h^T. \quad (2.36)$$

Inserting (2.36) into (2.35), we get

$$\begin{aligned}\dot{V} \le &-(1/2)S^T(K_d + d_0/\phi)S \\ &- (1/2)(|S|^T K_h - d_{00}^T K_h^{-1})(|S|^T K_h - d_{00}^T K_h^{-1})^T \\ &- (1/2)d_{00}^T K_h^{-1} K_h^T |S| - (1/2)|S|^T K_h (K_h^{-1})^T d_{00} \\ &+ |S|^T d_{00} + (1/2)(d_{00}^T K_h^{-1})(d_{00}^T K_h^{-1})^T \\ &- (\hat{P} - P)^T \sigma \hat{P}. \end{aligned} \quad (2.37)$$

Since $K_h$ is diagonal, it holds

$$K_h = K_h^T, \; (K_h^{-1})^T = K_h^{-1}. \quad (2.38)$$

Since $d_{00}^T|S|$ and $\hat{P}^T P$ are scalar, we then obtain

$$d_{00}^T|S| = |S|^T d_{00}, \; \hat{P}^T P = P^T \hat{P}. \quad (2.39)$$

Inserting (2.38) and (2.39) into (2.37), we then obtain

$$\begin{aligned}\dot{V} \le &-(1/2)S^T(K_d + d_0/\phi)S - (1/2)\| |S|^T K_h - d_{00}^T K_h^{-1}\|^2 \\ &+ (1/2)\|d_{00}^T K_h^{-1}\|^2 - (\hat{P} - P)^T \sigma \hat{P}. \end{aligned} \quad (2.40)$$

From (2.40), we get

$$\begin{aligned}\dot{V} \le &-(1/2)S^T(K_d + d_0/\phi)S - \sigma \hat{P}^T(\hat{P} - P) \\ &+ (1/2)\|d_{00}^T K_h^{-1}\|^2. \end{aligned} \quad (2.41)$$

From (2.41), we get

$$\limsup_{T\to\infty} (1/T)\int_{t_0}^{t_0+T} S^T(K_d + d_0/\phi)S\,dt \le \|d_{00}^T K_h^{-1}\|^2 - \lim_{T\to\infty} (2/T)\int_{t_0}^{t_0+T} (\hat{P} - P)^T \sigma \hat{P}\,dt \quad (2.42)$$

where $d_0/\phi$, $d_{00}$, and $K_h$ are defined in (2.33), (2.34), and (2.36). In [28] we derived the error bound of the single arm for switching-$\sigma$ adaptive control law as

$$\limsup_{T\to\infty} (1/T)\int_{t_0}^{t_0+T} S^T K_d S\,dt \le \|d_{00}^T K_e^{-1}\|^2 \quad (2.43)$$

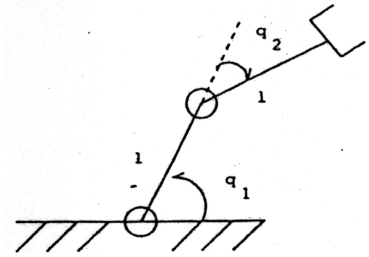


Fig. 2. Single-arm manipulator with two links.

where

$$K_e K_e^T = K_d, \; d_{00} = (d_{10}, d_{20}, \cdots, d_{n0})^T.$$

In [28] we also derived the error bound of the single arm for continuous VSC law as

$$\limsup_{T\to\infty} (1/T)\int_{t_0}^{t_0+T} S^T(K_d + d_0/\phi)S\,dt \le \|d_{00}^T K_h^{-1}\|^2 \quad (2.44)$$

where $d_0/\phi$, $d_{00}$, and $K_h$ are defined in (2.33), (2.34), and (2.36). We know from (2.30), (2.42), (2.43), and (2.44) that the error bound for the combination of the continuous VSC law and switching-$\sigma$ adaptive law is smaller than either continuous VSC law proposed in Slotine and Li [23]–[26] or robust adaptive control law proposed in Reed and Ioannou alone [19], [27].

## III. Single-Arm Simulation

In this section we use a single-arm manipulator with two rigid links as our simulation example (Fig. 2). The link lengths are both $l$, the first and second link's mass are $m_1$ and $m_2$, the first and second joint angles are $q_1$ and $q_2$, the first and second joint torques are $\tau_1$ and $\tau_2$, and $d_1$, $d_2$ are disturbances. The Lagrange-Euler equation of motion for this single-arm manipulator with two links is [1]–[4]. See (3.1) at the bottom of the page. In the simulation we let both $m_1$ and $m_2$ be 1 kilogram, the link length $l$ be one meter for simplicity, and the time-varying disturbances $d_1$, $d_2$ be

$$d_1 = d_2 = 2/(1 + t) \quad (3.2)$$

where $t$ represents time in seconds. We also let the desired trajectories which are twice differentiable for joints 1 and 2 be

$$q_{1d} = \sin(t) + 0.1\sin(3t)$$

$$q_{2d} = 0.1\sin(2t) + 0.1\sin(4t). \quad (3.3)$$

We simulate our robust composite adaptive control law mentioned in Section II, the continuous VSC law proposed in Slotine and Li [23]–[26] and the switching-$\sigma$ adaptive control law proposed in Reed and Ioannou [19], [27] both without unmodeled dynamics and with actuator unmodeled dynamics involved. We model the actuator unmodeled dynamics as a first-order low pass filter, where its cut-off frequency is 100 radians per second and dc gain is one. We can

$$\begin{aligned}\begin{pmatrix}\tau_1\\ \tau_2\end{pmatrix} = &\begin{pmatrix}\frac{1}{3}m_1 l^2 + \frac{4}{3}m_2 l^2 + m_2\cos(q_2)l^2 & \frac{1}{3}m_2 l^2 + \frac{1}{2}m_2 l^2\cos(q_2)\\ \frac{1}{3}m_2 l^2 + \frac{1}{2}m_2 l^2 \cos(q_2) & \frac{1}{3}m_2 l^2\end{pmatrix}\cdot\begin{pmatrix}\ddot{q}_1\\ \ddot{q}_2\end{pmatrix} \\ &+ \begin{pmatrix}-m_2\sin(q_2)l^2\dot{q}_2 & \frac{-1}{2}m_2\sin(q_2)l^2\dot{q}_2\\ \frac{1}{2}m_2\sin(q_2)l^2\dot{q}_1 & 0\end{pmatrix}\begin{pmatrix}\dot{q}_1\\ \dot{q}_2\end{pmatrix} + \begin{pmatrix}\frac{1}{2}m_1\,\mathrm{glcos}\,(q_1) + \frac{1}{2}m_2\,\mathrm{glcos}\,(q_1 + q_2) + m_2\,\mathrm{glcos}\,(q_1)\\ \frac{1}{2}m_2\,\mathrm{glcos}\,(q_1 + q_2)\end{pmatrix} - \begin{pmatrix}d_1\\ d_2\end{pmatrix}. \end{aligned} \quad (3.1)$$

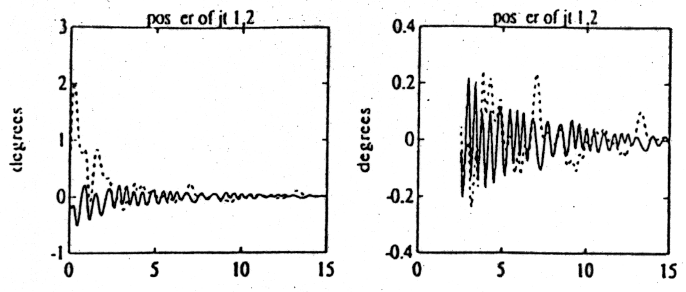


Fig. 3. The switching-$\sigma$ adaptive control law without unmodeled dynamics, solid line: joint 1, dot line: joint 2.

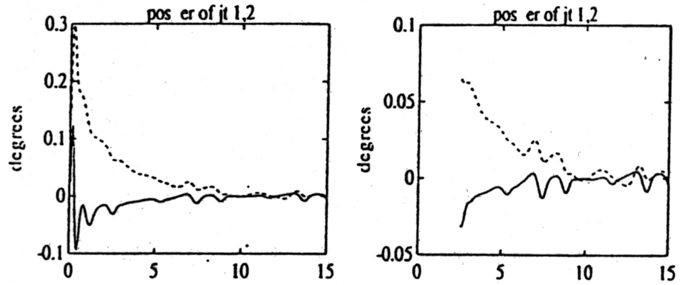


Fig. 4. The continuous VSC law without unmodeled dynamics, boundary layer 0.1, solid line: joint 1, dot line: joint 2.

express our composite robust adaptive controller $\tau_1$, $\tau_2$ as

$$\tau_1 = \hat{m}_1\left(\frac{1}{3}\ddot{q}_{1r} + \frac{1}{2}\,\mathrm{g}\cos(q_1)\right) + \hat{m}_2\left(\left(\frac{4}{3} + \cos(q_2)\right)\ddot{q}_{1r} + \left(\frac{1}{3} + \frac{1}{2}\cos(q_2)\right)\ddot{q}_{2r} - \sin(q_2)\dot{q}_2\dot{q}_{1r} + \frac{-1}{2}\sin(q_2)\dot{q}_2\dot{q}_{2r} + \frac{1}{2}\,\mathrm{g}\cos(q_1+q_2) + \mathrm{g}\cos(q_1)\right) - k_{d1}S_1 - d_{10}\,\mathrm{sat}(S_1/\phi_1)$$

$$\tau_2 = \hat{m}_2\left(\frac{1}{3}\ddot{q}_{1r} + \frac{1}{2}\cos(q_2)\ddot{q}_{1r} + \frac{1}{3}\ddot{q}_{2r} + \frac{1}{2}\sin(q_2)\dot{q}_1\dot{q}_{1r} + \frac{1}{2}\,\mathrm{g}\cos(q_1+q_2) - k_{d2}S_2 - d_{20}\,\mathrm{sat}(S_2/\phi_2)\right) \quad (3.4)$$

where

$$\mathrm{sat}(S_i/\phi_i) = \begin{cases} S_i/\phi_i, & S_i/\phi_i \le 1 \\ \mathrm{sgn}(S_i/\phi_i), & \text{otherwise} \end{cases} \quad i = 1, 2$$

and the switching-$\sigma$ adaptive law can be expressed as

$$\dot{\hat{m}}_1 = -\gamma_1^{-1}\left(\frac{1}{3}\ddot{q}_{1r} + \frac{1}{2}\,\mathrm{g}\cos(q_1)\right)S_1 - \sigma_1\gamma_1^{-1}\hat{m}_1$$

$$\dot{\hat{m}}_2 = -\gamma_2^{-1}\Bigg(\left(\left(\frac{4}{3} + \cos(q_2)\right)\ddot{q}_{1r} + \left(\frac{1}{3} + \frac{1}{2}\cos(q_2)\right)\ddot{q}_{2r} - \sin(q_2)\dot{q}_2\dot{q}_{1r} + \frac{-1}{2}\sin(q_2)\dot{q}_2\dot{q}_{2r} + \frac{1}{2}\,\mathrm{g}\cos(q_1+q_2) + \mathrm{g}\cos(q_1)S_1 + \left(\frac{1}{3}\ddot{q}_{1r} + \frac{1}{2}\cos(q_2)\ddot{q}_{1r} + \frac{1}{3}\ddot{q}_{2r} + \frac{1}{2}\sin(q_2)\dot{q}_1\dot{q}_{1r} + \frac{1}{2}\,\mathrm{g}\cos(q_1+q_2)\right)S_2\Bigg) - \sigma_2\gamma_2^{-1}\hat{m}_2 \quad (3.5)$$

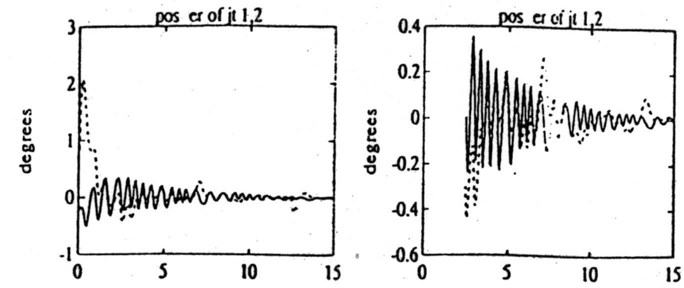


Fig. 5. The continuous VSC law without unmodeled dynamics, boundary layer 10, solid line: joint 1, dot line: joint 2.

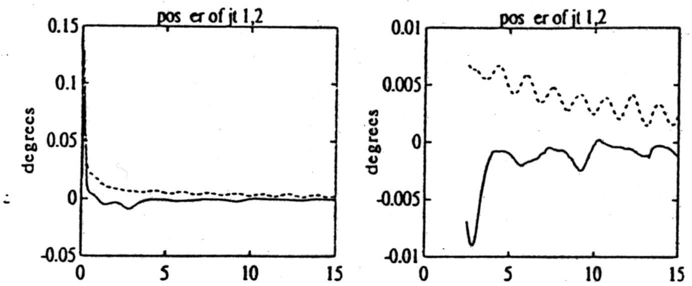


Fig. 6. The composite control law without unmodeled dynamics, boundary layer 0.1, solid line: joint 1, dot line: joint 2.

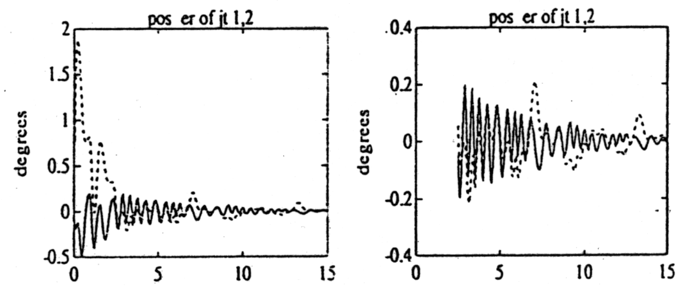


Fig. 7. The composite control law without unmodeled dynamics, boundary layer 10, solid line: joint 1, dot line: joint 2.

where

$$\sigma_i = \begin{cases} 0 & \|\hat{m}_i\| \le m_0 \\ \sigma_0\left(\frac{\|\hat{m}_i\|}{m_0} - 1\right) & m_0 < \|\hat{m}_i\| \le 2m_0 \\ \sigma_0 & 2m_0 < \|\hat{m}_i\| \end{cases} \quad i = 1, 2$$

where $\tau_1$, $\tau_2$, $\hat{m}_1$, and $\hat{m}_2$ for the continuous VSC law and switching-$\sigma$ adaptive control law as described by Chen and Papavassilopoulos [28]. We let initial values of the estimated parameters $\hat{m}_1 = 0.8$, $\hat{m}_2 = 1.2$, controller gains $k_{d1} = k_{d2} = 2$, and chose scalars $\lambda_1 = 20$, $\lambda_2 = 15$, where $\hat{m}_1$ and $\hat{m}_2$ ar the estimates of $m_1$ and $m_2$, respectively. We let $d_{10}$ and $d_{20}$ be the upper bounds of disturbances $d_1$ and $d_2$, thus $d_{10} = d_{20} = 2$. We also choose $\sigma_0 = 5$, $m_0 = 1.1$, $\gamma_1 = \gamma_2 = 1/2.3$. In the simulation, the different boundary layers 0.1 and 10 are used and Adams variable step-size predictor-corrector techniques [6] are used to solve these equations. Their plots are in Figs. 3 to 7 for the case without unmodeled dynamics and in Figs. 8 to 12 for the case with actuator unmodeled dynamics. In each figure, the solid line represents the position error of joint 1 and the dotted line is the position error of joint 2. Fig. 3 shows the joint position error of the single arm as the switching-$\sigma$ adaptive control law is used. Figs. 4 and 5 represent the joint position errors of the single arm as the continuous VSC law with the boundary layers 0.1 and

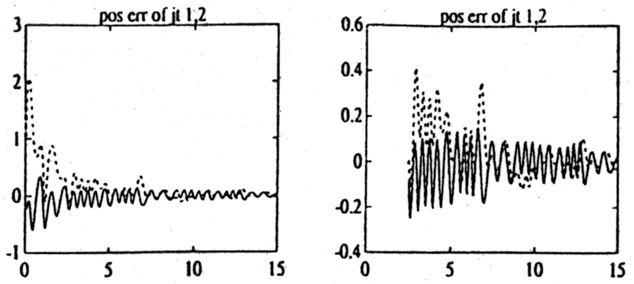


Fig. 8. The switching-$\sigma$ adaptive control law with actuator unmodeled dynamics, solid line: joint 1, dot line: joint 2.

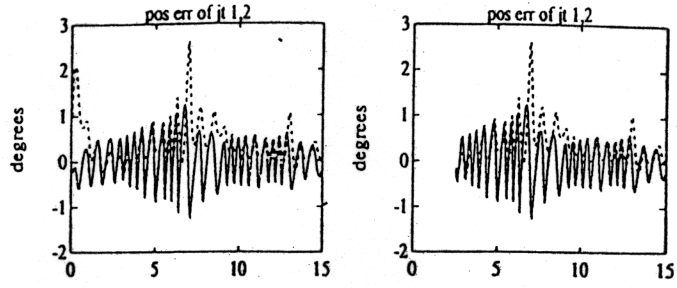


Fig. 10. The continuous VSC law with actuator unmodeled dynamics, boundary layer 10, solid line: joint 1, dot line: joint 2.

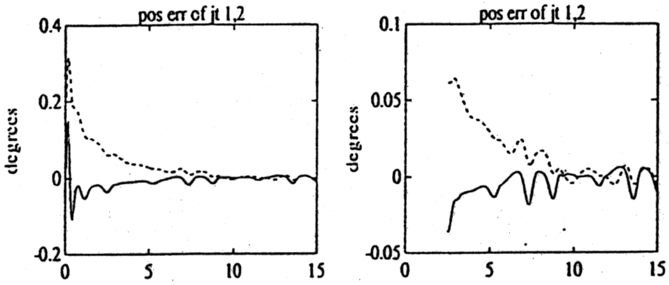


Fig. 9. The continuous VSC law with actuator unmodeled dynamics, boundary layer 0.1, solid line: joint 1, dot line: joint 2.

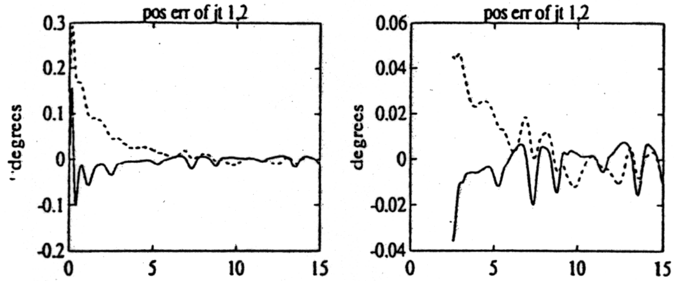


Fig. 11. The composite control law with actuator unmodeled dynamics, boundary layer 0.1, solid line: joint 1, dot line: joint 2.

10 is used. Figs. 6 and 7 show the joint position errors of the single arm as the composite control law with the boundary layers 0.1 and 10 is used. Figs. 8–12 are plotted in the same fashion as Figs. 3–7 where the actuator unmodeled dynamics are considered. For the sake of clarity, there are two plots in each figure. The time scale of the left one is from zero to 15 seconds and the right one is from 2.5 to 15. Comparing Figs. 6 and 7 for the composite control law with Figs. 4 and 5 for the continuous VSC law, we can see that the joint position errors of Figs. 6 and 7 are smaller than those in the corresponding Figs. 4 and 5. We also can see that the joint position errors in Figs. 6 and 7 for the composite control law are smaller than those for the switching-$\sigma$ adaptive control law in Fig. 3. Thus the tracking precision performance of the composite control law is better than in the switching-$\sigma$ adaptive control law and the continuous VSC law. Comparing Figs. 11 and 12 with Figs. 6 and 7, we see that the joint position error of the composite control law for the case without unmodeled dynamics is smaller than for the case with the actuator unmodeled dynamics, as was to be expected. Comparing Fig. 8 with Figs. 11 and 12 and comparing Figs. 9 and 10 with 11 and 12, we can see that the composite control law is more robust to actuator unmodeled dynamics than the continuous VSC law and switching-$\sigma$ adaptive control law.

## IV. Conclusions

In this paper, we have analyzed our composite robust adaptive control law and compared it with the continuous VSC law and the switching-$\sigma$ adaptive law for the single-arm dynamics with bounded disturbance, unknown parameters, and actuator unmodeled dynamics. We can see that the combination of the continuous VSC law and the switching-$\sigma$ adaptive control law has better tracking precision performance than either the VSC control law or the switching-$\sigma$ adaptive control law alone for both the cases without unmodeled dynamics and with actuator unmodeled dynamics. From the theoretical analysis, we know that a small boundary layer may excite high frequency unmodeled dynamics which will cause instabilities,

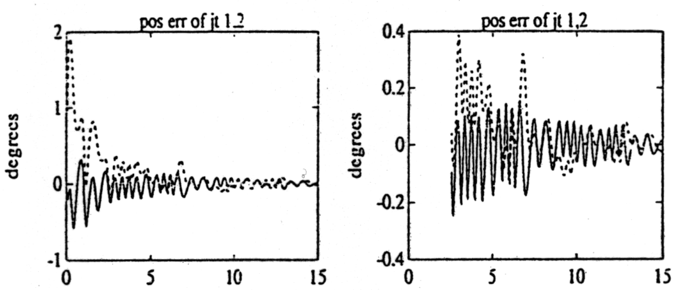


Fig. 12. The composite control law with actuator unmodeled dynamics, boundary layer 10, solid line: joint 1, dot line: joint 2.

but the simulations show that the system with a small boundary layer (Fig. 11) still has good tracking precision even if actuator unmodeled dynamics are considered. The reason is that the high-frequency amplitude of the control torque is small. The joint position error without unmodeled dynamics, however, is smaller than with actuator unmodeled dynamics. Although the simulations show that a small boundary layer can achieve better tracking even if actuator unmodeled dynamics are considered, such a boundary layer is limited by its physical attributes.

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# Set of Reachable Positions for a Car

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*Abstract*—This paper shows how to compute the reachable positions for a model of a car with a lower bounded turning radius that moves forward and backward with a constant velocity. First, we compute the shortest paths when the starting configuration (i.e., position and direction) is completely specified and the goal is only defined by the position with the direction being arbitrary. Then we compute the boundary of the region reachable by such paths. Such results are useful in motion planning for nonholonomic mobile robot.

## I. Introduction

Let us consider a car moving forward and backward with a lower-bounded turning radius $R$ (without any loss of generality, we assume $R = 1$) and an upper-bounded velocity. The position and the direction of the car are, respectively, defined by the coordinates $(x, y)$ of the reference point and the angle $\theta$ between the abscissa axis and the main axis of the car (see Fig. 1). So, the car is completely defined as a point $(x, y, \theta)$ in the configuration space $R^2 \times S^1$. If we assume that the linear velocity is constant, the motion is defined by the control system

$$(\Sigma) \qquad \begin{cases} \dot{x} = \cos\theta \cdot u_1 \\ \dot{y} = \sin\theta \cdot u_1 \\ \dot{\theta} = u_2 \end{cases}$$

with $|u_1(t)| = 1$ and $|u_2(t)| \leq 1$ where $u_1$ and $u_2$ are, respectively, the linear and angular velocity of the car. Such a differential system expresses kinematic constraints which characterize the nonholonomic nature of the car [4]. This is the Reeds and Shepp model.

Initially this model has been introduced by Dubins [3] for a car that moves only forward (i.e., $u_1 \equiv 1$). He determines a sufficient family of shortest paths.[1] Using this result and the result of Melzak [6], Robertson [10], and Cockayne and Hall [2] provide the set of accessible positions for the model of Dubins (i.e., $u_1 \equiv 1$).

The problem of finding a shortest path between two configurations when backward motions are allowed ($u_1 = \pm 1$) has been set by Reeds and Shepp in [9]; it has been completely solved after a sequence of different works [1], [9], [11], [12].

This paper solves the following problem:

> How do we compute the set of reachable positions from the origin by path of a given length when the final direction of the car is not specified?

It is solved in two steps: among all the paths linking an initial position with fixed direction of the car to a final position with free direction, we point out a shortest one. Then, from the shortest path expression, we obtain the complete analytical description in the plane $(O, x, y)$ of the set of positions reachable from the origin by a path of length lesser than some given value. Section II presents the state of the art on Reeds and Shepp's problem. Section III shows how to apply the Pontryagin Maximum Principle (PMP) to compute a sufficient family of optimal paths. This family is then reduced by

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[1] i.e., a family rich enough to always contain a shortest path to link any two configurations.