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A CLASS OF RISK-SENSITIVE NONCOOPERATIVE GAMES*

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Risk sensitivity is studied in connection to a class of noncooperative games with incomplete information. Specifically, we consider a two-player noncooperative stochastic game where each player maximizes the expected value of a utility function with constant absolute risk aversion. The approach generalizes more traditional models for economic policy evaluation, including the linear-quadratic stochastic Nash game studied by Papavassilopoulos (1981) and the exponential-quadratic function studied, in the context of single decision making, by Van der Ploeg (1984). Conditions for the existence of noncooperative equilibria are derived. The paper offers new insight on the influence of risk attitudes on equilibrium. It is shown, among other results, that in the assumption of Gaussian distribution of the random variables a Nash equilibrium may not exist when players risk attitudes are too conservative. The main results are illustrated with an example.

1. Introduction

In the static approach of expected utility, risk aversion is conceptualised as concavity of a Von Neumann–Morgenstern utility function and is related to the price an agent would be ready to pay to avoid participating in a fair lottery.

In a dynamic decision-making context, risk sensitivity entails lack of certainty equivalence, in the sense that replacement of the random variables by their first moments does not lead to optimal solutions. Indeed, optimal risk-sensitive decisions should depend at least on the second moment of the probability distribution [Caravani (1987)].

Both features – concave utility and distribution-dependent optimal decisions – are exhibited by the class of exponential-quadratic functions studied by Jacobson (1977), Whittle (1981), and Van der Ploeg (1984) in the respective contexts of control, statistics, and economics.

A different modelling context to study risk attitudes is that of noncooperative stochastic games. Two players with different goals may prefer not to

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cooperate and settle for a Nash equilibrium in which they fall short of their goals by a predictable (or expected) amount rather than cooperate and be given an uncertain reward. Again, as in the expected utility context, risk is akin to a costly preference for certainty.

Moving from the background of exponential-quadratic functions, this paper is a study of the equilibria prevailing when two conflicting agents with different risk aversions and incomplete information engage in a noncooperative game.

In the paper we derive sufficient conditions for the existence of a Nash equilibrium. In the assumption of Gaussian random variables, we also derive necessary and sufficient conditions for the existence of a Nash equilibrium when the strategies are restricted to the class of functions affine in the information, or when the information available to the players is identical. In either case, the nature of these conditions is constructive, i.e., they provide a computable solution to the game.

The results of the paper highlight the rather crucial role played by the information structure and the risk-aversion parameter in the existence of a Nash equilibrium. In particular it is shown that in the case of identical information and Gaussian random variables if a solution exists it has to be affine in the information. In the case of affine strategies, if a solution exists it is unique. The existence of such a solution is always guaranteed provided both players are either risk-neutral or risk-loving. Conversely, no solutions exist if both players are *enough* risk-averse, that is if their Arrow–Pratt indices go beyond a computable threshold. Hence caution and defensiveness not necessarily lead to noncooperative solutions of Nash type.

Finally, on the basis of a paradigmatic example, we demonstrate the existence of trade-offs between risk attitudes. There exist Nash equilibria with one player risk-averse and the other risk-loving. But, beyond a certain degree, risk aversion of one player enforces imitative behaviour in the other: for a Nash equilibrium to exist, the opponent's attitudes must *also* be risk-averse.

In section 2 we specify the basic model and the form of players' utilities. In section 3 we formulate a simple example showing the connection of our model to linear-quadratic games on one hand, and to exponential-quadratic single decision making on the other. Section 4 deals with Nash equilibria. We first consider the existence of a Nash equilibrium in the case of arbitrary strategies and incomplete information (Theorem 1). We then specialise our results to the case of identical (but incomplete) information (Theorem 2) and to the case of affine strategies (Theorem 3). In section 5 we use our results to discuss in some detail a particular case that naturally extends the examples of section 3. The main conclusions are summarised in section 6.

The proofs of the theorems require intermediate results. These are presented in section A.1 of the appendix as a set of self-contained Lemmata, which might be of independent interest. Section A.2 of the appendix contains the algebra required to carry out the example of section 5 in the main text.

2. Model specification

The model comprises a coupling constraint, a goal function, and an information structure.

We assume that two players, indexed by $i = 1, 2$, wish to influence to their advantage a vector $z \in R^n$ by exerting actions $u_1 \in R^{m_1}$ and $u_2 \in R^{m_2}$. Their influence on z is linear and uncertain, due to the presence of additive disturbances x,

$$z = D_1 u_1 + D_2 u_2 + x, \tag{1}$$

where D_1 and D_2 are $n \times m_1$ and $n \times m_2$ matrices. Vector x is a Gaussian random variable with respect to the probability space $(\Omega, \mathscr{F}, \mathscr{P})$ and is assumed to have zero mean and covariance matrix Σ. We express this by writing briefly $x \sim \mathrm{N}(0, \Sigma)$. If one wishes, eq. (1) can be thought of as the reduced form of an econometric model, in which z collects the objectives of economic policy while u_1 and u_2 are the instruments in the hands of two separate authorities.

Player 1 has a twofold objective. On one hand he would like vector z to attain a target value a_1. But to attain a_1 he must exert u_1 and he wishes u_1 to be small. So he perceives as a loss both the distance of z from a_1 and the magnitude of u_1. Similarly for player 2, and if $a_1 \neq a_2$, they can clearly engage in a noncooperative game. Assuming quadratic losses, the loss for player i is

$$\mathscr{L}_i = \tfrac{1}{2}(z - a_i)'Q_i(z - a_i) + \tfrac{1}{2}u_i'P_i u_i, \tag{2}$$

where a prime denotes transpose. Matrices $Q_i \sim n \times n$ and $P_i \sim m_i \times m_i$ allow to weight differently the various loss components. For technical reasons, we assume P_i to be symmetric positive definite and Q_i symmetric positive semidefinite.

By defining, in our context, welfare as minus loss ($W = -\mathscr{L}$) the welfare of each player will be negative or at best zero. As Von Neumann utility increases with welfare and is defined up to an additive constant, it can be assumed to be bounded above by zero. Specifically, we consider exponential utilities of the form [Van der Ploeg (1984)]

$$U(W_i) = -\frac{1}{\theta_i}(\mathrm{e}^{-\theta_i W_i} - 1). \tag{3}$$

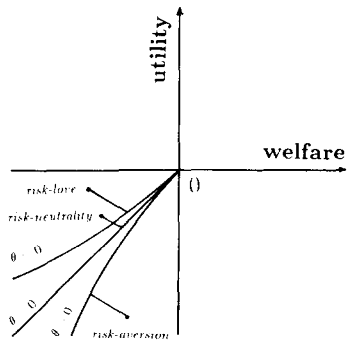

Fig. 1. Exponential utility parametrized by risk aversion.

Notice that for $\theta_i \to 0$, $U(W_i) \to W_i$ so quadratic functions are obtained as a special case. Note that $U(W_i)$ is always negative and strictly convex for $\theta_i < 0$ (or strictly concave for $\theta_i > 0$). Parameter θ_i is easily interpreted as the Arrow–Pratt index of absolute risk aversion, $\theta_i = -U''/U'$, where a prime denotes partial derivative with respect to W_i [Pratt (1964)]. Parameter θ_i is positive if player i is risk-averse (or negative if he is risk-loving). Risk neutrality is obtained for $\theta_i = 0$, the case of linear utility (see fig. 1).

Let E be the expected value operator with respect to the random variable x. For small $|\theta_i \mathrm{E} W_i^2|$ the expected utility of welfare tends to

$$\mathrm{E} W_i - \tfrac{1}{2}\theta_i \mathrm{E} W_i^2.$$

Therefore, variability of the welfare function decreases the expected utility of a risk-averse player and it increases that of a risk-loving player. Some advantage can accrue to the players by their information about the uncertain disturbance vector x. This information takes on the form of *private* observations of a random vector $y_i \in R^{p_i}$, linearly related to x by

$$y_i = C_i x, \tag{4}$$

where matrices C_i have rank $p_i \leq n$. This includes the case where x is perfectly observed ($p_i = n$) or is totally unobserved ($p_i = 0$). As special cases are also included the case of identical information, in which there is a one-to-one correspondence between y_1 and y_2 (for instance $C_1 = C_2$), and the case of disjoint information, in which y_1 and y_2 are statistically independent ($C_1 \Sigma C_2' = 0$). Each player is assumed to have complete knowledge of his own goal function (i knows θ_i, a_i, Q_i, P_i) and of the coupling constraint parameters D_1 and D_2.

We assume that players maximize expected utility, that is i chooses u_i as a function of y_i so as to minimize the conditional expectation of an exponential-quadratic loss function

$$\min \mathrm{E}\left[\frac{1}{\theta_i}(\mathrm{e}^{\theta_i \mathscr{L}_i} - 1) | y_i\right]. \tag{5}$$

In the sequel, -1 will be omitted since it plays no role in the minimization. In order to make this formulation meaningful, we must specify the set $\mathscr{M}$ in which the optimal solution is sought. Since each player is assumed to act purely on the basis of his own information, his real unknown is a function $u_i = \gamma_i(y_i)$.

Formally, γ_i is an element of the space $\mathscr{M}_i$ of $\mathscr{F}_i$-measurable functions where $\mathscr{F}_i$ is the minimal sub σ-field of $\mathscr{F}$ generated by y_i.

The complete model can now be put in so-called normal form [Basar and Olsder (1982, p. 207)] by eliminating z from (1)–(2) and replacing $\mathscr{L}_i$ in (5),

$$\min \mathrm{E}\left[\frac{1}{\theta_i}\exp\left(\theta_i\left(\tfrac{1}{2}\{x' \;\; u'\}\begin{bmatrix} Q_i & A_i' \\ A_i & R_i \end{bmatrix}\begin{Bmatrix} x \\ u \end{Bmatrix} + \{q_i' \;\; r_i'\}\begin{Bmatrix} x \\ u \end{Bmatrix}\right)\right)\middle| y_i\right], \tag{6}$$

where $u' = \{u_1' \;\; u_2'\}$, $u_i = \gamma_i(y_i)$, and minimization is with respect to γ_i. Furthermore,

$$q_i' = -a_i'Q_i, \quad r_i' = \{-a_i'Q_iD_1 \quad -a_i'Q_iD_2\}, \quad A_i' = [Q_iD_1 \quad Q_iD_2],$$

$$R_1 = \begin{bmatrix} D_1'Q_1D_1 + P_1 & D_1'Q_1D_2 \\ D_2'Q_1D_1 & D_2'Q_1D_2 \end{bmatrix}, \quad R_2 = \begin{bmatrix} D_1'Q_2D_1 & D_1'Q_2D_2 \\ D_2'Q_2D_1 & D_2'Q_2D_2 + P_2 \end{bmatrix}.$$

The dimensions of the five arrays above are, in the order, $1 \times n$, $1 \times m$, $n \times m$, $m \times m$, $m \times m$, with $m = m_1 + m_2$.

Each player's minimand, therefore, is a functional of γ_1 and γ_2 which will be denoted $J_{\theta_i}(\gamma_1, \gamma_2)$. We investigate now whether a solution to problem (6) exists. This is equivalent to searching for a pair (γ_1^*, γ_2^*) satisfying

$$\begin{aligned} J_{\theta_1}(\gamma_1^*, \gamma_2^*) &\le J_{\theta_1}(\gamma_1, \gamma_2^*), \qquad \forall \gamma_1 \in \mathscr{M}_1, \\ J_{\theta_2}(\gamma_1^*, \gamma_2^*) &\le J_{\theta_2}(\gamma_1^*, \gamma_2), \qquad \forall \gamma_2 \in \mathscr{M}_2. \end{aligned} \tag{7}$$

We recall that when such a pair exists it is termed a Nash equilibrium. The rationale for choosing γ_1^* is to force 2 to play γ_2^*, so as to secure himself the Nash payoff, a predictable (or expected) quantity. Clearly, he could gain more by *not* playing u_1^*, if his expectation that the other cooperates were fulfilled – but he could loose more if it were not. It is precisely in this sense that we say that Nash solution is risk-averse.[1]

[1]We slightly abuse of technical language, as the term is usually restricted to concavity of the utility function in single decision making.

We finally remark that the model considered is essentially a one-act, static game. It can also represent a multi-act dynamic game when lagged variables are suitably stacked into vector z. The information structure (4), however, is such that the resulting strategies are open-loop, that is they only depend on observed states of nature and not on previous actions. It is known that in this case the dynamic Nash game can be transformed into an equivalent static Nash game [Basar and Olsder (1982)].

3. Particular cases

As an example, consider what is perhaps the simplest nontrivial instance of model (1)–(4):

$$
\begin{aligned}
&\text{Coupling constraint} && s = d_1 u_1 + d_2 u_2 + v, \\
&\text{Loss function} && \mathscr{L}_i = \tfrac{1}{2} k_i (s - \alpha_i)^2 + \tfrac{1}{2} P_i u_i^2, \\
&\text{Player 1 information} && y_1 = v + w_1, \\
&\text{Player 2 information} && y_2 = v + w_2, \\
&\text{Minimand} && \mathrm{E}\left[\frac{1}{\theta_i} \mathrm{e}^{\theta_i \mathscr{L}_i} \middle| y_i \right] \to \min,
\end{aligned}
$$

where v, w_1, and w_2 are scalar-valued, Gaussian, zero-mean independent random variables with variances[2] f^2, σ_1^2, σ_2^2, and $k_i \geq 0$, $P_i > 0$. The information available to each player comprises a subjective noise component (w_i) and a common noise component (v). It is easy to recognize that this is an instance of model (1)–(4) if we make the identifications

$$
\begin{aligned}
x &= \{v \quad w_1 \quad w_2\}', \\
z &= \{s \quad w_1 \quad w_2\}', \\
a_i &= \{\alpha_i \quad 0 \quad 0\}', \\
D_i &= \{d_i \quad 0 \quad 0\}', \\
C_1 &= \{1 \quad 1 \quad 0\}, \\
C_2 &= \{1 \quad 0 \quad 1\}, \\
Q_i &= \begin{bmatrix} k_i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\end{aligned}
$$

[2] All of which are assumed known to both players.

From these we deduce the normal-form matrices

$$q_i = -\{\alpha_i k_i \quad 0 \quad 0\}',$$

$$r_i = -\{\alpha_i k_i d_1 \quad \alpha_i k_i d_2\}',$$

$$A_i' = \begin{bmatrix} k_i d_1 & k_i d_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$R_1 = \begin{bmatrix} d_1^2 k_1 + P_1 & d_1 d_2 k_1 \\ d_2 d_1 k_1 & d_2^2 k_1 \end{bmatrix},$$

$$R_2 = \begin{bmatrix} d_1^2 k_2 & d_1 d_2 k_2 \\ d_2 d_1 k_2 & d_2^2 k_2 + P_2 \end{bmatrix}.$$

The difficulty faced by player 1 is that, due to the coupling constraint, his own expectation depends on u_2 and this is an *unknown* function of the random variable y_2 (and symmetrically for player 2). The basic question he faces is whether or not a sort of noncooperative equilibrium is possible at all, that is, whether or not a Nash solution exists. The answer to this question brings into play, as we shall see, the agents' risk attitudes in a rather crucial way. To motivate our subsequent study, we discuss first three particular cases.

Case 1. Two players with linear utilities: $\theta_1 = \theta_2 = 0$

This is the stochastic quadratic Nash game studied in Papavassilopoulos (1981). The result is that the set of equilibria is generically nonvoid, in which case it includes a solution affine in the information. This is given by[3]

$$u_i^* = g_i \mathrm{E}[v|y_i] + \mu_i,$$

where

$$\begin{Bmatrix} g_1 \\ g_2 \end{Bmatrix} = -\begin{bmatrix} P_1 + k_1 d_1^2 & k_1 d_1 d_2 F(\sigma_2, f) \\ k_2 d_2 d_1 F(\sigma_1, f) & P_2 + k_2 d_2^2 \end{bmatrix}^{-1} \begin{Bmatrix} k_1 d_1 \\ k_2 d_2 \end{Bmatrix},$$

$$\begin{Bmatrix} \mu_1 \\ \mu_2 \end{Bmatrix} = \begin{bmatrix} P_1 + k_1 d_1^2 & k_1 d_1 d_2 \\ k_2 d_2 d_1 & P_2 + k_2 d_2^2 \end{bmatrix}^{-1} \begin{Bmatrix} k_1 d_1 \alpha_1 \\ k_2 d_2 \alpha_2 \end{Bmatrix},$$

$$\mathrm{E}[v|y_1] = F(\sigma_1, f) y_1, \qquad \mathrm{E}[v|y_2] = F(\sigma_2, f) y_2,$$

and $F(\sigma, f)$ is the function $(\sigma, f) \to 1/[1 + (\sigma/f)^2]$.

[3]Computation is straightforward in our case.

Case 2. One player with exponential utility: $\theta_1 \neq 0, \quad \theta_2 = d_2 = 0$
This is the exponential-quadratic problem studied (in a more general dynamic context) by Jacobson (1977), Whittle (1981), and Van der Ploeg (1984). Their solution yields in our case

$$u_1^* = g_1 \mathrm{E}[v|y_1] + \mu_1,$$

where

$$g_1 = -\frac{k_1 d_1}{P_1 + k_1 d_1^2 - \theta_1 f^2(1 - F(\sigma_1, f)) P_1 k_1},$$

$$\mu_1 = -\alpha_1 g_1.$$

A sufficient condition for the solution to exist is

$$1 - \theta_1 k_1 f^2 (1 - F(\sigma_1, f)) > 0, \tag{8}$$

since this avoids the denominator of g_1 to vanish. It turns out that this condition is also necessary to ensure bounded utility (see remark in section A.2 of the appendix).

Case 3. One player with linear utility: $\theta_1 = \theta_2 = d_2 = 0$
This is the classic linear-quadratic problem studied by Simon (1956) and Theil (1958). Its solution can be obtained by letting $d_2 = 0$ in case 1 or $\theta_1 = 0$ in case 2,

$$u_1^* = g_1 \mathrm{E}[v|y_1] + \mu_1,$$

where

$$g_1 = -\frac{k_1 d_1}{P_1 + k_1 d_1^2},$$

$$\mu_1 = -\alpha_1 g_1.$$

Inspection of these cases shows that when utilities are linear in the loss (cases 1 and 3) the solutions depend on f, σ_1, and σ_2 only via the function $F(\sigma_i, f)$. Since this is homogeneous of degree zero, the solutions do not change if f, σ_1, and σ_2 are multiplied by a scalar factor, that is if the uncertainty level is proportionally increased or decreased. The circumstance in which second and higher moments do not alter the optimal solution is known as so-called certainty equivalence [for example, Bertsekas (1976, p. 18)]. Although evidenced here in a particular case, certainty equivalence is peculiar of quadratic functions and it reflects a risk-neutrality assumption [Van der Ploeg (1984)]. It is, of course, questionable that an optimal strategy should remain so *regardless* of the uncertainty level affecting the decisions.

We propose to amend this modelling defect by using the results of case 2, where the solution does depend explicitly on f^2. As a natural extension of case 2, we consider the case of two players with exponential utilities, as announced in the introduction.

4. Nash equilibria

In this section we characterize Nash equilibria. Some intermediate results are necessary in the proofs of the theorems. These are contained in section A.1 of the appendix.

Theorem 1 (*General case*). *Let* $x \in R^n$ *be* $\mathrm{N}(0, \Sigma)$. *Let* $y' = \{y_1' \quad y_2'\}$ *with* $y_i \in R^{p_i}$, $y_i = C_i x$ *with* $\mathrm{rank}(C_i) = p_i \le n$. *Let* $\gamma_i \in \mathcal{M}_i$, *the space of* m_i*-valued measurable functions of* y_i, *and let* $\gamma' = \{\gamma_1' \quad \gamma_2'\}$. *Let scalars* θ_i *and matrices* Q_i *and* R_i *be defined such that* $\Sigma^{-1} - \theta_i Q_i$ *and* R_i *are positive definite. Let*

$$J_{\theta_i}(\gamma_1, \gamma_2)$$
$$= \mathrm{E}\left[\frac{1}{\theta_i}\exp\left(\theta_i\left(\tfrac{1}{2}\{x' \quad \gamma'\}\begin{bmatrix} Q_i & A_i' \\ A_i & R_i \end{bmatrix}\begin{Bmatrix} x \\ \gamma \end{Bmatrix} + \{q_i' \quad r_i'\}\begin{Bmatrix} x \\ \gamma \end{Bmatrix}\right)\right)\middle| y_i\right].$$

A sufficient condition for the pair (γ_1^*, γ_2^*) *to satisfy the (Nash) conditions (7) is that* (γ_1^*, γ_2^*) *satisfy, for* $i = 1, 2$,

$$\mathrm{E}\Bigg[(A_{ii}x + R_{ii}\gamma + r_{ii})$$
$$\times \exp\left(\theta_i\left(\tfrac{1}{2}\{x' \quad \gamma'\}\begin{bmatrix} Q_i & A_i' \\ A_i & R_i \end{bmatrix}\begin{Bmatrix} x \\ \gamma \end{Bmatrix} + \{q_i' \quad r_i'\}\begin{Bmatrix} x \\ \gamma \end{Bmatrix}\right)\right)\Bigg| y_i\Bigg] = 0,$$

where

$$A_i' = [A_{i1}' \quad A_{i2}'], \quad A_{i1}' \sim n \times m_1, \quad A_{i2}' \sim n \times m_2,$$
$$R_i = [R_{i1}' \quad R_{i2}'], \quad R_{i1}' \sim m \times m_1, \quad R_{i2}' \sim m \times m_2, \quad m = m_1 + m_2,$$
$$r_i' = \{r_{i1}' \quad r_{i2}'\}, \quad r_{i1}' \sim 1 \times m_1, \quad r_{i2}' \sim 1 \times m_2.$$

This condition is also necessary for $\theta_i \le 0$.

Proof. Let

$$f_{\theta_i}(\gamma_1, \gamma_2) = \frac{1}{\theta_i}\exp\left(\theta_i\left(\tfrac{1}{2}\{x' \quad \gamma'\}\begin{bmatrix} Q_i & A_i' \\ A_i & R_i \end{bmatrix}\begin{Bmatrix} x \\ \gamma \end{Bmatrix} + \{q_i' \quad r_i'\}\begin{Bmatrix} x \\ \gamma \end{Bmatrix}\right)\right).$$

This is a continuously differentiable function of γ_1, γ_2. Setting equal to zero the first derivatives with respect to γ_i and taking conditional expectations, we obtain

$$\mathrm{E}\left[\frac{\partial f_{\theta_i}(\gamma_1^*, \gamma_2^*)}{\partial \gamma_i}\bigg| y_i\right] = 0, \qquad i = 1, 2,$$

which is the condition of the Theorem, evaluated at the point (γ_1^*, γ_2^*). Assume this condition holds and distinguish three cases.

Case 1: $\theta_i > 0, \quad i = 1, 2$
Since R_i is positive definite, $f_{\theta_i}(\gamma_1, \gamma_2)$ is strictly convex in each of its arguments, therefore

$$f_{\theta_1}(\gamma_1, \gamma_2^*) \geq f_{\theta_1}(\gamma_1^*, \gamma_2^*) + \frac{\partial f_{\theta_1}(\gamma_1^*, \gamma_2^*)}{\partial \gamma_1}(\gamma_1 - \gamma_1^*),$$

$$f_{\theta_2}(\gamma_1^*, \gamma_2) \geq f_{\theta_2}(\gamma_1^*, \gamma_2^*) + \frac{\partial f_{\theta_2}(\gamma_1^*, \gamma_2^*)}{\partial \gamma_2}(\gamma_2 - \gamma_2^*).$$

Taking conditional expectations the second terms on the right-hand side vanish and we obtain (7).

Case 2: $\theta_i = 0, \quad i = 1, 2$
The condition of the Theorem is readily interpreted as the necessary condition for the minimization of $J_0(\gamma_1, \gamma_2)$. Since this is a quadratic function of γ_1, γ_2 and R_i is positive definite, the minimand is strictly convex and the condition is also sufficient.

Case 3: $\theta_i < 0, \quad i = 1, 2$
The condition of the Theorem is again interpreted as the necessary condition for the minimization of $J_{\theta_i}(\gamma_1, \gamma_2)$. Although this function is nonconvex, by completing the square it can be put in the form

$$J_\theta(\gamma) = \mathrm{E}\left[\frac{1}{\theta}\exp(\theta\{\gamma + g(x)\}'R\{\gamma + g(x)\})f(x)|y\right],$$

with $f(x) > 0$. For $\theta < 0$, $J_\theta(\gamma)$ has a unique global minimum and the condition is also sufficient for (7).

For cases which are mixtures of case 2 and case 3 (say, $\theta_1 < 0$, $\theta_2 = 0$) the condition of the Theorem is clearly necessary and sufficient. QED

The reason why this Theorem does not always provide a necessary condition is that, for $\theta_i > 0$, the derivatives $\partial J_{\theta_i}(\gamma_1, \gamma_2)/\partial \gamma_i$ may fail to exist. At the end of this section we shall illustrate this point in more detail.

In the case of identical information, however, the situation simplifies considerably, because both γ_1 and γ_2 depend on the same random variable $y = y_1 = y_2$. We fall in the class of problems where $\gamma' = \{\gamma_1' \quad \gamma_2'\}$ is a function of a single conditioning variable y, for which the following holds.

Theorem 2 (Identical information). Let $x \in R^n$ be $\mathrm{N}(0, \Sigma)$. Let $y \in R^p$ and $y = Cx$ with $\operatorname{rank}(C) = p \le n$. Let $\gamma_i \in \mathcal{M}_i$, $i = 1, 2$, be the space of m_i-valued measurable functions of y and let $\gamma' = \{\gamma_1' \quad \gamma_2'\}$. Let scalars θ_i and matrices Q_i and R_i be defined such that $\Sigma^{-1} - \theta_i Q_i$ and R_i are positive definite. Let

$$J_{\theta_i}(\gamma_1, \gamma_2)$$

$$= \mathrm{E}\left[\frac{1}{\theta_i}\exp\left(\theta_i\left(\tfrac{1}{2}\{x' \quad \gamma'\}\begin{bmatrix} Q_i & A_i' \\ A_i & R_i \end{bmatrix}\begin{Bmatrix} x \\ \gamma \end{Bmatrix} + \{q_i' \quad r_i'\}\begin{Bmatrix} x \\ \gamma \end{Bmatrix}\right)\right)\middle| y\right].$$

A necessary and sufficient condition for the pair (γ_1^*, γ_2^*) to satisfy the (Nash) conditions (7) is that (γ_1^*, γ_2^*) satisfy, for $i = 1, 2$,

$$A_{ii}[I - \theta_i S Q_i]^{-1}(\mathrm{E}[x|y] + \theta_i S(q_i + A_i'\gamma)) + R_{ii}\gamma + r_{ii} = 0,$$

where

$$\mathrm{E}[x|y] = \Sigma C'[C\Sigma C']^{-1} y,$$

$$S = \mathrm{E}[xx'|y] = \Sigma - \Sigma C'[C\Sigma C']^{-1} C\Sigma,$$

and

$$A_i' = [A_{i1}' \quad A_{i2}'], \quad A_{i1}' \sim n \times m_1, \quad A_{i2}' \sim n \times m_2,$$

$$R_i = [R_{i1}' \quad R_{i2}'], \quad R_{i1}' \sim m \times m_1, \quad R_{i2}' \sim m \times m_2, \quad m = m_1 + m_2,$$

$$r_i' = \{r_{i1}' \quad r_{i2}'\}, \quad r_{i1}' \sim 1 \times m_1, \quad r_{i2}' \sim 1 \times m_2.$$

Proof. Since $J_{\theta_i}(\gamma_1, \gamma_2)$ is a continuously differentiable function of γ_i, a necessary condition for (7) to hold is that $\partial J_{\theta_i}(\gamma_1, \gamma_2)/\partial \gamma_i$ vanish at γ_i^*. By the argument used in the proof of Theorem 1, this condition is also sufficient. Since

$$\partial J_{\theta_1}/\partial \gamma_1 = [I \quad 0]\, \partial J_{\theta_1}/\partial \gamma,$$

$$\partial J_{\theta_2}/\partial \gamma_2 = [0 \quad I]\, \partial J_{\theta_2}/\partial \gamma,$$

we have

$$[I \quad 0]\mathrm{E}\big[(A_1 x + R_1\gamma + r_1)$$
$$\times \exp\big(\theta_1\big(\tfrac{1}{2}x'Q_1x + (\gamma'A_1 + q_1')x\big)\big) | Cx\big] = 0,$$

$$[0 \quad I]\mathrm{E}\big[(A_2 x + R_2\gamma + r_2)$$
$$\times \exp\big(\theta_2\big(\tfrac{1}{2}x'Q_2x + (\gamma'A_2 + q_2')x\big)\big) | Cx\big] = 0,$$

and, by Lemma 5 (appendix), the result follows. QED

Notice that if the condition of the Theorem is satisfied, then γ^* must be an affine function of y. Thus, in the case of identical information if a Nash equilibrium strategy exists, it must be an affine function of the information.

When the assumption of identical information is dropped, it may still be worthwhile to check whether a solution affine in the information exists. Suppose $\mathscr{M}_i$ is restricted to the class $\mathscr{A}_i$ of functions affine in the information. We have in this case:

Theorem 3 (Affine solutions). Let $x \in R^n$ be $\mathrm{N}(0, \Sigma)$. Let $y' = \{y_1' \quad y_2'\}$ with $y_i \in R^{p_i}$, $y_i = C_i x$ with $\mathrm{rank}(C_i) = p_i \le n$. Let $\gamma_i \in \mathscr{A}_i$, $i = 1, 2$, be the space of m_i-valued affine functions of y_i and let $\gamma' = \{\gamma_1' \quad \gamma_2'\}$. Let

$$J_{\theta_i}(\gamma_1, \gamma_2)$$
$$= \mathrm{E}\left[\frac{1}{\theta_i}\exp\left(\theta_i\left(\tfrac{1}{2}\{x' \quad \gamma'\}\begin{bmatrix} Q_i & A_i' \\ A_i & R_i\end{bmatrix}\begin{Bmatrix} x \\ \gamma\end{Bmatrix} + \{q_i' \quad r_i'\}\begin{Bmatrix} x \\ \gamma\end{Bmatrix}\right)\right)\middle| y_i\right].$$

A necessary and sufficient condition for the pair (γ_1^*, γ_2^*) to satisfy the (Nash) conditions (7) (with $\mathscr{M}_1$, $\mathscr{M}_2$ replaced by $\mathscr{A}_1$, $\mathscr{A}_2$) is that, for $i = 1, 2$, there exist matrices L_i and vectors μ_i satisfying

$$\gamma_i^* = L_i y_i + \mu_i,$$

$$\tilde{A}_{ij}\big[I - \theta_i S_i \tilde{Q}_{ij}\big]^{-1}\big(\mathrm{E}[x | y_i] + \theta_i S_i\big(\tilde{q}_{ij} + \tilde{A}_{ij}'\gamma_i^*\big)\big) + \tilde{R}_{ii}\gamma_i^* + \tilde{r}_{ij} = 0,$$

where, for $i, j \in (1,2)$ *and* $i \neq j$,

$$\tilde{Q}_{ij} = Q_i + C_j' L_j' R_{ijj} L_j C_j + A_{ij}' L_j C_j + C_j' L_j' A_{ij},$$

$$\tilde{A}_{ij} = A_{ii} + R_{iij} L_j C_j,$$

$$\tilde{R}_{ii} = R_{iii},$$

$$\tilde{q}_{ij} = q_i + C_j' L_j' (r_{ij} + R_{ijj} \mu_j) + A_{ij}' \mu_j,$$

$$\tilde{r}_{ij} = r_{ii} + R_{iij} \mu_j,$$

$$\mathrm{E}[x|y_i] = \Sigma C_i' [C_i \Sigma C_i']^{-1} y_i,$$

$$S_i = \mathrm{E}[xx'|y_i] = \Sigma - \Sigma C_i' [C_i \Sigma C_i']^{-1} C_i \Sigma,$$

with

$$A_i' = [A_{i1}' \quad A_{i2}'], \qquad A_{i1}' \sim n \times m_1, \qquad A_{i2}' \sim n \times m_2,$$

$$r_i' = \{r_{i1}' \quad r_{i2}'\}, \qquad r_{i1}' \sim 1 \times m_1, \qquad r_{i2}' \sim 1 \times m_2,$$

$$R_i = \begin{bmatrix} R_{i11} & R_{i12} \\ R_{i21} & R_{i22} \end{bmatrix}, \quad R_{ii1}' \sim m_i \times m_1, \quad R_{ii2}' \sim m_i \times m_2,$$

and the matrix $\Sigma^{-1} - \theta_i \tilde{Q}_{ij}$ *is positive definite.*

Proof. In terms of the definitions of the Theorem, the exponent for player one can be expressed as

$$\tfrac{1}{2}[x'Q_1 x + \gamma_1' R_{111} \gamma_1 + \gamma_2' R_{122} \gamma_2] + x' A_{11}' \gamma_1 + x' A_{12}' \gamma_2 + \gamma_1' R_{112} \gamma_2$$
$$+ q_1' x + r_{11}' \gamma_1 + r_{12}' \gamma_2,$$

and, for a fixed $\gamma_2(y_2) = L_2 C_2 x + \mu_2$, this is a quadratic function of x and γ_1

$$\tfrac{1}{2} x' [Q_1 + C_2' L_2' R_{122} L_2 C_2 + A_{12}' L_2 C_2 + C_2' L_2' A_{12}] x + \tfrac{1}{2} \gamma_1' R_{111} \gamma_1$$
$$+ \gamma_1' [A_{11} + R_{112} L_2 C_2] x + [q_1' + (r_{12}' + \mu_2' R_{122}) L_2 C_2 + \mu_2' A_{12}] x$$
$$+ [r_{11}' + \mu_2' R_{112}'] \gamma_1 + K_1$$
$$= \tfrac{1}{2} \{x' \quad \gamma_1'\} \begin{bmatrix} \tilde{Q}_{12} & \tilde{A}_{12}' \\ \tilde{A}_{12} & \tilde{R}_{11} \end{bmatrix} \begin{Bmatrix} x \\ \gamma_1 \end{Bmatrix} + \{\tilde{q}_{12}' \quad \tilde{r}_{12}'\} \begin{Bmatrix} x \\ \gamma_1 \end{Bmatrix} + K_1,$$

where K_1 is a scalar independent of x and γ_1. The problem faced by player 1 is to find a strategy γ_1 minimizing

$$J_{\theta_1}(\gamma_1) = \mathrm{E}\left[\frac{1}{\theta_1}\exp\left(\theta_1\left(\tfrac{1}{2}\{x' \quad \gamma_1'\}\begin{bmatrix}\tilde{Q}_{12} & \tilde{A}_{12}' \\ \tilde{A}_{12} & \tilde{R}_{11}\end{bmatrix}\begin{Bmatrix}x \\ \gamma_1\end{Bmatrix}\right.\right.\right.$$
$$\left.\left.\left.+\{\tilde{q}_{12}' \quad \tilde{r}_{12}'\}\begin{Bmatrix}x \\ \gamma_1\end{Bmatrix}+K_1\right)\right)\middle| y_1\right].$$

By the argument used in the proof of Theorem 1, the minimum γ_1^* is unique and satisfies (necessarily, in this case)

$$\mathrm{E}\left[\left(\tilde{A}_{12}x+\tilde{R}_{11}+\tilde{r}_{12}\right)\exp\left(\theta_1\left(\tfrac{1}{2}x'\tilde{Q}_{12}x+\left(\gamma_1'\tilde{A}_{12}+\tilde{q}_{12}'\right)x\right)\right)|C_1x\right]=0,$$

and similarly for γ_2^*, with 1 and 2 interchanged. By Lemma 5 (appendix), the result follows. QED

Before closing this section, we discuss in greater detail the difficulty encountered in Theorem 1 for the case $\theta_i > 0$. Observe that Theorem 1, for $\theta_i > 0$, does not offer a straightforward procedure to evaluate the solution. Indeed, it is not even clear in what class of functions the search for a solution is bound to have success. The difficulty here is linked to the fact that $\gamma' = \{\gamma_1' \quad \gamma_2'\}$ is a function of both y_1 and y_2 and, from the viewpoint of player 1, who forms his own expectations on the basis of y_1 only, γ is a random vector whose statistic is not known *a priori* since it depends on what strategy γ_2 his opponent will adopt. In principle, player 1 could wonder what Nash solutions are available to him when the class of strategies of the other player is totally unrestricted, and choose the best for him. But this, in practice, is a formidable task for $\theta_i > 0$, leading as it does to the serious mathematical difficulty which we now describe.

Consider the set $\mathscr{F}_i$ of m_i-valued measurable functions of y_i for which J_{θ_i} is finite. In the case $\theta_i = 0$, this is a well-defined set, indeed is the Hilbert space of square-integrable functions with respect to the Gaussian measure. The same applies when $\theta_i < 0$ since finiteness of J_0 implies that of J_{θ_i} for $\theta_i < 0$.

When $\theta_i > 0$, however, $\mathscr{F}_i$ is no longer a linear space and the definition of a derivative encounters the difficulty of characterizing feasible perturbations. Suppose $\theta_1 > 0$ and, for a particular choice of the parameters, γ_2^* has the form $\phi(x)$ so that, in order to find γ_1^*, one has to minimize with respect to γ_1

$$J_{\theta_1}(\gamma_1, \gamma_2^*) = \mathrm{E}\exp\left(\phi(x)^2 + 2\gamma_1\phi(x) + P_1\gamma_1^2\right),$$

where x is a normally distributed scalar random variable with zero mean and unit variance. Let γ_1^* be a point where J_{θ_1} attains a *finite* minimum. Consider now perturbations $\gamma_1^+ = \gamma_1^* + \varepsilon$ and $\gamma_1^- = \gamma_1^* - \varepsilon$ with $\varepsilon > 0$. If $\phi(x) \leq 0$, at $\gamma_1 = \gamma_1^+$ we have

$$\exp\big(\phi(x)^2 + 2(\gamma_1^* + \varepsilon)\phi(x)\big) \leq \exp\big(\phi(x)^2 + 2\gamma_1^*\phi(x)\big).$$

Since the expected value of the right-hand side is finite, so is that of the left-hand side, and it is easy to check that $J_{\theta_1}(\gamma_1^+, \gamma_2^*)$ exists. However, at $\gamma_1 = \gamma_1^-$ the inequality is violated, the expectation of the left-hand side may well have an infinite value, and $J_{\theta_1}(\gamma_1^-, \gamma_2^*)$ may fail to exist. Actually, if

$$\phi(x) = \begin{cases} -\sqrt{\frac{1}{2}x^2 - \delta \ln|x|} & \text{if} \quad x \leq -1, \\ 0 & \text{if} \quad -1 < x < 1, \\ +\sqrt{\frac{1}{2}x^2 - \delta \ln|x|} & \text{if} \quad x \geq +1, \end{cases}$$

we can compute the expectation appearing in J_{θ_1} and obtain

$$\mathrm{e}^{P_1\gamma_1^2}\left[\int_{-\infty}^{-1} \frac{\exp\left(-2\gamma_1\sqrt{\frac{1}{2}x^2 - \delta \ln|x|}\right)}{\sqrt{2\pi}\,|x|^\delta}\mathrm{d}x + \int_{-1}^{1} \frac{\mathrm{e}^{-1/2x^2}}{\sqrt{2\pi}}\mathrm{d}x \right.$$

$$\left. + \int_1^{+\infty} \frac{\exp\left(2\gamma_1\sqrt{\frac{1}{2}x^2 - \delta \ln|x|}\right)}{\sqrt{2\pi}\,|x|^\delta}\mathrm{d}x\right],$$

a quantity which is finite only for $\gamma_1 = 0$. We conclude that the minimum of J_{θ_1} is finite at γ_1^* but any perturbation of γ_1^* yields an infinite value of J_{θ_1}. For a similar difficulty in the context of team decision theory, see Radner (1962) and Speyer (1980).

Notice that the derivative with respect to γ_1 of the quantity *inside* the expectation is

$$[\phi(x) + 2P_1\gamma_1]\exp\big(\phi(x)^2 + 2\gamma_1\phi(x) + P_1\gamma_1^2\big),$$

and, at $\gamma_1 = \gamma_1^* = 0$, its expectation is certainly finite, a fact used in the proof of Theorem 1.

5. An example

As an example, let us return to the model of section 3 and complete the analysis by considering:

Case 4. Two players with exponential utilities
Assuming affine solutions

$$u_i^* = g_i \mathrm{E}[v|y_i] + \mu_i$$

and using the results of Theorem 3, we obtain

$$\begin{Bmatrix} g_1 \\ g_2 \end{Bmatrix} = -\begin{bmatrix} P_1 + k_1 d_1^2 - \theta_1 P_1 k_1 f^2 h_1(g_2) & k_1 d_1 d_2 F(\sigma_2, f) \\ k_2 d_2 d_1 F(\sigma_1, f) & P_2 + k_2 d_2^2 - \theta_2 P_2 k_2 f^2 h_2(g_1) \end{bmatrix}^{-1} \begin{Bmatrix} k_1 d_1 \\ k_2 d_2 \end{Bmatrix},$$

$$\begin{Bmatrix} \mu_1 \\ \mu_2 \end{Bmatrix} = \begin{bmatrix} P_1 + k_2 d_1^2 - \theta_1 P_1 k_1 f^2 h_1(g_2) & k_1 d_1 d_2 \\ k_2 d_2 d_1 & P_2 + k_2 d_2^2 - \theta_2 P_2 k_2 f^2 h_2(g_1) \end{bmatrix}^{-1} \begin{Bmatrix} k_1 d_1 \alpha_1 \\ k_2 d_2 \alpha_2 \end{Bmatrix},$$

with

$$h_1(g_2) = (1 + d_2 g_2 F(\sigma_2, f))^2 (1 - F(\sigma_1, f)) + d_2^2 g_2^2 F(\sigma_2, f)(1 - F(\sigma_2, f)),$$

$$h_2(g_1) = (1 + d_1 g_1 F(\sigma_1, f))^2 (1 - F(\sigma_2, f)) + d_1^2 g_1^2 F(\sigma_1, f)(1 - F(\sigma_1, f)),$$

$$\mathrm{E}[v|y_1] = F(\sigma_1, f) y_1, \qquad \mathrm{E}[v|y_2] = F(\sigma_2, f) y_2,$$

and $F(\sigma, f)$ is the function $(\sigma, f) \to 1/(1 + (\sigma/f)^2)$.

The solution is derived in detail in section A.2 (appendix). In particular, we have

$$g_1 = -\frac{k_1 d_1 (1 + d_2 g_2 F(\sigma_2, f))}{P_1 + k_1 d_1^2 - \theta_1 P_1 k_1 f^2 h_1(g_2)},$$
$$\mu_1 = -\frac{d_2 \mu_2 - a_1}{1 + d_2 g_2 F(\sigma_2, f)} g_1, \tag{9}$$

provided (see remark in section A.2 of the appendix)

$$1 - \theta_1 k_1 f^2 h_1(g_2) > 0, \tag{10}$$

and similarly for g_2, μ_2 with the indices 1 and 2 interchanged. Notice that, for

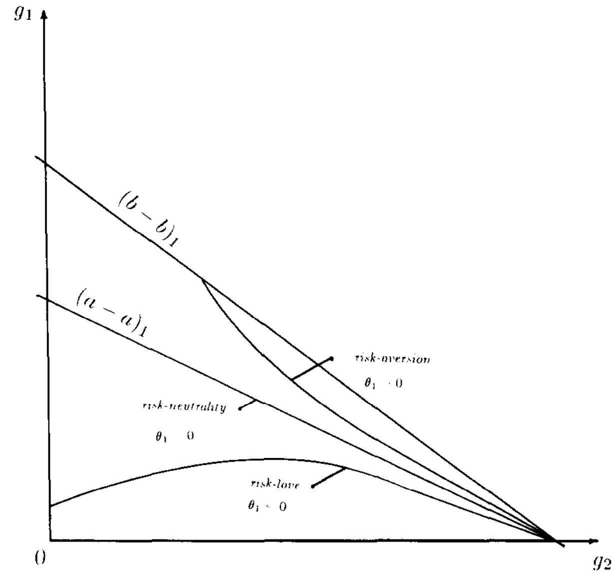

Fig. 2. Reaction curve of player 1.

$\theta_i = 0$, the solutions become those of case 1 in section 3, while for $\theta_1 \neq 0$ and $\theta_2 = d_2 = 0$ we get case 2. We shall now concentrate on the relationship between the reaction multipliers g_1 and g_2.

Assume, with no loss of generality, that $d_1 < 0$, $d_2 < 0$ and consider the plane $(g_1 - g_2)$. For $\theta_1 = 0$, eq. (9) is the straight line $(a-a)_1$ in fig. 2. For $\theta_1 < 0$ condition (10) always holds, (9) lies between $(a-a)_1$ and the g_2 axis, and asymptotically goes to zero as $g_2 \to \pm\infty$. It also has a unique minimum and a unique maximum. The closer θ_1 is to zero, the closer is this curve to $(a-a)_1$, above it for $\theta_1 > 0$, below it for $\theta_1 < 0$. In addition, by (10) the admissible part of this curve is up to the point where $1 = \theta_1 k_1 f^2 h_1(g_2)$. Where this equality holds, eq. (9) yields

$$g_1 = -\big(1 + d_2 g_2 F(\sigma_2, f)\big)/d_1,$$

which is line $(b-b)_1$. For $\theta_1 > 0$, (9) goes to $+\infty$ when the denominator becomes zero, so it will hit line $(b-b)_1$. As $\theta_1 > 0$ increases, curve (9) intersects $(b-b)_1$ closer to the right. Moreover, all the mentioned curves and lines mutually intersect at the point $g_1 = 0$, $g_2 = -1/d_2 F(\sigma_2, f)$. A similar diagram can be drawn in this plane for g_2 as a function of g_1.

Suppose both players are risk-loving ($\theta_i < 0$). Condition (10), and the corresponding condition for player 2 with 1 and 2 interchanged, are always

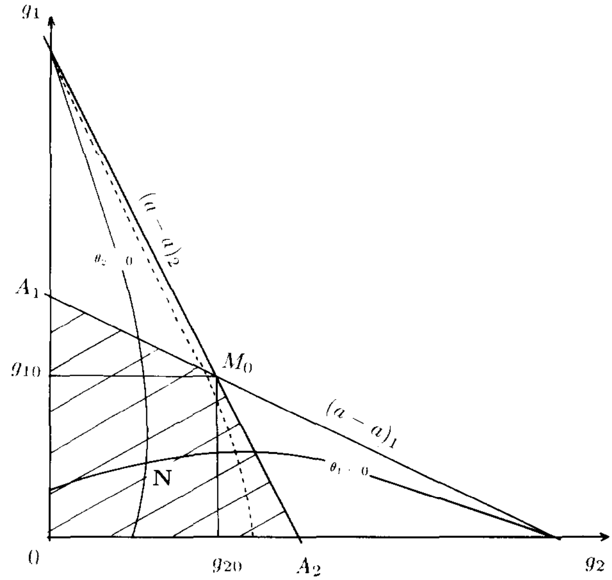

Fig. 3. Reaction curves when both players are risk-loving ($\theta_1, \theta_2 > 0$). The shaded area: the set of Nash equilibrium points.

satisfied and there is always a Nash equilibrium point N determined by the intersection of the reaction curves inside $0A_1M_0A_2$ in fig. 3. In particular, $N = M_0$ in the risk-neutral case ($\theta_i = 0$), and $N \to 0$ when both players are *very* risk-loving ($\theta_i \to -\infty$).

Players reactions tend to become weaker as their attitudes become more daring. They feel nature, although unpredictable, will ultimately act in their favour. This is true when their attitudes towards risk are equally daring, better, when point N is in $0g_{10}M_0g_{20}$.

But if one is much less daring than the other, he is likely to have a stronger reaction than in the risk-neutral case, that is, point N is in $M_0g_{10}A_1$ or in $M_0g_{20}A_2$. For instance, the intersection of the dotted reaction curve of player 2 with the reaction curve of player 1 inside $M_0g_{20}A_2$ indicates that player 2 is much less daring than player 1.

When both players are risk-averse ($\theta_i > 0$) curve (9) can have one of the three shapes $(\alpha - \alpha_1)$, $(\alpha - \alpha_2)$, or $(\alpha - \alpha_3)$ shown in fig. 4. The reaction curve of player 2 is $(\beta - d)$ and, for θ_1, θ_2 sufficiently close to zero, the curves intersect. When one of the players' risk aversion, say θ_1, increases, a Nash equilibrium may fail to exist, as in the cases $(\alpha - \alpha_2)$ and $(\alpha - \alpha_3)$.

The point $\overline{M}$ is the intersection of the lines defined by the boundary of the admissible strategy sets, determined (for $\theta_i > 0$) by inequality (10) and by a

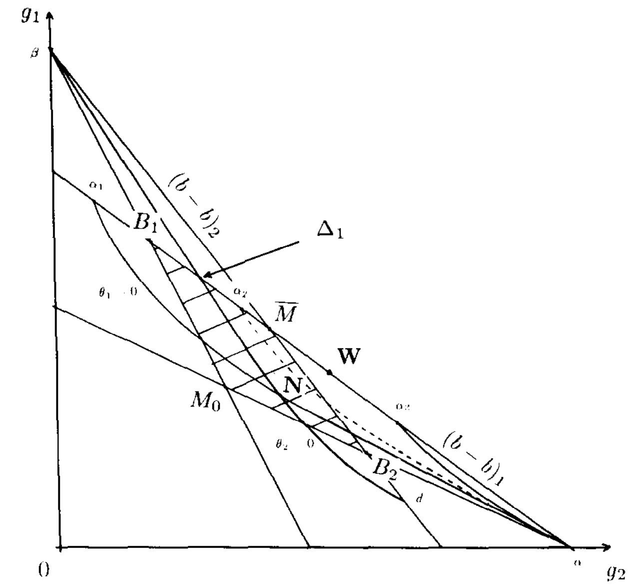

Fig. 4. Reaction curves when both players are risk-averse ($\theta_1, \theta_2 < 0$). The shaded area: the set of Nash equilibrium points.

corresponding inequality for player 2 (obtained by interchanging indices 1 and 2). Thus intersections are only possible in the area $M_0B_2\bar{M}B_1$ which determines the set of Nash equilibria.

A solution never exists as long as θ_1 is above the value at which (9) hits $(b-b)_2$ at $\bar{M}$ (for instance, see point W) and/or θ_2 is above the value at which (9) hits $(b-b)_2$ at $\bar{M}$. On the other hand, a solution always exists, as long as θ_1 is below the value at which (9) hits $(b-b)_1$ at point B_1 and, at the same time, θ_2 is below the value at which (9) hits $(b-b)_2$ at point B_2. An example where the above conditions are not satisfied is given by curve $(\alpha-\alpha_2)$.

Generally, risk aversion makes reactions stronger than risk neutrality, but again only when attitudes are similar. Otherwise, the less cautious has a weaker reaction.

When one player is risk-averse ($\theta_1 > 0$) and the other risk-loving ($\theta_2 < 0$), possible Nash equilibria are inside $M_0B_1C_1A_1$ in fig. 5 (or inside $M_0B_2C_2A_2$, if $\theta_1 < 0$, $\theta_2 > 0$). A solution always exists, as long as θ_1 is below the value at which (9) hits $(b-b)_1$ at C_1, and never exists if θ_1 is above the value at which (9) hits $(b-b)_1$ at B_1 (for instance, see point W). Compared to the case of risk neutrality, reaction should be tightened for the cautious player and weakened for the daring.

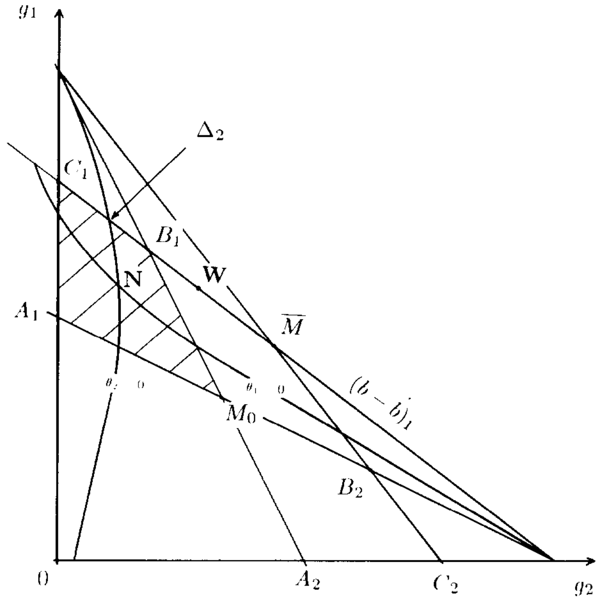

Fig. 5. Reaction curves when player 1 is risk-averse, player 2 is risk-loving. The shaded area: the set of Nash equilibrium points.

From the discussion above, it is apparent that Nash equilibria are possible only if the risk-aversion parameters stand in a certain relationship. In a $(\theta_1 - \theta_2)$ plane, Nash regions are easy to find. To do this, identify four points:

θ_{11}, the value at which curve (9) hits $(b-b)_1$ at point C_1 (fig. 5)
By inspection of fig. 5, a solution always exists for $\theta_1 \le \theta_{11}$, and similarly with 1 replaced by 2. This point is obtained by solving

$$g_2 = 0,$$

$$g_1 = -1/d_1,$$

$$1 = \theta_1 k_1 f^2 h_1(g_2),$$

from which

$$\theta_{11} = \frac{1}{k_1 f^2 (1 - F(\sigma_1, f))}.$$

Notice that θ_{11} is the limit value of θ_1 in eq. (8), that is, the highest degree of risk aversion allowed to player 1 acting alone.

θ_{12}, the value at which curve (9) hits $(b-b)_1$ at point B_1 (fig. 5)
No solutions exist if the opponent has a negative θ, i.e., if he is risk-loving. Solutions can exist if the opponent has a nonnegative θ. This point is obtained by solving

$$g_1 = -(1 + d_2 g_2 F(\sigma_2, f))/d_1,$$

$$g_2 = -\frac{k_2 d_2 (1 + d_1 g_1 F(\sigma_1, f))}{P_2 + k_2 d_2^2},$$

$$1 = \theta_1 k_1 f^2 h_1(g_2).$$

θ_{13}, the value at which curve (9) hits $(b-b)_1$ at $\overline{M}$ (fig. 5)
No solutions exist for $\theta_1 \geq \theta_{13}$, and similarly with 1 replaced by 2. This point is obtained by solving

$$g_1 = -(1 + d_2 g_2 F(\sigma_2, f))/d_1,$$

$$g_2 = -(1 + d_1 g_1 F(\sigma_1, f))/d_2,$$

$$1 = \theta_1 k_1 f^2 h_1(g_2).$$

θ_{14}, the value at which curve (9) hits $(b-b)_1$ at Δ_1 (fig. 4) or Δ_2 (fig. 5)
This characterizes the limit solutions obtaining when θ_2 is inside the feasible region and θ_1 at the border of it, that is, when N lies on $(b-b)_1$. This is obtained by solving

$$g_1 = -(1 + d_2 g_2 F(\sigma_2, f))/d_1,$$

$$g_2 = -\frac{k_2 d_2 (1 + d_1 g_1 F(\sigma_1, f))}{P_2 + k_2 d_2^2 - \theta_2 P_2 k_2 f^2 h_2(g_1)},$$

$$1 = \theta_1 k_1 f^2 h_1(g_2).$$

Obviously, for $\theta_2 = 0$, $\Delta_1 \to B_1$, $\Delta_2 \to B_1$; and $\theta_{14} \to \theta_{12}$.

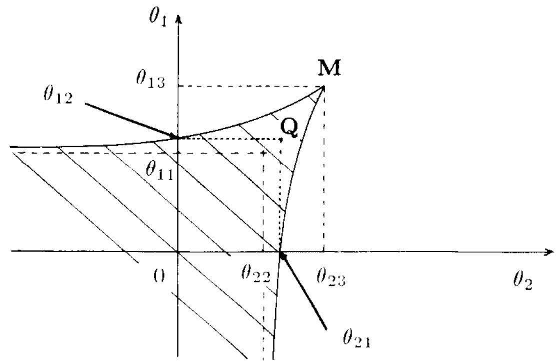

Fig. 6. Nash equilibria in the risk-parameter plane (shaded area). Imitative equilibria inside $\theta_{21}M\theta_{12}Q$; M = critical point of mistrust.

We end up with the diagram of fig. 6, where the shaded area contains all pairs (θ_1, θ_2) for which Nash solutions exist. The curved part in fig. 6 is found by eliminating g_1, g_2 from the equations of θ_{14}.

Notice that if a player acting alone, as in case 2, eq. (8), has a maximum θ, say θ_{11} for player 1, in the game set up he can have a θ_1 greater than θ_{11}, provided the other player has a θ_2 sufficiently close to zero.

Notice also that if one of the players, say player 1, is risk-averse beyond threshold θ_{12}, then there is no way to reach a Nash equilibrium by compensation, that is by a risk-loving attitude of the other. For θ_1 between θ_{12} and θ_{13} caution of player 1 must induce caution in player 2 before a Nash equilibrium can be reached. We call these *imitative equilibria* (see fig. 6).

This sets a clear asymmetry with respect to the risk-loving case, where risk-loving behavior of one player can always be matched, if one wishes, by a (moderately) cautious behavior of the other in a Nash equilibrium.

However, equilibria are only possible up to point M in fig. 6. Overcautiousness beyond this point makes it impossible to reach a Nash equilibrium. We call this point a *critical point of mistrust*.

Finally notice that, for $f^2 \to 0$, the reaction multipliers become those of the de-coupled single-objective quadratic-cost case and the shaded area of fig. 6 becomes the whole plane.

6. Conclusions

Generalizing results known for linear-quadratic games, we have shown connections between Nash equilibria and risk-aversion parameters when two players with exponential utilities engage in a noncooperative game with incomplete information. We have shown that when the disturbances are

Gaussian, solutions affine in the information exist, provided the Arrow–Pratt parameters satisfy a condition essentially prohibiting excessive risk aversion. Unlike the linear-quadratic case, players' optimal reactions are sensitive to proportional changes in the noise covariances. The case in which players have a single instrument and a single target has been discussed in detail. The exercise shows that in a Nash equilibrium the relationship between a player's reaction to the perceived state of nature and his own risk attitude is rather complex. Generally, risk aversion makes reactions stronger than risk neutrality (or risk love) but only when attitudes are similar. When both players are risk-averse but by very different degrees, the less cautious of the two can have a *weaker* reaction than in the risk-neutral case.

The existence of a Nash equilibrium is conditional upon risk parameters. When both players are risk-loving or neutral, a Nash equilibrium always exists. When one of the players is risk-averse beyond a computable threshold, a Nash equilibrium only exists if the other player is *also* risk-averse. A sort of imitative equilibrium is thereby established. There are cases in which the reaction curves of the players fail to intersect. This occurs when one player's risk aversion far exceeds that of the other. Hence we have reasons to believe that excessive risk aversion can be a factor of instability in the resolution of a conflict.

Appendix

This appendix comprises two sections. Section A.1 contains analytical results which are necessary to the proofs of Theorems 1, 2, and 3. Section A.2 develops the algebra needed to solve case 4 in section 5.

A.1. Analytical results

To make the exposition self-contained, the results are presented as a chain of lemmata. We make repeated use of the estimation formulas

$$\mathrm{E}[x|y] = \Sigma C'[C\Sigma C']^{-1}y,$$

$$\mathrm{E}[xx'|y] = \Sigma - \Sigma C'[C\Sigma C']^{-1}C\Sigma,$$

valid for $x \sim \mathrm{N}(0, \Sigma)$ and $y = Cx$, where C is a full-rank matrix.

Lemma 1. Let $x \in R^n$ be $\mathrm{N}(0, \Sigma)$ and Q a matrix such that $\Sigma^{-1} - Q$ is positive definite. Then

$$J = \mathrm{E}\exp\left(\tfrac{1}{2}x'Qx + q'x\right) = \sqrt{\frac{|\Gamma|}{|\Sigma|}}\exp\left(\tfrac{1}{2}q'\Gamma q\right),$$

where $\Gamma = [\Sigma^{-1} - Q]^{-1}$.

Proof.

$$J=\int_{-\infty}^{+\infty}\frac{\exp\left(\frac{1}{2}x'Qx+q'x-\frac{1}{2}x'\Sigma^{-1}x\right)}{\sqrt{(2\pi)^n|\Sigma|}}\mathrm{d}x$$

$$=\int_{-\infty}^{+\infty}\frac{\exp\left(-\frac{1}{2}(x-\bar{x})'\Gamma^{-1}(x-\bar{x})+M\right)}{\sqrt{(2\pi)^n|\Sigma|}}\mathrm{d}x$$

$$=\sqrt{\frac{|\Gamma|}{|\Sigma|}}\,\mathrm{e}^{M}\int_{-\infty}^{+\infty}\frac{\exp\left(-\frac{1}{2}(x-\bar{x})'\Gamma^{-1}(x-\bar{x})\right)}{\sqrt{(2\pi)^n|\Gamma|}}\mathrm{d}x,$$

where Γ, $\bar{x}$, and M, by equating coefficients of equal powers of x in the exponent, are given by

$$\Gamma^{-1}=\Sigma^{-1}-Q,\qquad \bar{x}=\Gamma q,\qquad M=\tfrac{1}{2}q'\Gamma q.$$

Since Γ is positive definite, the integrand on the right-hand side is a probability density function and the result follows. QED

Lemma 2. Let $x\in R^n$ *be* $\mathrm{N}(0,\Sigma)$ *and* Q *a matrix such that* $\Sigma^{-1}-Q$ *is positive definite. Then*

$$J=\mathrm{E}x\exp\left(\tfrac{1}{2}x'Qx+q'x\right)=\sqrt{\frac{|\Gamma|}{|\Sigma|}}\exp\left(\tfrac{1}{2}q'\Gamma q\right)\Gamma q,$$

where $\Gamma=[\Sigma^{-1}-Q]^{-1}$.

Proof.

$$J=\int_{-\infty}^{+\infty}\frac{x\exp\left(\frac{1}{2}x'Qx+q'x-\frac{1}{2}x'\Sigma^{-1}x\right)}{\sqrt{(2\pi)^n|\Sigma|}}\mathrm{d}x$$

$$=\int_{-\infty}^{+\infty}\frac{x\exp\left(-\frac{1}{2}(x-\bar{x})'\Gamma^{-1}(x-\bar{x})+M\right)}{\sqrt{(2\pi)^n|\Sigma|}}\mathrm{d}x$$

$$=\sqrt{\frac{|\Gamma|}{|\Sigma|}}\,\mathrm{e}^{M}\int_{-\infty}^{+\infty}\frac{x\exp\left(-\frac{1}{2}(x-\bar{x})'\Gamma^{-1}(x-\bar{x})\right)}{\sqrt{(2\pi)^n|\Gamma|}}\mathrm{d}x,$$

where, as in Lemma 1,

$$\Gamma^{-1} = \Sigma^{-1} - Q, \qquad \bar{x} = \Gamma q, \qquad M = \tfrac{1}{2} q' \Gamma q.$$

Since Γ is positive definite, the integral on the right-hand side is $\bar{x}$ and the result follows. QED

Lemma 3. Let $x_2 \in R^n$ be $\mathrm{N} \sim (0, \Sigma_{22})$ *and let $x' = \{x_1' \quad x_2'\}$. Let*

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}, \qquad \dim(Q_{22}) = n \times n, \qquad Q_{12} = Q_{21}',$$

$$q' = \{q_1' \quad q_2'\}, \qquad \dim(q_2) = n.$$

Assume $\Sigma_{22}^{-1} - Q_{22}$ is positive definite. Then if x_1 is independent of x_2,

$$J = \mathrm{E}\left[\exp\left(\tfrac{1}{2} x' Q x + q' x\right) | x_1\right] = \sqrt{\frac{|\Gamma_{22}|}{|\Sigma_{22}|}} \exp\left(\tfrac{1}{2} \bar{q}' \Gamma \bar{q}\right),$$

where the expectation is with respect to x_2 and

$$\bar{q}' = \{x_1' \quad q_1' \quad q_2'\},$$

$$\Gamma = \begin{bmatrix} \Gamma_{11} & I & \Gamma_{12} \\ I & 0 & 0 \\ \Gamma_{21} & 0 & \Gamma_{22} \end{bmatrix},$$

with

$$\Gamma_{22} = \left[\Sigma_{22}^{-1} - Q_{22}\right]^{-1},$$

$$\Gamma_{11} = Q_{11} + Q_{12} \Gamma_{22} Q_{21},$$

$$\Gamma_{12} = Q_{12} \Gamma_{22} = \Gamma_{21}'.$$

Proof.

$$J = \mathrm{E}\left[\exp\left(\tfrac{1}{2}\{x_1' \quad x_2'\} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \{q_1' \quad q_2'\} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}\right) \Bigg| x_1\right]$$

$$= \exp\left(\tfrac{1}{2} x_1' Q_{11} x_1 + q_1' x_1\right) \mathrm{E}\left[\exp\left(\tfrac{1}{2} x_2' Q_{22} x_2 + (x_1' Q_{12} + q_2') x_2\right) | x_1\right].$$

Conditioning can be ignored since x_1 and x_2 are independent. By Lemma 1,

$$J = \sqrt{\frac{|\Gamma_{22}|}{|\Sigma_{22}|}} \exp\left(\tfrac{1}{2}x_1'Q_{11}x_1 + q_1'x_1 + \tfrac{1}{2}(x_1'Q_{12} + q_2')\Gamma_{22}(Q_{21}x_1 + q_2)\right),$$

and the result follows. QED

Lemma 4. Let $x_2 \in R^n$ *be* $\mathrm{N} \sim (0, \Sigma_{22})$ *and let* $x' = \{x_1' \quad x_2'\}$. *Let*

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}, \qquad \dim(Q_{22}) = n \times n, \qquad Q_{12} = Q_{21}',$$

$$q' = \{q_1' \quad q_2'\}, \qquad \dim(q_2) = n.$$

Assume $\Sigma_{22}^{-1} - Q_{22}$ *is positive definite. Then if* x_1 *is independent of* x_2,

$$J = \mathrm{E}\left[x_2 \exp\left(\tfrac{1}{2}x'Qx + q'x\right)|x_1\right]$$

$$= \sqrt{\frac{|\Gamma_{22}|}{|\Sigma_{22}|}} \exp\left(\tfrac{1}{2}\bar{q}'\Gamma\bar{q}\right)(\Gamma_{21}x_1 + \Gamma_{22}q_2),$$

where the expectation is with respect to x_2 *and*

$$\bar{q}' = \{x_1' \quad q_1' \quad q_2'\},$$

$$\Gamma = \begin{bmatrix} \Gamma_{11} & I & \Gamma_{12} \\ I & 0 & 0 \\ \Gamma_{21} & 0 & \Gamma_{22} \end{bmatrix},$$

with

$$\Gamma_{22} = \left[\Sigma_{22}^{-1} - Q_{22}\right]^{-1},$$

$$\Gamma_{11} = Q_{11} + Q_{12}\Gamma_{22}Q_{21},$$

$$\Gamma_{12} = Q_{12}\Gamma_{22} = \Gamma_{21}'.$$

Proof.

$$J = \mathrm{E}\left[x_2 \exp\left(\tfrac{1}{2}\{x_1' \quad x_2'\}\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22}\end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \{q_1' \quad q_2'\}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}\right)\middle| x_1\right]$$

$$= \exp\left(\tfrac{1}{2}x_1'Q_{11}x_1 + q_1'x_1\right)\mathrm{E}\left[x_2 \exp\left(\tfrac{1}{2}x_2'Q_{22}x_2 + (x_1'Q_{12} + q_2')x_2\right)|x_1\right].$$

Conditioning can be ignored because x_1 and x_2 are independent. By Lemma 2,

$$J = \sqrt{\frac{|\Gamma_{22}|}{|\Sigma_{22}|}}\exp\left(\tfrac{1}{2}x_1'Q_{11}x_1 + q_1'x_1 + \tfrac{1}{2}(x_1'Q_{12} + q_2')\Gamma_{22}(Q_{21}x_1 + q_2)\right)$$

$$\times \Gamma_{22}(Q_{21}x_1 + q_2),$$

and the result follows. QED

Lemma 5. Let $x \in R^n$ be N$(0, \Sigma)$ *and $y \in R^p$ be a vector satisfying $y = Cx$ with* rank$(C) = p \leq n$. *Let Q be a matrix such that $\Sigma^{-1} - Q$ is positive definite. Then*

$$J = \mathrm{E}\left[(Ax + d)\exp\left(\tfrac{1}{2}x'Qx + q'x\right)|Cx\right] = 0,$$

if and only if

$$A[I - SQ]^{-1}(\mathrm{E}[x|y] + Sq) + d = 0,$$

where

$$S = \Sigma - \Sigma C'[C\Sigma C']^{-1}C\Sigma = \mathrm{E}[xx'|y].$$

Proof. Let $\xi = \Sigma^{-\frac{1}{2}}x$ so $\xi \sim$ N$(0, I)$. In terms of ξ,

$$J = \mathrm{E}\left[\left(A\Sigma^{\frac{1}{2}}\xi + d\right)\exp\left(\tfrac{1}{2}\xi'\Sigma^{\frac{1}{2}}Q\Sigma^{\frac{1}{2}}\xi + q'\Sigma^{\frac{1}{2}}\xi\right)|C\Sigma^{\frac{1}{2}}\xi\right].$$

Decompose the $p \times n$ matrix $C\Sigma^{\frac{1}{2}}$ in singular values. Since $p \leq n$,

$$C\Sigma^{\frac{1}{2}} = U[\Lambda \quad 0]V',$$

where U and V are $p \times p$ and $n \times n$ unitary matrices whose columns are eigenvectors of $C\Sigma C'$ and $\Sigma^{\frac{1}{2}}C'C\Sigma^{\frac{1}{2}}$. Λ is a diagonal matrix containing the p

singular values of $C\Sigma^{\frac{1}{2}}$. Since this matrix has rank p, its singular values are nonzero.

Consider the partition $V = [V_1 \quad V_2]$ with $V_1 \sim n \times p$.

$$y = C\Sigma^{\frac{1}{2}}\xi = U[\Lambda \quad 0]\begin{bmatrix} V_1' \\ V_2' \end{bmatrix}\xi = U\Lambda V_1'\xi. \tag{A.1}$$

Since $U\Lambda$ is invertible, conditioning on Cx is the same as conditioning on $V_1'\xi$. Using the identity $I = VV'$, we have

$$J = \mathrm{E}\left[\left(A\Sigma^{\frac{1}{2}}VV'\xi + d\right)\exp\left(\tfrac{1}{2}\xi'VV'\Sigma^{\frac{1}{2}}Q\Sigma^{\frac{1}{2}}VV'\xi + q'\Sigma^{\frac{1}{2}}VV'\xi\right)|V_1'\xi\right].$$

Let now

$$z_i = V_i'\xi, \quad r_i = V_i'\Sigma^{\frac{1}{2}}q, \quad A_i = A\Sigma^{\frac{1}{2}}V_i, \quad R_{ij} = V_i'\Sigma^{\frac{1}{2}}Q\Sigma^{\frac{1}{2}}V_j,$$

$$z' = \{z_1' \quad z_2'\}, \qquad r' = \{r_1 \quad r_2\}.$$

Then

$$J = \mathrm{E}\left[\left(A_1z_1 + A_2z_2 + d\right)\exp\left(\tfrac{1}{2}z'Rz + r'z\right)|z_1\right] = J_1 + J_2,$$

where, by Lemma 3,

$$J_1 = \left(A_1z_1 + d\right)\mathrm{E}\left[\exp\left(\tfrac{1}{2}z'Rz + r'z\right)|z_1\right]$$

$$= \left(A_1z_1 + d\right)\sqrt{|\Gamma_{22}|}\exp\left(\tfrac{1}{2}\bar{r}'\Gamma\bar{r}\right),$$

and, by Lemma 4,

$$J_2 = A_2\mathrm{E}\left[z_2\exp\left(\tfrac{1}{2}z'Rz + r'z\right)|z_1\right]$$

$$= A_2\sqrt{|\Gamma_{22}|}\exp\left(\tfrac{1}{2}\bar{r}'\Gamma\bar{r}\right)\left(\Gamma_{21}z_1 + \Gamma_{22}r_2\right),$$

where

$$\bar{r}' = \{z_1' \quad r_1' \quad r_2'\},$$

$$\Gamma = \begin{bmatrix} \Gamma_{11} & I & \Gamma_{12} \\ I & 0 & 0 \\ \Gamma_{21} & 0 & \Gamma_{22} \end{bmatrix},$$

with

$$\Gamma_{22} = [I - R_{22}]^{-1},$$

$$\Gamma_{11} = R_{11} + R_{12}\Gamma_{22}R_{21},$$

$$\Gamma_{12} = R_{12}\Gamma_{22} = \Gamma_{21}'.$$

Since Γ_{22} is nonsingular, the condition

$$J = \sqrt{|\Gamma_{22}|}\exp\left(\tfrac{1}{2}\bar{r}'\Gamma\bar{r}\right)\left[(A_1 + A_2\Gamma_{21})z_1 + A_2\Gamma_{22}r_2 + d\right] = 0$$

holds if and only if the quantity in square bracket is zero, or

$$A\Sigma^{\frac{1}{2}}\left[\left(I + V_2\Gamma_{22}V_2'\Sigma^{\frac{1}{2}}Q\Sigma^{\frac{1}{2}}\right)V_1V_1'\xi + V_2\Gamma_{22}V_2'\Sigma^{\frac{1}{2}}q\right] + d = 0. \tag{A.2}$$

Now

$$V_2\Gamma_{22}V_2' = V_2\left[I - V_2'\Sigma^{\frac{1}{2}}Q\Sigma^{\frac{1}{2}}V_2\right]^{-1}V_2' = \left[I - V_2V_2'\Sigma^{\frac{1}{2}}Q\Sigma^{\frac{1}{2}}\right]^{-1}V_2V_2',$$

where the last step follows from an easily verifiable matrix identity and the fact that $I - AB$ is invertible if and only if $I - BA$ is. Furthermore, we have the identity

$$I + \left[I - V_2V_2'\Sigma^{\frac{1}{2}}Q\Sigma^{\frac{1}{2}}\right]^{-1}V_2V_2'\Sigma^{\frac{1}{2}}Q\Sigma^{\frac{1}{2}} = \left[I - V_2V_2'\Sigma^{\frac{1}{2}}Q\Sigma^{\frac{1}{2}}\right]^{-1},$$

therefore (A.2) becomes

$$A\Sigma^{\frac{1}{2}}\left[I - V_2V_2'\Sigma^{\frac{1}{2}}Q\Sigma^{\frac{1}{2}}\right]^{-1}\left(V_1V_1'\xi + V_2V_2'\Sigma^{\frac{1}{2}}q\right) + d = 0. \tag{A.3}$$

Now, from $y = U\Lambda V_1'\xi$ and $\xi \sim \mathrm{N}(0, I)$ it follows

$$V_1V_1'\xi = \mathrm{E}[\xi|y] = \Sigma^{-\frac{1}{2}}\mathrm{E}[x|y]. \tag{A.4}$$

Also

$$V_2V_2' = I - V_1V_1' = I - \Sigma^{\frac{1}{2}}C'[C\Sigma C']^{-1}C\Sigma^{\frac{1}{2}}. \tag{A.5}$$

Using (A.4) and (A.5) in (A.3) the result follows. QED

Notice that the assumption of positive definiteness in Lemmata 1–5 is crucial. It is not difficult to see that when this assumption is violated, the integrals involved in the expectations do not have a finite sum.

A.2. Solution of case 4 in section 5

$$s = d_1 u_1 + d_2 u_2 + v,$$

$$y_1 = v + w_1,$$

$$y_2 = v + w_2,$$

$$\mathscr{L}_1 = \tfrac{1}{2} k_1 (s - a_1)^2 + \tfrac{1}{2} P_1 u_1^2$$

$$J_1 = \mathrm{E}\left[\frac{1}{\theta_1} \exp(\theta_1 \mathscr{L}_1) | y_1 \right] \to \min.$$

Assume player 2 adopts an affine strategy,

$$u_2 = L_2 y_2 + \mu_2 = g_2 \mathrm{E}[v | y_2] + \mu_2 = g_2 \frac{f^2}{f^2 + \sigma_2^2} y_2 + \mu_2 = g_2 F_2 y_2 + \mu_2.$$

Then,

$$\begin{aligned} \mathscr{L}_1 &= \tfrac{1}{2} k_1 (d_1 u_1 + d_2 L_2 y_2 + d_2 \mu_2 + v - a_1)^2 + \tfrac{1}{2} P_1 u_1^2 \\ &= \tfrac{1}{2} k_1 (d_1 u_1 + (1 + d_2 L_2) v + d_2 L_2 w_2 + d_2 \mu_2 - a_1)^2 + \tfrac{1}{2} P_1 u_1^2. \end{aligned}$$

Let

$$\alpha = 1 + d_2 L_2, \qquad \beta = d_2 L_2,$$

and

$$x_1 = v + w_1 = y_1,$$

$$x_2 = \alpha v + \beta w_2,$$

$$x' = \{ x_1 \quad x_2 \},$$

$$h_{12} = d_2 \mu_2 - a_1.$$

Player 1 seeks u_1 that minimizes

$$\mathrm{E}\Big[\frac{1}{\theta_1} \exp\big(\theta_1 \big(\tfrac{1}{2} (k_1 d_1^2 + P_1) u_1^2 + \tfrac{1}{2} k_1 x_2^2 + \tfrac{1}{2} k_1 h_{12}^2$$

$$+ k_1 d_1 x_2 u_1 + k_1 d_1 h_{12} u_1 + k_1 x_2 h_{12} \big) \big) | y_1 \Big].$$

The first-order condition is

$$\mathrm{E}\left[\left([0 \quad k_1 d_1]x + \left(k_1 d_1^2 + P_1\right)u_1 + k_1 d_1 h_{12}\right)\right.$$

$$\left.\times \exp\left(\theta_1\left(\tfrac{1}{2}x'\begin{bmatrix} 0 & 0 \\ 0 & k_1 \end{bmatrix}x + \{0 \quad k_1 d_1 u_1 + k_1 h_{12}\}x\right)\right) \middle| [1 \quad 0]x\right] = 0.$$

By letting

$$A = [0 \quad k_1 d_1],$$

$$d = \left(k_1 d_1^2 + P_1\right)u_1 + k_1 d_1 h_{12},$$

$$Q = \theta_1\begin{bmatrix} 0 & 0 \\ 0 & k_1 \end{bmatrix},$$

$$q' = \theta_1\{0 \quad k_1 d_1 u_1 + k_1 h_{12}\},$$

$$C = [1 \quad 0],$$

this can be written

$$\mathrm{E}\left[(Ax + d)\exp\left(\tfrac{1}{2}x'Qx + q'x\right)|Cx\right] = 0.$$

By Lemma 5,

$$A[I - SQ]^{-1}(\mathrm{E}[x|Cx] + Sq) + d = 0,$$

where

$$S = \mathrm{E}[xx'|y_1] = \Sigma - \Sigma C'[C\Sigma C']^{-1}C\Sigma,$$

$$\Sigma = \begin{bmatrix} f^2 + \sigma_1^2 & \alpha f^2 \\ \alpha f^2 & \alpha^2 f^2 + \beta^2\sigma_2^2 \end{bmatrix}.$$

A brief computation yields

$$[I - SQ]^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{1}{1 - \Theta_1 k_1} \end{bmatrix}, \tag{A.6}$$

where

$$\Theta_1 = \theta_1\left(\alpha^2 f^2(1 - F_1) + \beta^2\sigma_2^2\right),$$

$$A[I - SQ]^{-1} = \{0 \quad k_1 d_1\}[I - SQ]^{-1} = \left\{0 \quad \frac{k_1 d_1}{1 - \Theta_1 k_1}\right\},$$

$$\mathrm{E}[x|Cx] = \Sigma C'[C\Sigma C']^{-1} y_1 = \begin{Bmatrix} f^2 + \sigma_1^2 \\ \alpha f^2 \end{Bmatrix} \frac{1}{f^2 + \sigma_1^2} y_1 = \begin{Bmatrix} 1 \\ \alpha F_1 \end{Bmatrix} y_1,$$

$$Sq = \begin{bmatrix} 0 & 0 \\ 0 & \Theta_1/\theta_1 \end{bmatrix} \begin{Bmatrix} 0 \\ \Theta_1(k_1 d_1 u_1 + k_1 h_{12}) \end{Bmatrix},$$

$$\mathrm{E}[x|y] + Sq = \begin{Bmatrix} y_1 \\ \alpha F_1 y_1 + \Theta_1(k_1 d_1 u_1 + k_1 h_{12}) \end{Bmatrix}.$$

Assembling the various terms,

$$k_1 d_1 \frac{\alpha F_1 y_1 + \Theta_1(k_1 d_1 u_1 + k_1 h_{12})}{1 - \Theta_1 k_1} + \left(k_1 d_1^2 + P_1\right) u_1 + k_1 d_1 h_{12} = 0.$$

From this we get

$$u_1 = -\frac{k_1 d_1 \alpha F_1}{k_1 d_1^2 + P_1 - \Theta_1 P_1 k_1} y_1 + \frac{d_1 k_1 h_{12}}{k_1 d_1^2 + P_1 - \Theta_1 P_1 k_1}$$

or

$$u_1 = g_1 F_1 y_1 + \mu_1 = g_1 \mathrm{E}[v|y_1] + \mu_1,$$

with

$$g_1 = -\frac{k_1 d_1 \alpha}{k_1 d_1^2 + P_1 - \theta_1\left[\alpha^2 f^2(1 - F_1) + \beta^2\sigma_2^2\right] P_1 k_1}$$

$$= -\frac{k_1 d_1(1 + d_2 L_2)}{k_1 d_1^2 + P_1 - \theta_1\left[(1 + d_2 L_2)^2 f^2(1 - F_1) + d_2^2 L_2^2 \sigma_2^2\right] P_1 k_1}.$$

Since

$$L_2^2\sigma_2^2 = g_2^2 F_2^2 \sigma_2^2 = g_2^2 \frac{f^2}{f^2+\sigma_2^2}\frac{\sigma_2^2}{f^2+\sigma_2^2} f^2 = g_2^2 F_2(1-F_2) f^2,$$

$$g_1 = -\frac{k_1 d_1(1+d_2 g_2 F_2)}{k_1 d_1^2 + P_1 - \theta_1 f^2\left[(1+d_2 g_2 F_2)^2(1-F_1) + d_2^2 g_2^2 F_2(1-F_2)\right] P_1 k_1},$$

$$\mu_1 = \frac{d_1 k_1(d_2\mu_2 - a_1)}{k_1 d_1^2 + P_1 - \theta_1 f^2\left[(1+d_2 g_2 F_2)^2(1-F_1) + d_2^2 g_2^2 F_2(1-F_2)\right] P_1 k_1},$$

and similarly for player 2 with indices 1 and 2 interchanged. These are eqs. (9) of section 5.

Remark. It has been observed at the end of section A.1 that if the assumption of positive definiteness of $\Sigma^{-1} - Q$ is violated, the expectations in Lemmata 1–5 yield an infinite value. It is not difficult to see that positive definiteness of $\Sigma^{-1} - Q$ implies that of $I - SQ$ in Lemma 5 and this in turn implies $1 - \Theta_1 k_1 > 0$ in eq. (A.6) which, after proper substitutions, is condition (10) of section 5. Condition (8) in section 3 is the same as (10) for $d_2 = 0$.

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