JOURNAL OF OPTIMIZATION THEORY AND APPLICATIONS: Vol 62, No. 3, SEPTEMBER 1989

Learning Algorithms for Repeated Bimatrix Nash Games with Incomplete Information[1]

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Communicated by J. B. Cruz, Jr.

Abstract. The purpose of this paper is to study a particular recursive scheme for updating the actions of two players involved in a Nash game, who do not know the parameters of the game, so that the resulting costs and strategies converge to (or approach a neighborhood of) those that could be calculated in the known parameter case. We study this problem in the context of a matrix Nash game, where the elements of the matrices are unknown to both players. The essence of the contribution of this paper is twofold. On the one hand, it shows that learning algorithms which are known to work for zero-sum games or team problems can also perform well for Nash games. On the other hand, it shows that, if two players act without even knowing that they are involved in a game, but merely thinking that they try to maximize their output using the learning algorithm proposed, they end up being in Nash equilibrium.

Key Words. Nash games, learning algorithms, bimatrix games, games with incomplete information.

1. Introduction

The purpose of this paper is to study a particular recursive scheme for updating the actions of two players involved in a Nash game, who do not know the parameters of the game, so that the resulting costs and strategies converge to (or approach a neighborhood of) those that could be calculated in the known parameter case. We study this problem in the context of a matrix Nash game, where the elements of the matrices are unknown to both players. The essence of the contribution of this paper is twofold. On the

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0022-3239/89/0900-0467$06.00/0

one hand, it shows that learning algorithms which are known to work for zero-sum games or team problems can also perform well for Nash games. On the other hand, it shows that, if two players act without even knowing that they are involved in a game, but merely thinking that they try to maximize their output using the learning algorithm proposed, they end up being in Nash equilibrium.

In the context of zero-sum games and team problems, schemes similar to the one examined here have been studied in Refs. 1 and 2. In Ref. 3 a Nash game and in Ref. 4 a Stackelberg game with unknown coefficients are considered. The costs in Refs. 3 and 4 are quadratic, whereas in Refs. 1 and 2, as well as in the present work, the matrix game models are employed. These types of studies can find applications in situations of conflict or decentralized decision making where several parameters are unknown, as for example in economic systems where the agents do not know each other's parameters, in traffic routing of telephone or computer messages (see Refs. 1 and 2), in military systems, and elsewhere.

The structure of the paper is the following. In Section 2, a complete analysis of the 2×2 matrix case is provided. It is shown that, if the static game with known parameters has a unique solution, then the proposed schemes for the unknown parameter case converge in strategy and value within ϵ to those of the known parameter case, where ϵ is controlled by a certain parameter. It should be pointed out that the strategy pair considered is not a solution of the constructed dynamic game, but that it approximates, as time goes to infinity, the solution of the static game with known parameters. If the unique solution is a mixed one, the rapidity of convergence decreases as ϵ does. The study of the proposed schemes is reduced to the study of an associated differential equation, which is essentially our central subject of scrutiny. For the 2×2 case, the differential equation is analyzed in Section 2.1. The possibility of limit cycles (an issue overlooked in Ref. 1) is ruled out.

The more general $M \times N$ matrix case is introduced and briefly discussed in Section 3. Although a relatively satisfactory analysis of the equation is possible for the 2×2 case, the $M \times N$ case is considerably harder. Some results are given, and some important issues concerning further study associated with the differential equation pertaining to the $M \times N$ case are also delineated in Section 3.

As was mentioned in the previous paragraph, the study of a differential equation is quite crucial to this study. It should be pointed out that a related differential equation, which represents an algorithm for solving zero-sum games, was first introduced and studied in Ref. 5. The differential equation of Ref. 5 pertains to the known parameter case and does not cover the differential equation considered here.

2. The 2×2 Matrix Case

Let us first review some known facts concerning the simple 2×2 Nash matrix game with known parameters. Consider the two matrices

$$A_1 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \qquad A_2 = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix},$$

and let

$$S = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_1 + x_2 = 1, x_1 \geq 0, x_2 \geq 0 \right\}.$$

Player 1 chooses rows, while player 2 chooses columns. A pair $(x^*, y^*) \in S \times S$ is a Nash equilibrium if

$$V_1 = x^* A_1 y^* \geq x A_1 y^*, \qquad \forall x \in S, \tag{1a}$$

$$V_2 = x^* A_2 y^* \geq x^* A_2 y, \qquad \forall y \in S. \tag{1b}$$

The pure solutions are:

(I) $x^* = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $y^* = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, if $a_{11} \geq a_{21}$ and $b_{11} \geq b_{12}$;

(II) $x^* = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $y^* = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, if $a_{12} \geq a_{22}$ and $b_{12} \geq b_{11}$;

(III) $x^* = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $y^* = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, if $a_{21} \geq a_{11}$ and $b_{21} \geq b_{22}$;

(IV) $x^* = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $y^* = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, if $a_{22} \geq a_{12}$ and $b_{22} \geq b_{21}$.

The mixed solutions are:

$$x^* = [1/(b_{11} - b_{12} + b_{22} - b_{21})] \begin{bmatrix} b_{22} - b_{21} \\ b_{11} - b_{12} \end{bmatrix}, \tag{2}$$

$$y^* = [1/(a_{11} - a_{21} + a_{22} - a_{12})] \begin{bmatrix} a_{22} - a_{12} \\ a_{11} - a_{21} \end{bmatrix}, \tag{3}$$

if

$$(a_{11} - a_{21})(a_{22} - a_{12}) \geq 0, \qquad (b_{11} - b_{12})(b_{22} - b_{21}) \geq 0,$$

and the denominators in (2) and (3) are nonzero. It is easy to see that pure and mixed solutions can coexist. The only cases where there exists a unique solution which has to be mixed are

$$a_{11} > a_{21}, \qquad a_{22} > a_{12}, \qquad b_{11} < b_{12}, \qquad b_{22} < b_{21}, \tag{4a}$$

or

$$a_{11} < a_{21}, \qquad a_{22} < a_{12}, \qquad b_{11} > b_{12}, \qquad b_{22} > b_{21}. \tag{4b}$$

2.1. Problem Statement. Let us consider the matrix game where the 2×2 matrices A_1, A_2 have elements

$$0 \leq a_{ij} \leq 1, \qquad 0 \leq b_{ij} \leq 1.$$

If player 1 chooses row i and player 2 chooses column j, then player 1 receives a gain equal to 1 with probability a_{ij} and pays 1 with probability $1-a_{ij}$, whereas player 2 receives a gain equal to 1 with probability b_{ij} and pays 1 with probability $1-b_{ij}$. Let us assume that the players do not know the elements a_{ij}, b_{ij}. The game is played repeatedly and, at each instant of time, the players choose a row and a column respectively and all that they learn is whether they win or lose one unit. The question is how they should choose a row or a column so that, as time goes by, things work as they would in the case where A_1, A_2 were completely known to them.

One way of going about this problem is the following. At time k, player 1 chooses a probability vector

$$p(k) = \begin{bmatrix} p_1(k) \\ p_2(k) \end{bmatrix}, \qquad p_1(k) \geq 0, p_2(k) \geq 0, p_1(k) + p_2(k) = 1,$$

and player 2 chooses a probability vector

$$q(k) = \begin{bmatrix} q_1(k) \\ q_2(k) \end{bmatrix}, \qquad q_1(k) \geq 0, q_2(k) \geq 0, q_1(k) + q_2(k) = 1,$$

meaning that row i and column j are chosen with probability $p_i(k)q_j(k)$, and thus player 1 receives gain 1 with probability a_{ij} or loses 1 with probability $1-a_{ij}$, given that row i and column j were chosen. Player 1 knows $p_1(k)$, $p_2(k)$, knows whether row 1 or 2 was chosen, and also knows whether he gained or lost one unit. Similar things hold for player 2. Based on this knowledge, player 1 updates the vector $p_1(k)$, $p_2(k)$ by

$$\begin{bmatrix} p_1(k+1) \\ p_2(k+1) \end{bmatrix} = F_1\left(\begin{bmatrix} p_1(k) \\ p_2(k) \end{bmatrix}, \alpha_{1,k}, e_{1,k}\right),$$

where

$$\alpha_{1k} = \begin{cases} 1, & \text{if row 1 was chosen,} \\ 2, & \text{if row 2 was chosen,} \end{cases}$$

$$e_{1k} = \begin{cases} 1, & \text{if 1 unit was gained,} \\ -1, & \text{if 1 unit was lost.} \end{cases}$$

Similarly, player 2 updates $q_1(k)$, $q_2(k)$ by

$$\begin{bmatrix} q_1(k+1) \\ q_2(k+1) \end{bmatrix} = F_2\left(\begin{bmatrix} q_1(k) \\ q_2(k) \end{bmatrix}, \alpha_{2,k}, e_{2,k}\right),$$

where

$$\alpha_{2k} = \begin{cases} 1, & \text{if column 1 was chosen,} \\ 2, & \text{if column 2 was chosen,} \end{cases}$$

$$e_{2k} = \begin{cases} 1, & \text{if 1 unit was gained,} \\ -1, & \text{if 1 unit was lost.} \end{cases}$$

For a given pair of functions F_1, F_2, the vector $[p_1(k+1), p_2(k+1), q_1(k+1), q_2(k+1)]'$ is a random vector, the probability distribution of which depends solely on $[p_1(k), p_2(k), q_1(k), q_2(k)]'$, and thus we deal with a Markovian process. The issue now is how to choose F_1, F_2, so that this Markovian vector behaves, as $k \to +\infty$, in an appropriate way. For example, it would be desirable to have that

$$[p_1(k), p_2(k)]' \to x^*, \qquad [q_1(k), q_2(k)] \to y^*,$$
$$E[p(k)'A_1q(k)] \to x^*Ay^*, \qquad E[p(k)'A_2q(k)] \to x^*A_2y^*,$$

where x^*, y^* are as in Section 2. In the next section, we consider a particular choice of F_1, F_2 and study the resulting behavior of $p(k)$, $q(k)$.

2.2. Analysis. Since we deal with the 2×2 matrix case, we need to consider only $p_1(k)$, $q_1(k)$, which we from now on denote by $p(k)$, $q(k)$. The following type of $F_1(k)$, $F_2(k)$ is chosen:

$$p(k+1) = p(k) + \begin{cases} \theta\lambda_1(1-p(k)), & \text{if } \alpha_{1k}=1,\ E_{1k}=1, \\ -\theta\lambda_2 p(k) & \text{if } \alpha_{1k}=1,\ E_{1k}=-1, \\ -\theta\lambda_1 p(k), & \text{if } \alpha_{1k}=2,\ E_{1k}=1, \\ \theta\lambda_2(1-p(k)), & \text{if } \alpha_{1k}=2,\ E_{1k}=-1, \end{cases} \tag{5}$$

$$q(k+1) = q(k) + \begin{cases} \theta\mu_1(1-q(k)), & \text{if } \alpha_{2k}=1,\ e_{2k}=1, \\ -\theta\mu_2 q(k), & \text{if } \alpha_{2k}=1,\ e_{2k}=-1, \\ -\theta\mu_1 q(k), & \text{if } \alpha_{2k}=2,\ e_{2k}=1, \\ \theta\mu_2(1-q(k)), & \text{if } \alpha_{2k}=2,\ e_{2k}=-1, \end{cases} \tag{6}$$

where θ, λ_1, λ_2, μ_1, μ_2 are nonnegative constants, with

$$1 \geq \theta\lambda_i, \qquad 1 \geq \theta\mu_i.$$

Thus, by construction,

$$0 \leq p(k+1) \leq 1, \qquad p \leq q(k+1) \leq 1.$$

This scheme is called linear reward penalty ergodic (LRPE) algorithm; see Ref. 1. Each of the four cases in (5) and of the four cases in (6) occurs with the following respective probabilities:

$$p(k)[a_{11}q(k)+a_{12}(1-q(k)],$$
$$p(k)[(1-a_{11})q(k)+(1-a_{12})(1-q(k)],$$
$$[1-p(k)][a_{21}q(k)+a_{22}(1-q(k)],$$
$$[1-p(k)][(1-a_{21})q(k)+(1-a_{22})(1-q(k)],$$
$$q(k)[b_{11}p(k)+b_{21}(1-p(k)],$$
$$q(k)[(1-b_{11})p(k)+(1-b_{21})(1-p(k))],$$
$$[1-q(k)][b_{12}p(k)+b_{22}(1-p(k))],$$
$$[1-q(k)][(1-b_{12})p(k)+(1-b_{22})(1-p(k))].$$

It holds that

$$E\left[\begin{bmatrix} p(k+1)-p(k) \\ q(k+1)-q(k) \end{bmatrix} \middle| \begin{bmatrix} p(k) \\ q(k) \end{bmatrix}\right] = \theta \begin{bmatrix} W_1(p(k), q(k)) \\ W_2(p(k), q(k)) \end{bmatrix}, \tag{7}$$

where

$$\begin{aligned} W_1(p, q) = {} & \lambda_1 p(1-p)[a_{11}q + a_{12}(1-q)] \\ & -\lambda_2 p^2[(1-a_{11})q + (1-a_{12})(1-q)] \\ & -\lambda_1 p(1-p)[a_{21}q + a_{22}(1-q)] \\ & +\lambda_2(1-p)^2[(1-a_{21})q + (1-a_{22})(1-q)], \end{aligned} \tag{8a}$$

$$\begin{aligned} W_2(p, q) = {} & \mu_1 q(1-q)[b_{11}p + b_{21}(1-p)] - \mu_2 q^2[(1-b_{11})p \\ & +(1-b_{21})(1-p)] - \mu_1 q(1-q)[b_{12}p + b_{22}(1-p)] \\ & +\mu_2(1-q)^2[(1-b_{12})p + (1-b_{22})(1-p)]. \end{aligned} \tag{8b}$$

For θ very small, we anticipate (and this will be substantiated later) that the evolution of $[p(k), q(k)]'$ will follow the evolution of the differential equation

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} W_1(x, y) \\ W_2(x, y) \end{bmatrix}. \tag{9}$$

The following theorem is an easy extension of Theorems 2.2, 2.3 and 6.1 of Ref. 1.

Theorem 2.1. Assume that the differential equation (9) has a unique equilibrium point in $[0, 1]\times[0, 1]$ which is globally asymptotically stable there. Let $x(0)=p(0)$, $y(0)=q(0)$. Then,

$$E\left[\left\|\begin{bmatrix} p(k)-x(k) \\ q(k)-x(k) \end{bmatrix}\right\|^2 \left[\begin{bmatrix} p(0) \\ q(0) \end{bmatrix}\right]\right]=O(\theta),$$

$$E\left[\begin{bmatrix} p(k) \\ q(k) \end{bmatrix}\middle|\begin{bmatrix} p(0) \\ q(0) \end{bmatrix}\right]=\begin{bmatrix} x(k) \\ y(k) \end{bmatrix}+O(\theta),$$

$$\mathrm{Var}\left[\begin{bmatrix} p(k) \\ q(k) \end{bmatrix}\middle|\begin{bmatrix} p(0) \\ q(0) \end{bmatrix}\right]=O(\theta),$$

$$\lim|\eta_i(k)-V_i|=O(\theta), \qquad k\to+\infty, i=1, 2,$$

where

$$\eta_i(k)=\left(E\begin{bmatrix} p(k) \\ 1-p(k) \end{bmatrix}\right)^{T} A_i\left(E\begin{bmatrix} p(k) \\ 1-p(k) \end{bmatrix}\right), \qquad i=1, 2,$$

uniformly for all $k\geq 0$ and $p(0)$, $q(0)$. Here, V_i = the value for player i in the known parameter case; see (1). □

It is clear that all the results of Theorem 2.1 hinge upon the behavior of the differential equation (9). Let

$$\lambda=\lambda_2/\lambda_1, \qquad \mu=\mu_2/\mu_1, \qquad \hat{\theta}=\theta\cdot\lambda_1, \hat{y}(t)=y((\lambda_1/\mu_1)\cdot t).$$

Letting $\theta\to 0$ corresponds to $\hat{\theta}\to 0$, and thus (9) becomes

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{y}} \end{bmatrix}=\begin{bmatrix} W_1(x, \hat{y}) \\ W_2(x, \hat{y}) \end{bmatrix}, \tag{10}$$

where

$$W_1(x, \hat{y})=x(1-x)[(a_{11}+a_{22}-a_{12}-a_{21})\hat{y}+a_{12}-a_{22}]$$
$$+\lambda\{x^2[(a_{11}-a_{12})\hat{y}+a_{12}-1]+(1-x)^2[(a_{22}-a_{21})\hat{y}+1-a_{22}]\},$$
$$W_2(x, \hat{y})=\hat{y}(1-\hat{y})[(b_{11}+b_{22}-b_{12}-b_{21})x+b_{21}-b_{22}]$$
$$+\mu\{\hat{y}^2[(b_{11}-b_{21})x+b_{21}-1]+(1-\hat{y})^2[(b_{22}-b_{12})x+1-b_{22}]\}.$$

Next, we will study (10), where for notational simplicity we will use y instead of $\hat{y}$.

If $x=0$, then

$$\dot{x}=\lambda[(a_{22}-a_{21})y+1-a_{22}]\geq 0, \qquad \text{for } y\in[0, 1].$$

If $x=1$, then

$$\dot{x}=\lambda[(a_{11}-a_{12})y+a_{12}-1]\leq 0, \qquad \text{for } y\in[0, 1].$$

Similarly,

$$\dot{y} \geq 0, \qquad \text{if } y = 0,$$

and

$$\dot{y} \leq 0, \qquad \text{if } y = 1.$$

Thus, the solution of (9) remains always within the square $[0, 1] \times [0, 1]$ if it starts there. The phase portrait of (10) can be obtained equivalently from

$$dx/dy = F_1(x, y)/F_2(x, y),$$

where

$$\begin{aligned} F_1(x, y) &= [(a_{11} + a_{22} - a_{12} - a_{21})y + a_{12} - a_{22}] \\ &\quad + \lambda\{1/[y(1-y)]\}[x/(1-x)][(a_{11} - a_{12})y + a_{12} - 1] \\ &\quad + [(1-x)/x][(a_{22} - a_{21})y + 1 - a_{22}], \\ F_2(x, y) &= [(b_{11} + b_{22} - b_{12} - b_{21})x + b_{21} - b_{22}] \\ &\quad + \mu\{1/[x(1-x)]\}[y(1-y)][(b_{11} - b_{21})x + b_{21} - 1] \\ &\quad + [(1-y)/y][(b_{22} - b_{12})x + 1 - b_{22}]. \end{aligned}$$

It holds that

$$\begin{aligned} &\partial F_2/\partial x + \partial F_2/\partial y \\ &= -\lambda\{1/[y(1-y)]\}[[1/(1-x)^2][(a_{12} - a_{11})y + a_{12}] + (1/x^2) \\ &\times [(a_{22} - a_{21})y + 1 - a_{22}]] \\ &- \mu\{1/[x(1-x)]\}[1/(1-y)^2][(b_{21} - b_{11})x + 1 - b_{21}] + (1/y^2) \\ &\times [(b_{22} - b_{12})x + 1 - b_{22}]]; \end{aligned}$$

and, since $\lambda > 0$, $\mu > 0$,

$$\partial F_1/\partial x + \partial F_2/\partial y < 0, \qquad \text{for } (x, y) \in (0, 1) \times (0, 1).$$

Thus, by the criterion of Bendixson (see Ref. 6, p. 238), the differential equation (10) has no limit cycle in $(0, 1) \times (0, 1)$. It should be noticed that λ does not need to equal μ.

Next, we investigate the equilibrium points of (10); to this end, we consider the equation

$$W_1(x, y) = 0.$$

Let us assume that $1 > a_{ij}$. It holds that

$$W_1(0,0)=\lambda(1-a_{22})>0, \qquad W_1(0,1)=\lambda(1-a_{21})>0,$$

$$W_1(1,0)=-\lambda(1-a_{12})<0, \qquad W_1(1,1)=-\lambda(1-a_{11})<0,$$

$$W_1(0,y)=\lambda[(a_{22}-a_{21})y+1-a_{22}]>0,$$

$$W_1(1,y)=\lambda[(a_{11}-a_{12})y+a_{12}-1]<0,$$

$$W_1(0,\bar{y})=0, \qquad \text{where } \bar{y}=(1-a_{22})/(a_{21}-a_{22}),$$

$$W_1(1,\bar{\bar{y}})=0, \qquad \text{where } \bar{\bar{y}}=(1-a_{12})/(a_{11}-a_{12}),$$

$$W_1(x,y)=0 \Leftrightarrow y[N(x,\lambda)/D(x,\lambda)], \tag{11a}$$

where

$$N(x,\lambda)=((1-\lambda)(a_{12}-a_{22})x^2+[a_{12}-a_{22}+2\lambda(a_{22}-1)]x+\lambda(1-a_{22})), \tag{11b}$$

$$D(x,\lambda)=((1-\lambda)(-a_{11}+a_{22}+a_{12}+a_{21})x^2 + [-a_{11}-a_{22}+a_{12}+a_{21}+2\lambda(a_{22}-a_{21})]x+\lambda(a_{21}-a_{22})). \tag{11c}$$

Since

$$W_1(0,y)>0, \qquad W_1(1,y)<0,$$

and since $W_1(x,y)$ is quadratic in x for each y, there is a unique x for which $W_1(x,y)=0$, with $0\le x\le 1$. $D(x,0)$ has roots $x=0, 1$ and

$$D(0,\lambda)=\lambda(a_{22}-a_{21}),$$

$$D(1,\lambda)=\lambda(a_{11}-a_{22}).$$

Thus the denominator of (11) becomes zero, for x close to 0 or to 1, if λ is very small. $N(x,0)$ has roots $x=0, 1$ and

$$N(0,\lambda)=-\lambda(1-a_{22})<0,$$

$$N(1,\lambda)=\lambda(1-a_{12})>0.$$

Thus, $N(x,\lambda)$ has a unique root between 0 and 1 which is close to 0 if

$$a_{11}+a_{22}-a_{12}-a_{21}>0.$$

Finally, notice that, if λ is very small, then the curve $W_1(p,q)=0$ follows closely the line

$$q^*=(a_{22}-a_{12})/(a_{11}+a_{22}-a_{12}-a_{21}),$$

for $p\in(0,1)$. Using all these facts, we conclude that the curve $W_1(p,q)=0$, for $\lambda>0$, λ sufficiently small, is as in Fig. 1 or Fig. 2, where the asymptotes

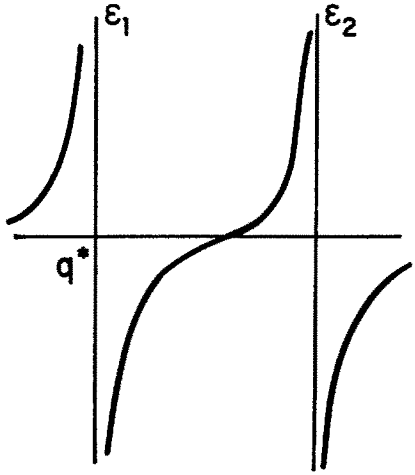

Fig. 1. Case where $\alpha = a_{11} + a_{22} - a_{12} - a_{21} > 0$.

ϵ_1, ϵ_2 are close to $x=0$, $x=1$ respectively. The point q^* could be anywhere in the real line. Similar results hold for the curve $W_2(p, q) = 0$, under the assumption $1 > b_{ij}$, where the role of α will be assumed by

$$\beta = b_{11} + b_{12} - b_{12} - b_{21}$$

and that of q^* by

$$p^* = (b_{22} - b_{21})/(b_{11} + b_{22} - b_{12} - b_{21}).$$

Considering the signs of α, β, and considering whether $p^* > 1$, $p^* < 0$,

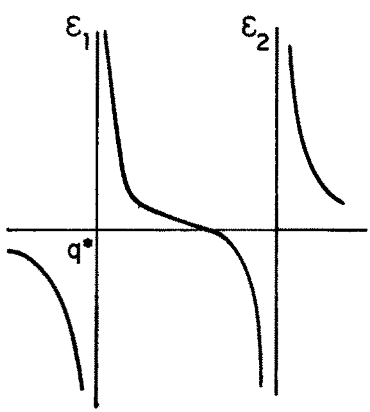

Fig. 2. Case where $\alpha = a_{11} + a_{22} - a_{12} - a_{21} < 0$.

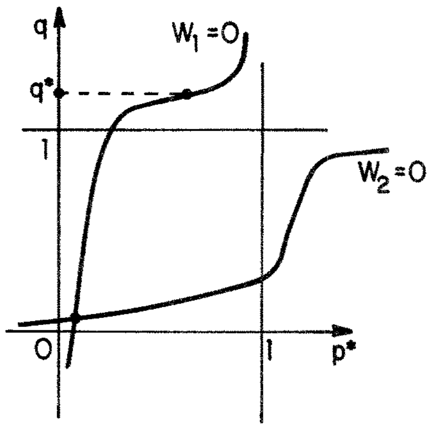

Fig. 3. Case where $\alpha>0$, $\beta>0$ and $q^*>1$, $p^*>1$.

$0<p^*<1$, $q^*>1$, $q^*<0$, $0<q^*<1$, several situations, concerning the intersections of $W_1=0$, $W_2=0$, can arise. Some of them are delineated in Figs. 3-6. Twelve similar cases can appear by considering the $\alpha<0$ case.

The general conclusion is that, if $\lambda>0$, $\mu>0$ are sufficiently small, then:

(i) if $\alpha\beta<0$ and $0<p^*$, $q^*<1$, then the curves $W_1=0$, $W_2=0$ intersect at a unique point which is close to (p^*, q^*); the point (p^*, q^*) corresponds to the unique Nash solution of (1), which has to be mixed; see Fig. 6;

(ii) if $\alpha\beta>0$ and $0<p^*$, $q^*<1$, then the curves $W_1=0$, $W_2=0$ intersect at three points, one of which is close to (p^*, q^*), a mixed solution of (1), and the other two are close to the pure solutions of (1); see Fig. 5;

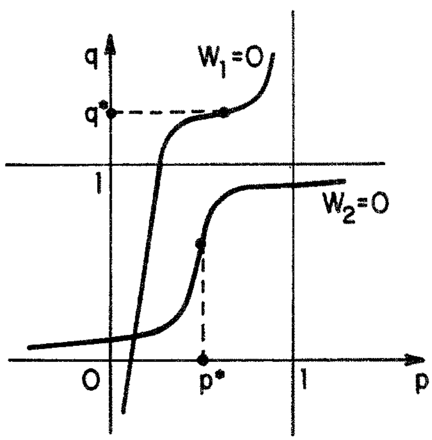

Fig. 4. Case where $\alpha>0$, $\beta>0$ and $q^*>1$, $0<p^*<1$.

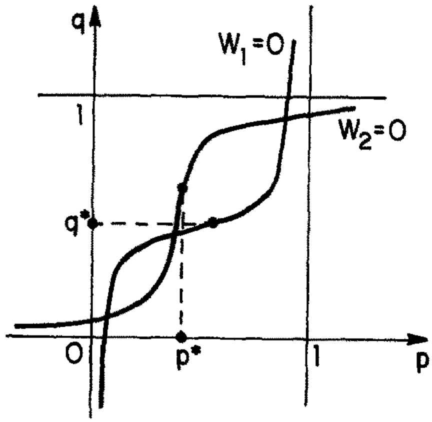

Fig. 5. Case where $\alpha > 0$, $\beta > 0$ and $0 < q^* < 1$, $0 < p^* < 1$.

(iii) in all other cases, W_1, W_2 intersect at a unique point which is close to the pure solution of (1).

In terms of studying the stability behavior of (10), only the cases of Figs. 3, 4, 5 and 6 need to be considered, since the study of the other 20 cases can be seen to be equivalent to the cases of Fig. 3, or Fig. 4, or Fig. 5, or Fig. 6.

Case A. See Figs. 3 and 7. In this case,

$$(\alpha > 0, q^* > 1, \beta > 0, p^* > 1)$$
$$\Leftrightarrow (a_{22} - a_{12} > a_{21} - a_{11} > 0, b_{22} - b_{21} > b_{12} - b_{11} > 0).$$

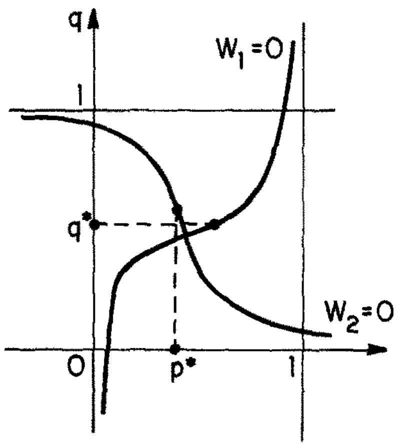

Fig. 6. Case where $\alpha > 0$, $\beta > 0$ and $0 < q^* < 1$, $0 < p^* < 1$.

In Fig. 7, the arrows indicate the direction of $(\dot{x}, \hat{y})$. The Jacobian of $(W_1, W_2)'$ is

$$J = \begin{bmatrix} (1-2p)(\alpha \cdot q + a_{12} - a_{22}) & \alpha p(1-p) \\ \beta q(1-q) & (1-2q)(\beta p + b_{21} - b_{22}) \end{bmatrix} + O(\lambda, \mu);$$

at the point $p = p_1 \cong 0$, $q = q_1 \cong 0$, the point of intersection of $W_1 = W_2 = 0$, the above Jacobian equals approximately

$$J = \begin{bmatrix} a_{12} - a_{22} & 0 \\ 0 & b_{21} - b_{22} \end{bmatrix} + O(\lambda, \mu). \tag{12}$$

Thus, the Jacobian has two negative eigenvalues, and (p_1, q_1) is locally asymptotically stable. Since there is no limit cycle, it is globally asymptotically stable in $(0, 1) \times (0, 1)$.

Case B. See Fig. 4. In this case,

$$(\alpha > 0, q^* > 1, \beta > 0, 0 < p^* < 1$$
$$\Leftrightarrow (a_{22} - a_{12} > a_{21} - a_{11} > 0, b_{11} - b_{12} > 0, b_{22} - b_{21} > 0).$$

The local stabiliy analysis is the same as for Fig. 3.

Case C. See Figs. 5 and 8. In this case,

$$(\alpha > 0, 0 < q^* < 1, \beta > 0, 0 < p^* < 1)$$
$$\Leftrightarrow (a_{22} > a_{12}, a_{11} > a_{21}, b_{11} > b_{12}, b_{22} > b_{21}).$$

The directions of $(\dot{x}, \hat{y})$ in the several areas are shown in Fig. 8. The pairs (p_1, q_1), (p_2, q_2), which are close to the pure strategy solutions, are locally asymptotically stable, whereas $(\bar{p}, \bar{q})$, which is close to the mixed solution, in unstable since the Jacobian at $\bar{p}, \bar{q}$ has a positive eigenvalue; see (12).

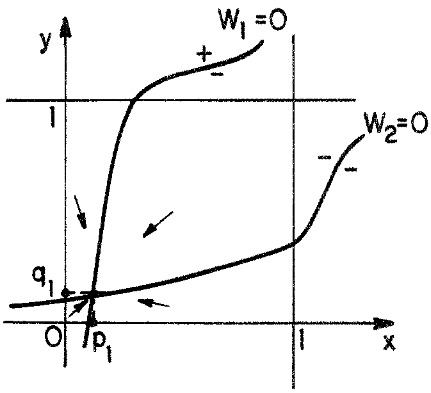

Fig. 7. Case where $\alpha > 0$, $\beta > 0$ and $q^* > 1$, $p^* > 1$.

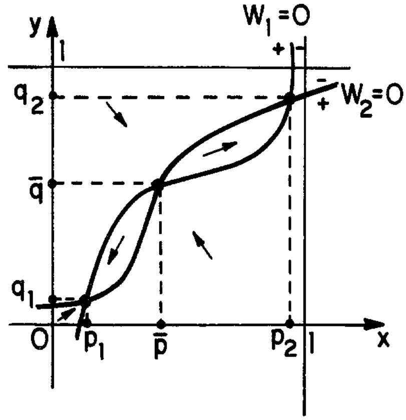

Fig. 8. Case where $\alpha>0$, $\beta>0$ and $0<q^*<1$, $0<0^*<1$.

Case D. See Fig. 6. In this case,

$$(\alpha>0, 0<q^*<1, \beta<0, 0<p^*<1)\Leftrightarrow\begin{pmatrix} a_{11}>a_{21}, & b_{11}<b_{12} \\ a_{22}>a_{12}, & b_{22}<b_{21}\end{pmatrix}.$$

There is a unique solution of $W_1=0$, $W_2=0$, which tends to (p^*, q^*) as $\mu, \lambda\to 0$. In order to study the local stability properties of $(\bar{p}, \bar{q})$, we consider the Jacobian of W_1, W_2. For convenience, let $\lambda=\mu$. It holds that

$$J=\begin{bmatrix} \partial W_1(p(\lambda), q(\lambda), \lambda)/\partial p & \partial W_1(p(\lambda), q(\lambda), \lambda)/\partial q \\ \partial W_2(p(\lambda), q(\lambda), \lambda)/\partial p & \partial W_2(p(\lambda), q(\lambda), \lambda)/\partial q \end{bmatrix}$$

$$=O(\lambda^2)+\begin{bmatrix} \partial W_1(p(\lambda), q(\lambda), \lambda)/\partial p & \partial W_1(p(\lambda), q(\lambda), \lambda)/\partial q \\ \partial W_2(p(\lambda), q(\lambda), \lambda)/\partial p & \partial W_2(p(\lambda), q(\lambda), \lambda)/\partial q \end{bmatrix}_{\lambda=0}$$

$$+\lambda\left\{\begin{bmatrix} \partial^2 W_1(p(\lambda), q(\lambda), \lambda/\partial\lambda\,\partial p & \partial^2 W_1(p(\lambda), q(\lambda), \lambda)/\partial\lambda\,\partial q \\ \partial^2 W_2(p(\lambda), q(\lambda), \lambda)/\partial\lambda\,\partial p & \partial^2 W_2(p(\lambda), q(\lambda), \lambda)/\partial\lambda\,\partial q \end{bmatrix}_{\lambda=0}\right.$$

$$\left.\times\begin{bmatrix} \left(\frac{\partial^2 W_1}{\partial p^2}\right)\left(\frac{\partial p}{\partial\lambda}\right)+\left(\frac{\partial^2 W_1}{\partial p\,\partial q}\right)\left(\frac{\partial q}{\partial\lambda}\right) & \left(\frac{\partial^2 W_1}{\partial q\,\partial p}\right)\left(\frac{\partial p}{\partial\lambda}\right)+\left(\frac{\partial^2 W_1}{\partial q^2}\right)\left(\frac{\partial q}{\partial\lambda}\right) \\ \left(\frac{\partial^2 W_2}{\partial p^2}\right)\left(\frac{\partial p}{\partial\lambda}\right)+\left(\frac{\partial^2 W_2}{\partial p\,\partial q}\right)\left(\frac{\partial q}{\partial\lambda}\right) & \left(\frac{\partial^2 W_2}{\partial p\,\partial q}\right)\left(\frac{\partial p}{\partial\lambda}\right)+\left(\frac{\partial^2 W_2}{\partial q^2}\right)\left(\frac{\partial q}{\partial\lambda}\right) \end{bmatrix}\right\}_{\lambda=0} \tag{13}$$

Since $W_i(p(\lambda), q(\lambda), \lambda)\equiv 0$, we also have the relations

$$(\partial W_1/\partial p)(\partial p/\partial\lambda)+(\partial W_1/\partial q)(\partial q/\partial\lambda)+\partial W_1/\partial\lambda\equiv 0, \tag{14a}$$

$$(\partial W_2/\partial p)(\partial p/\partial\lambda)+(\partial W_2/\partial q)(\partial q/\partial\lambda)+\partial W_2/\partial\lambda\equiv 0. \tag{14b}$$

Using the fact that $p(0)=p^*$, $q(0)=q^*$, (13) and (14) become

$$J=\begin{bmatrix}\partial W_1/\partial p & \partial W_1/\partial q\\ \partial W_2/\partial p & \partial W_2/\partial q\end{bmatrix}$$

$$=\begin{bmatrix}0 & \alpha(p^*(1-p^*)\\ \beta q^*(1-q^*) & 0\end{bmatrix}$$

$$+\lambda\begin{bmatrix}-2[(a_{22}-a_{21})q^*+1-a_{22}] & 0\\ 0 & -2[(b_{22}-b_{12})p^*+1-b_{22})]\end{bmatrix}$$

$$+\lambda\left\{\begin{bmatrix}0 & p^{*2}(a_{11}-a_{12})+(1-p^*)^2(a_{22}-a_{21})\\ q^{*2}(b_{11}-b_{21})+(1-q^*)^2(b_{22}-b_{12}) & 0\end{bmatrix}\right.$$

$$\left.+\begin{bmatrix}0 & -2\alpha\\ 0 & -2\beta\end{bmatrix}[\partial p(0)/\partial\lambda]+\begin{bmatrix}-2\alpha & 0\\ -2\beta & 0\end{bmatrix}[\partial q(0)/\partial\lambda]\right\}+O(\lambda), \qquad (15)$$

where

$$\begin{bmatrix}0 & ap^*(1-p^*)\\ bq^*(1-q^*) & 0\end{bmatrix}\begin{bmatrix}\partial p(0)/\partial\lambda\\ \partial q(0)/\partial\lambda\end{bmatrix}$$

$$+\begin{bmatrix}p^{*2}[(a_{11}-a_{12})q^*+a_{12}-1][1-p^*]^2[(a_{22}-a_{21})q^*+1-a_{22})]\\ q^{*2}[(b_{11}-b_{21})p^*+b_{21}-1]+(1-q^*)^2[(b_{22}-b_{12})p^*+1-b_{22})]\end{bmatrix}$$

$$=\begin{bmatrix}0\\0\end{bmatrix}. \qquad (16)$$

Thus, the Jacobian is equal to

$$J=\begin{bmatrix}0 & \alpha p^*(1-p^*)\\ \beta q^*(1-q^*) & 0\end{bmatrix}+\lambda\begin{bmatrix}-\mu_1 & +\sigma_1\\ +\alpha_2 & -\mu_2\end{bmatrix}+O(\lambda^2), \qquad (17)$$

where

$$\mu_1=[(a_{22}-a_{21})q^*+(1-a_{22})][2+(1-2p^*)^2/p^*(1-p^*)]>0,$$

$$\mu_2=[(b_{22}-b_{12})p^*+(1-b_{22})][2+(1-2q^*)^2/q^*(1-q^*)]>0,$$

$$\sigma_1=-(\alpha/\beta)\{(1-2p^*)(1-2q^*)/[q^*(1-q^*)]\}$$
$$\times[(b_{22}-b_{12})p^*+1-b_{22}]+p^{*2}[a_{11}-a_{12}]+(1-p^*)^2[a_{22}-a_{21}],$$

$$\sigma_2=-(\alpha/\beta)\{(1-2p^*)(1-2q^*)/[p^*(1-p^*)\}$$
$$\times[(a_{22}-a_{21})q^*+1-a_{22}]+q^{*2}(b_{11}-b_{21})+(1-q^*)^2[b_{22}-b_{12}],$$

$$\alpha=a_{11}+a_{22}-a_{12}-a_{21},$$

$$\beta=b_{11}+b_{22}-b_{12}-b_{21}.$$

The eigenvalues of (17) are

$$-\lambda[(\mu_1+\mu_2)/2]$$
$$\pm\{\alpha\beta+\lambda(\alpha\sigma_2+\beta\sigma_1)+\lambda^2[(\mu_1-\mu_2)/2]^2\}+o(\lambda). \tag{18}$$

Thus, in order that $p(\lambda)$, $q(\lambda)$ be locally stable for λ small, we need

$$\alpha\beta<0,$$

in which case the eigenvalues are complex with real part $-\lambda(\mu_1+\mu_2)/2+o(\lambda)$ and imaginary part $j(\sqrt{(|\alpha\beta|)}+O(\lambda))$. For λ sufficiently small, the solutions of (10) will spiral toward $p(\lambda)$, $q(\lambda)$. The condition that λ is small guarantees convergence, but makes it very slow. Actually, if $\lambda=\mu=0$, then the solutions of (10) are the closed curves

$$G(x,y)=x^{b_{21}-b_{22}}(1-x)^{b_{12}-b_{11}}y^{a_{22}-a_{12}}(1-y)^{a_{11}-a_{21}}=\text{const},$$

which are closely followed by the solution of (10) as λ, $\mu\to 0$.

The above analysis proves the following lemma.

Lemma 2.1. Let $\alpha\beta<0$. Then, for λ sufficiently small, $\lambda>0$, $p(\lambda)$, $q(\lambda)$ is a locally asymptotic stable equilibrium point of (10).

Let us formalize the whole preceding discussion in the following theorem.

Theorem 2.2. Let $1>a_{ij}$, b_{ij}. The following results hold:

(i) If $W_1=0$, $W_2=0$ have a unique intersection, then it is a globally asymptotically stable equilibrium of (10) in $(0,1)\times(0,1)$.

(ii) If λ, μ are sufficiently small, then the differential equation (10) has as many equilibrium points as the solutions of the game (1); and, as $\lambda\to 0$, $\mu\to 0$, these equilibrium points tend to the solutions of the game (1).

(iii) If the game (1) has three solutions, two pure and one mixed, then the equilibrium points of (10) which tend to the pure solutions are locally asymptotically stable, whereas the equilibrium point of (10) which tends to the mixed solution is unstable.

(iv) If the game has a unique solution that is mixed, then the differential equation (10) has a unique equilibrium point, for λ, μ sufficiently small, which tends to the mixed solution of (1) as $\lambda,\mu\to 0$ and to which all trajectories of (10) spiral asymptotically in $(0,1)\times(0,1)$. The speed of convergence decreases linearly in λ, μ as λ, $\mu\to 0$; see (18).

(v) If the game has a unique solution that is pure, then the differential equation has a unique equilibrium point, for λ, μ sufficiently small, which tends to the pure solution as λ, $\mu\to 0$ and to which the trajectories of (10) converge asymptotically as λ, $\mu\to 0$. The speed of convergence is given by the eigenvalues of (12).

3. The $M \times N$ Matrix Case

The purpose of this section is to provide some results and to delineate several issues associated with the $M \times N$ matrix case. Let A and B be two $N \times M$ matrices with elements $0 \leq a_{ij}$, $b_{ij} \leq 1$. If the elements a_{ij}, b_{ij} are known, the players find $x^* \in S_M$, $y^* \in S_N$,

$$S_M = \{(x_1, \ldots, x_M)' \in R^M : x_i \geq 0, x_1 + \cdots + x_M = 1\},$$

$$S_N = \{(y_1, \ldots, y_N)' \in R^N : y_i \geq 0, y_1 + \cdots + y_N = 1\},$$

which satisfy

$$x^{*\prime} A y^* \geq x A y^*, \qquad \forall x \in S_M, x \neq x^*, \tag{19a}$$

$$y^{*\prime} B x^* \geq y N x^*, \qquad \forall y \in S_N, y \neq y^*. \tag{19b}$$

If the elements a_{ij}, b_{ij} are unknown, a rationale similar to the one described in Section 2.1 results in the following updating scheme for player 1:

$$p_i(k+1) = p_i(k) + \theta\lambda_1(1 - p_i(k)),$$

$$p_j(k+1) = p_j(k) - \theta\lambda_1 p_j(k), \qquad j \neq i,$$

if row i was chosen and success resulted, and

$$p_i(k+1) = p_i(k) - \theta\lambda_2 p_i(k),$$

$$p_j(k+1) = p_j(k) - \theta\lambda_2[p_i(k)/(M-1)], \qquad j \neq i,$$

if row i was chosen and failure resulted.

A similar updating scheme exists for the $q_1(k), \ldots, q_N(k)$ of player 2, with μ_1, μ_2, N assuming the roles of λ_1, λ_2, M.

It holds that

$$E[p_i(k+1) - p_i(k)] \left| \begin{bmatrix} p_1(k), \ldots, p_M(k) \\ q_1(k), \ldots, q_N(k) \end{bmatrix} \right.$$

$$= \theta\lambda_1(1 - p_i(k)) p_i(k) \sum_j a_{ij} q_j(k) - \theta\lambda_2 p_i(k) p_i(k) \sum_j (1 - a_{ij}) q_j(k)$$

$$- \theta\lambda_1 p_i(k) \sum_{j \neq i} p_j(k)[a_{j1} q_1(k) + \cdots + a_{jN} q_N(k)]$$

$$+ \theta\lambda_2 \sum_{j \neq i} [p_j(k)/(M-1)] p_j(k)$$

$$\times [(1 - a_{j1}) q_1(k) + \cdots + (1 - a_{jN}) q_N(k)].$$

Let $\lambda_1 = 1$, $\lambda_2 = \lambda$; let

$$p(k) = \begin{bmatrix} p_1(k) \\ \vdots \\ p_M(k) \end{bmatrix}, \qquad q(k) = \begin{bmatrix} q_1(k) \\ \vdots \\ q_N(k) \end{bmatrix},$$

$$A = \begin{bmatrix} a_1 \\ \vdots \\ a_M \end{bmatrix}, \qquad B = \begin{bmatrix} b_1 \\ \vdots \\ b_N \end{bmatrix}.$$

The associated differential equation [see (10)] is

$$\begin{aligned} \dot{x}_i &= a_i(a_i y - x^{\mathrm{T}} A y) + [\lambda/(M-1)] \\ &\times \left\{ \sum_j x_j^2[1 - a_j y] - M x_i^2 [1 - a_i y] \right\}, \qquad i = 1, \ldots, M, \end{aligned} \tag{20}$$

$$\begin{aligned} \dot{y}_i &= y_i(b_i x - y^{\mathrm{T}} B x) + [\mu/(N-1)] \\ &\times \left\{ \sum_j y_j^2[1 - b_j x] - N y_i^2 [1 - b_i x] \right\}, \qquad i = 1, \ldots, N. \end{aligned} \tag{21}$$

It holds that

$$(d/dt)(x_1 + \cdots + x_M) = 0, \qquad (d/dt)(y_1 + \cdots + y_N) = 0;$$

thus, starting with $x(0), y(0)$ such that

$$\sum x_i(0) = 1, \qquad \sum y_i(0) = 1,$$

it will be

$$\sum x_i(t) \equiv 1, \qquad \sum y_i(t) \equiv 1.$$

It is also easily seen that, if $x(0) \in S_M$, $y(0) \in S_N$, then

$$x(t) \in S_M, \quad y(t) \in S_N, \qquad \forall t \geq 0.$$

Thus, we actually need to consider $(M-1) \times (N-1)$ equations only. The Jacobian of the first $M-1$ equations of (20) and of the first $N-1$ equations of (21), with respect to $x_1, \ldots, x_{M-1}, y_1, \ldots, y_{N-1}$, calculated at $\lambda = \mu = 0$, is given by

$$J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix},$$

$$J_{11} = \begin{bmatrix} a_1 y - x^{\mathrm{T}} A y & & 0 \\ & \ddots & \\ 0 & & a_{M-1} y - x^{\mathrm{T}} A y \end{bmatrix}$$

$$+ \begin{bmatrix} x_1(a_M y - a_1 y) & x_1(a_M y - a_2 y) & \cdots & x_1(a_M y - a_{M-1} y) \\ x_2(a_M y - a_1 y) & x_2(a_M y - a_2 y) & \cdots & x_2(a_M y - a_{M-1} y) \\ \vdots & \vdots & & \vdots \\ x_{M-1}(a_M y - a_1 y) & x_{M-1}(a_M y - a_2 y) & \cdots & x_{M-1}(a_M y - a_{M-1} y) \end{bmatrix},$$

$$J_{12}=\begin{bmatrix} x_1(a_1-x^{\mathrm{T}}A)L_1 \\ x_2(a_2-x^{\mathrm{T}}A)L_1 \\ \vdots \\ x_{M-1}(a_{M-1}-x^{\mathrm{T}}A)L_1 \end{bmatrix},$$

$$L_1=\begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \\ \hline -1 & -1 & -1 \end{bmatrix}_{[N\times(N-1)]}$$

J_{21}, J_{22} are given by similar formulas, where the roles of x_i, a_i, y, N, L_1 are assumed by y_i, b_i, x, M, L_2, respectively.

Case 1. At a point (x^*, y^*) with

$$x_1^*=1, \qquad x_2^*=\cdots=x_M^*=0,$$

$$y_1^*=1, \qquad y_2^*=\cdots=y_N^*=0,$$

the Jacobian is given by

$$J=\left[\begin{array}{cccc|cccc} a_{M1}-a_{11} & a_{M1}-a_{21} & \cdots & a_{M1}-a_{M1,1} & & & & \\ & a_{21}-a_{11} & & 0 & & 0 & & \\ 0 & & & a_{M-1,1}-a_{11} & & & & \\ \hline & & & & b_{N1}-b_{11} & b_{N1}-b_{21} & \cdots & b_{N1}-b_{N-1,1} \\ & 0 & & & & b_{21}-b_{11} & & 0 \\ & & & & 0 & & & b_{N-1,1}-b_{11} \end{array}\right],$$

which is invertible and has negative eigenvalues if

$$a_{11}>a_{21}, \qquad a_{31},\ldots,a_{M1}, \tag{22a}$$

$$b_{11}>b_{21}, \qquad b_{31},\ldots,b_{N1}. \tag{22b}$$

We can thus conclude that, if (22) holds, then (x^*, y^*) is a pure Nash equilibrium of (19) and that, for λ, μ sufficiently small, (20), (21) has an equilibrium point $x(\lambda, \mu)$, $y(\lambda, \mu)$, by the implicit function theorem, which is a locally asymptotically stable equilibrium of (20), (21) and $x(\lambda, \mu)\to x^*$, $y(\lambda, \mu)\to y^*$ as $\lambda, \mu\to 0$.

Case 2. At a point (x^*, y^*) where

$$a_1 y^*=\cdots=a_M y^*, \qquad b_1 x^*=\cdots=b_N x^*, \qquad \sum x_i^*=1, \sum y_i^*=1,$$

which is a mixed solution of (19), the Jacobian is given by

$$J = \left[\begin{array}{c|c} 0 & \begin{array}{c} x_1^*(a_1 - x^{*\mathrm{T}}A)L_1 \\ \vdots \\ x_{M-1}^*(a_{M-1} - x^{*\mathrm{T}}A)L_1 \end{array} \\ \hline \begin{array}{c} y_1^*(b_1 - y^{*\mathrm{T}}B)L_2 \\ \vdots \\ y_{N-1}^*(b_{N-1} - y^{*\mathrm{T}}B)L_2 \end{array} & 0 \end{array}\right].$$

If it is singular, then there exist $(w_1, \ldots, w_M)$, $(v_1, \ldots, v_N)$ such that

$$x_1^*(a_1 - x^{\mathrm{T}}A)w = x_2^*(a_2 - x^{*\mathrm{T}}A)w$$
$$= \cdots = x_{M-1}^*(a_{M-1} - x^{*\mathrm{T}}A)w = 0, \tag{23a}$$

$$w_1 + \cdots + w_M = 0, \tag{23b}$$

$$(w_1, \ldots, w_{M-1}) \neq (0, \ldots, 0), \tag{23c}$$

$$y_1^*(b_1 - y^{*\mathrm{T}}B)v = y_2^*(b_2 - y^{*\mathrm{T}}B)v$$
$$= \cdots = y_{N-1}^*(b_{N-1} - y^{*\mathrm{T}}B)v = 0, \tag{24a}$$

$$v_1 + \cdots + v_N = 0, \tag{24b}$$

$$(v_1, \ldots, v_{N-1}) \neq (0, \ldots, 0). \tag{24c}$$

If $x_1^* > 0, \ldots, x_M^* > 0$, (23) yields

$$a_1 w = a_2 w = \cdots = a_{M-1} w = \sum_{i=1}^{N} x_i^* a_i w = (1 - x_M^*) a_1 w + x_M^* a_M w,$$

and thus

$$a_M w = a_1 w.$$

We conclude that, if $x_1^* > 0, \ldots, x_M^* > 0$, (23) is equivalent to

$$a_1 w = a_2 w = \cdots = a_M w,$$
$$w_1 + \cdots + w_M = 0,$$
$$(w_1, \ldots, w_{M-1}) \neq (0, \ldots, 0);$$

and similarly, if $y_1^* > 0, \ldots, y_N^* > 0$, (24) is equivalent to

$$b_1 v = b_2 v = \cdots = b_N v,$$
$$v_1 + \cdots + v_N = 0,$$
$$(v_1, \ldots, v_{N-1}) \neq (0, \ldots, 0).$$

If such w and v exist, and if $x_i^*>0$, $y_i^*>0$ [i.e., (x^*, y^*) is strictly mixed], then $(x^*+\epsilon v, y^*+\delta w)$ are also strictly mixed solutions of (19) for ϵ, δ sufficiently small. We thus conclude that, if (x^*, y^*), with $x_i^*>0$, $y_i^*>0$, is the unique strictly mixed solution of (19), then, for λ, μ sufficiently small, (20), (27) has an equilibrium point $x(\lambda, \mu)$, $y(\lambda, \mu)$, by the implicit function theorem, with $x(\lambda, \mu)\to x^*$, $y(\lambda, \mu)\to y^*$, as $\lambda, \mu\to 0$. The local stability properties of this point can be determined by a process analogous to that of Section 2.1, but the notational burden is heavy.

It should be evident by now that the study of the differential equation (20), (21) should address the following issues.

(i) The equilibrium points of (20), (21). In Cases 1 and 2, the conditions are local and pertain to either pure strategies where strict inequality holds in (19) or strictly mixed strategies which are unique. There are, of course, many other cases.

(ii) The local stability properties of these equilibria. In case of nonuniqueness of equilibria, several will be unstable and several stable as the analysis of the 2×2 case indicates. In the 2×2 case, it turned out that the equilibria which are close to the pre solutions are stable, whereas those which are close to the mixed ones are stable only if they are unique and no pure ones exist. Whether this generalizes to the $M\times N$ case is not clear.

(iii) The structure of the Jacobian in Case 2, as well as the analysis of the 2×2 case, indicates that the solution of (20), (21) will spiral toward the equilibrium, if a unique strictly mixed equilibrium exists. Further analysis is needed to verify that.

(iv) The possibility of limit cycles was easily ruled out in the 2×2 case, by employing the criterion of Bendixson. There is no generalization of this criterion for differential equations with dimension higher than two. Thus, the examination of the possibility of existence of limit cycles or even of chaotic behavior remains an open issue for the $M\times N$ case.

4. Conclusions

A learning algorithm for matrix Nash games where the matrices are unknown to both players was introduced and studied. A relatively complete analysis was presented for the 2×2 case. Important issues pertaining to the $M\times N$ case were also discussed. It is felt that further research is needed for the $M\times N$ case.

Further research related to the subject matter of this paper may include the following topics.

(i) In the model considered, all the parameters were unknown to both players. It would be reasonable to assume that player 1 knows matrix

A but not B, whereas player 2 knows B but not A. Although the scheme examined here can be employed, other schemes which are better should be sought. By better, we mean schemes that converge faster and/or require less computational effort.

(ii) The many-player case can be considered. Here, problems similar to those delineated in Section 3 for the $M \times N$ two-player case will surface, since the associated differential equation will have dimension greater than two.

(iii) Instead of the Nash equilibrium concept, the Stackelberg equilibrium concept can be considered. The updating formulas for $p(k)$, $q(k)$ should then capture the rationale of the Stackelberg concept.

(iv) Although the $p(k)$'s, $q(k)$'s are in the limit close to the equilibrium strategies of the static game, they are not necessarily in near equilibrium for the constructed dynamic one. It would be important to find rules of updating the $p(k)$'s, $q(k)$'s, so that they are in near equilibrium for the constructed dynamic game, whereas their limits are close to some Pareto solution of the static one; see Ref. 4.

(v) Finally, one may consider a finite state, finite action dynamic, infinite horizon, Nash or Stackelberg game with average or discounted costs, where the cost and transition matrix parameters are unknown to the players. Such problems were examined in Ref. 1 for the team case.

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