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TECHNICAL NOTE

On the Uniqueness of Nash Strategies for a Class of Analytic Differential Games[1]

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Abstract. The uniqueness of Nash equilibria is shown for the case where the data of the problem are analytic functions and the admissible strategy spaces are restricted to analytic functions of the current state and time.

Key Words. Nash games, partial differential equations, differential games.

1. Introduction

Nonzero-sum Nash differential games have attracted considerable interest during the last few years. Despite the many results available in this area, those concerning existence and uniqueness of optimal strategies are far from being satisfactory. This holds true especially if the strategies take into account information about the present and past values of the state of the system. In this context, Refs. 1 and 2 can be pointed out. In these papers, the nonuniqueness of the Nash equilibrium strategies was demonstrated when the current state $x(t)$ and the initial state x_0 are available to at least one of the

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players. It was also shown that, in the case of linear–quadratic games, if noise is introduced to the state equation, then the Nash equilibrium strategies linear in the current state $x(t)$ (assuming that they exist) are the unique solution without restricting a priori the admissible strategies to be linear in the current state.

In the present paper, the problem of uniqueness of the Nash equilibrium strategies for a continuous-time differential game is examined, when both players have only the current state available, i.e., the admissible strategies are of the closed-loop, no-memory type. It is shown that, if the strategy spaces are restricted to analytic functions of the state and time and if the data f, L_1, L_2, g_1, g_2 [see (3) and (4)] are analytic functions, then the Nash equilibrium pair is unique, if it exists. In particular, for a linear–quadratic game, where the matrices involved are analytic functions of time, it is shown that, if the coupled Riccati differential equations have a solution, then the Nash equilibrium strategies which are affine functions of the state constitute the unique analytic solution pair.

Although the result given here is proven under the strong analyticity assumptions, it provides at least a partial answer to the question of uniqueness for a certain class of problems. It provides also an additional characterization of Nash equilibrium strategies which are affine functions of the state in the context of linear–quadratic games with analytic matrices, since it shows that these strategies constitute the solution over strategy spaces much larger than those which are a priori restricted to be affine in the state strategies.

2. Nash Game

Consider the sets U_1 and U_2 defined as follows:

$$I = (t_0', t_f') \subseteq R, \qquad \text{fixed}, \tag{1-1}$$

$$\Sigma = \{S \mid S \subseteq R^n \times I,\ S \text{ open, connected and projection of } S \text{ on } I = I\}, \tag{1-2}$$

$$U_i = \{u_i \mid u_i\colon S_i \to R^{m_i}, \text{for some } S_i \in \Sigma,\ u_i \text{ analytic on } S_i\}, \qquad i = 1, 2. \tag{2}$$

U_1 and U_2 will be called the strategy spaces. Consider also the fixed time interval $[\bar{t}_0, t_f] \subseteq I$ and the functions

$$f\colon R^n \times R^{m_1} \times R^{m_2} \times I \to R^n, \qquad g_i\colon R^n \to R,$$
$$L_i\colon R^n \times R^{m_1} \times R^{m_2} \times I \to R,\ i = 1, 2,$$

which are analytic everywhere in all their arguments.

For a given $(u_1, u_2) \in U_1 \times U_2$ with $S_1 \cap S_2 \in \Sigma$, we consider the dynamic system

$$\dot{x}(t) = f(x(t), u_1(x(t), t), u_2(x(t), t), t),$$
$$x(t_0) = x_0, \qquad (x_0, t_0) \in S_1 \cap S_2, \qquad \bar{t}_0 \leq t_0 \leq t_f. \tag{3}$$

Definition 2.1. A pair $(u_1, u_2) \in U_1 \times U_2$ is called playable at (x_0, t_0) if $S_1 \cap S_2 \in \Sigma$ and the solution of (3) exists over $[t_0, t_f]$. Thus, it is true that $(x(t), t) \in S_1 \cap S_2$, $\forall t \in [t_0, t_f]$.

For a related definition of playability, see Ref. 7. If (u_1, u_2) is playable at (x_0, t_0), we consider the functionals

$$J_i(u_1, u_2) = g_i(x(t_f)) + \int_{t_0}^{t_f} L_i(x(t), u_1(x(t), t), u_2(x(t), t), t))\, dt, \qquad i = 1, 2. \tag{4}$$

Definition 2.2. A pair $(u_1^*, u_2^*) \in U_1 \times U_2$ is said to be a Nash equilibrium pair for the Nash game associated with (3) and (4) on a set

$$S_0 = S \cap (R^n \times [t_0, t_f])$$

for some $S \in \Sigma$ iff

(i) it is playable at all $(x_0, t_0) \in S_0$

and

(ii) $$J_1(u_1^*, u_2^*) \leq J_1(u_1, u_2^*), \qquad \forall (u_1, u_2^*) \in U_1 \times U_2 \text{ playable at } (x_0, t_0), \tag{5}$$

$$J_2(u_1^*, u_2^*) \leq J_2(u_1^*, u_2), \qquad \forall (u_1^*, u_2) \in U_1 \times U_2 \text{ playable at } (x_0, t_0), \tag{6}$$

for all $(x_0, t_0) \in S_0$.

The following theorem concerns the existence and uniqueness of Nash equilibria.

Theorem 2.1. Assume that there exist two analytic[4] functions $\bar{u}_1$ and $\bar{u}_2$, $\bar{u}_i$: $R^n \times I \times R^n \times R^n \to R^{m_i}$, $i = 1, 2$, such that $\bar{u}_1(y, s, q_1, q_2)$ and $\bar{u}_2(y, s, q_1, q_2)$ are the unique global solutions[4] of the minimization problems

$$\min_{u_i \in R^{m_i}} q_i' f(y, u_1, u_2, s) + L_i(y, u_1, u_2, s), \tag{7}$$

[4] In relation to this, see for example Ref. 5, p. 152.

where

$$(y, s, q_1, q_2) \in R^n \times I \times R^n \times R^n.$$

Then, a necessary and sufficient condition for $(u_1^*, u_2^*) \in U_1 \times U_2$ to be a Nash equilibrium pair is that

$$u_i^*(y, s) = \bar{u}_i(y, s, \partial V_1(y, s)/\partial y, \partial V_2(y, s)/\partial y), \qquad i = 1, 2, \tag{8}$$

where $V_1(y, s)$, $V_2(y, s)$ are the unique real-valued analytic solutions of the system of partial differential equations

$$\begin{aligned}\partial V_i/\partial s + (\partial V_i'/\partial y) f(y, \bar{u}_1(y, s, \partial V_1/\partial y, \partial V_2/\partial y), \bar{u}_2(y, s, \partial V_1/\partial y, \partial V_2/\partial y), s)\\ + L_i(y, \bar{u}_1(y, s, \partial V_1/\partial y, \partial V_2/\partial y), \bar{u}_2(y, s, \partial V_1/\partial y, \partial V_2/\partial y), s) = 0,\\ i = 1, 2, \end{aligned}\tag{9}$$

with initial values

$$V_i(y, t_f) = g_i(y), \qquad i = 1, 2, \qquad \forall y \in R^n. \tag{10}$$

If such (u_1^*, u_2^*) exists, then it is unique in $U_1 \times U_2$.

Proof. Let $(u_1^*, u_2^*) \in U_1 \times U_2$ be a Nash pair. Then, the functions

$$\bar{V}_i(y, s) = g_i(x(t_f)) + \int_s^{t_f} L_i(x(t), u_1^*(x(t), t), u_2^*(x(t), t), t)\, dt, \qquad i = 1, 2, \tag{11}$$

where

$$\dot{x}(t) = f(x(t), u_1^*(x(t), t), u_2^*(x(t), t), t), \qquad x(s) = y, \qquad t \in [s, t_f], \tag{12}$$

are analytic in y, s (see Ref. 4, p. 44 and Ref. 8, p. 87, Theorem 4.3) and are the solutions of (9)–(10). This is true, since (9)–(10) are just the Hamilton–Jacobi partial differential equations for the two control problems (5) and (6) (see Ref. 8, p. 83, Theorem 4.1 or Ref. 6, Theorem 1). The sufficiency part follows from Theorem 4.4, p. 87 of Ref. 8. The uniqueness of the solution of (9)–(10) within the analytic class is an immediate consequence of the Cauchy–Kowalewsky theorem (see Ref. 9, p. 40). □

Next, we apply the above theorem to a linear–quadratic game. Consider the game described by (see Ref. 6)

$$\dot{x} = Ax + B_1 u_1 + B_2 u_2 + f(t), \qquad x(t_0) = x_0, \qquad t \in [t_0, t_f], \tag{13-1}$$

$$y_1 = C_1 x, \qquad y_2 = C_2 x, \tag{13-2}$$

$$J_i = \tfrac{1}{2}\left\{ \int_{t_0}^{t_f} [(z_i - y_i)' Q_i (z_i - y_i) + u_i' R_{ii} u_i + u_j' R_{ij} u_j]\, dt + x(t_f)' K_{if} x(t_f)' \right\}, \qquad i = 1, 2, \qquad i \neq j, \tag{13-3}$$

where A, B_i, f, C_i, Q_i, R_{ij}, z_i are analytic functions of t over all (t_0', t_f') and

$$Q_i = Q_i' \geq 0, \qquad R_{ij} = R_{ij}' \geq 0, \qquad R_{ii} > 0, \qquad \forall t \in [t_0, t_f].$$

$K_{if} = K_{if}' \geq 0$ are constant matrices. All the matrices are assumed to be of appropriate dimensions.

Proposition 2.1. Assume that $K_1(t), K_2(t), g_1(t), g_2(t), \Phi_1(t), \Phi_2(t)$ are solutions of the differential equations

$$\dot{K}_i = K_1S_1K_i + K_iS_1K_1 + K_2S_2K_i - K_iS_2K_2 - K_iS_{ij}K_j - K_jS_{ij}K_i - A'K_i - K_iA - K_iS_iK_i - C_i'Q_iC_i, \tag{14-1}$$

$$\dot{g}_i = K_iS_1g_1 + K_1S_1g_i + K_iS_2g_2 + K_2g_2g_i - \tfrac{1}{2}(K_iS_ig_i + K_jS_{ij}g_i + K_iS_{ij}g_i) - A'g_i - C_i'Q_iz_i + K_if \tag{14-2}$$

$$\dot{\Phi}_i = g_1'S_1g_i + g_2'S_2g_i - \tfrac{1}{2}g_i'S_ig_i - g_iS_{ij}g_j + \tfrac{1}{2}z_i'Q_iz_i - f'g_i, \tag{14-3}$$

$$K_i(t_f) = K_{if}, \qquad g_i(t_f) = 0, \qquad \Phi_i(t_f) = 0, \qquad i = 1, 2, \tag{14-4}$$

where

$$S_i = B_iR_{ii}^{-1}B_i', \qquad i = 1, 2,$$

$$S_{ij} = B_jR_{jj}^{-1}R_{ij}R_{jj}^{-1}B_j', \qquad i \neq j, \qquad i, j = 1, 2,$$

$$S_{ij} = 0, \qquad i = j.$$

Then,

$$u_i(x, t) = -R_{ii}^{-1}B_i'[K_ix - g_i], \qquad i = 1, 2,$$

constitute the unique Nash equilibrium pair in $U_1 \times U_2$ for the Nash game associated with (13), for any $(x_0, t_0) \in R^n \times [\bar{t}_0, t_f]$.

Proof. K_i, g_i, Φ_i, are clearly analytic functions of t. The functions

$$V_i(y, s) = \tfrac{1}{2}x'K_ix - g_i'x + \Phi_i$$

are solutions of (9) and (10) [in the form that (9) and (10) assume for the problem (13)]. Thus, the previously stated theorem applies. □

3. Conclusions

In the present paper, the uniqueness of the closed, no-memory analytic Nash strategies is shown for a differential game with analytic data. If, in addition to the analyticity of the data, the game is a linear-quadratic one, then the affine Nash strategies are the unique analytic solutions, assuming that they exist. The introduction of analyticity assumptions removes the nonuniqueness of the Nash solution for deterministic differential games, which is analogous to the removal of nonuniqueness of Nash solutions by introduction of noise (Refs. 1 and 2). The extension of these results to the N-player case is straightforward.

One can obtain the system (9)–(10) of partial differential equations under assumptions much weaker than ours. Nonetheless, the results available concerning existence and uniqueness of solutions of general systems of partial differential equations are complicated. Also, they usually assume boundedness of the range spaces of the sought solutions and of the domains of the independent variables, assumptions not employed here.

References

1. BASAR, T., *On the Uniqueness of the Nash Solution in Linear–Quadratic Differential Games*, International Journal of Game Theory, Vol. 5, pp. 65–90, 1975.
2. BASAR, T., *Informationally Nonunique Equilibrium Solutions in Differential Games*, SIAM Journal on Control, Vol. 15, pp. 636–660, 1977.
3. JOHN, F., *Partial Differential Equations*, Springer-Verlag, New York, New York, 1975.
4. LEFSCHETZ, S., *Differential Equations: Geometric Theory*, John Wiley and Sons (Interscience Publishers), New York, New York, 1957.
5. DETTMAN, J. W., *Applied Complex Variables*, McGraw Hill Book Company, New York, New York, 1970.
6. CASE, J. H., *Toward a Theory of Many Player Differential Games*, SIAM Journal on Control, Vol. 7, pp. 179–197, 1969.
7. LEITMANN, G., *Cooperative and Non-Cooperative Many Players Differential Games*, CISM Courses and Lectures, Springer-Verlag, New York, New York, 1974.
8. FLEMING, W. H., and RISHEL, R. W., *Deterministic and Stochastic Optimal Control*, Springer-Verlag, New York, New York, 1975.
9. COURANT, R., and HILBERT, D., *Methods of Mathematical Physics*, Vol. II, John Wiley and Sons (Interscience Publishers), New York, New York, 1962.