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Performance versus Informativeness in Linear-Quadratic Gaussian Noncooperative Games[1]

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Abstract. In this paper, we study the impact of informativeness on the performance of linear quadratic Gaussian Nash and Stackelberg games. We first show that, in two-person static Nash games, if one of the players acquires more information, then this extra information is beneficial to him, provided that it is orthogonal to both players' information. A special case is that when one of the players is informationally stronger than the other, then any new information is beneficial to him. We then show that a similar result holds for dynamic Nash games. In the dynamic games, the players use strategies that are linear functions of the current estimates of the state, generated by two Kalman filters. The same properties are proved to hold in static and feedback Stackelberg games as well.

Key Words. Nash games, Stackelberg games, information structure, informational properties, informationally stronger players, Kalman filter.

1. Introduction

In multi-person control and game problems, the information structure of the decision makers plays an important role which makes such problems interesting and challenging both conceptually and mathematically. Generally speaking, the impact of information structure on such problems has been studied in three aspects, namely, interactions with

(i) control (e.g., Refs. 1-5), which focuses attention on the existence, uniqueness, and linearity of the solutions under different information structures;

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(ii) incentives (e.g., Refs. 6–9), which focuses attention on the ability one decision maker has in influencing the other decision makers under different information structures;

(iii) performance (e.g., Refs. 10–13), which focuses attention on the impact of informativeness on the performance under different information structures.

The research in the first two areas has been relatively more fruitful than the third one, in the sense that general and assertive results are derived. In contrast, we have few general results in the third area, although several examples show that the impact of such interactions could be drastic. In this paper, we study this very aspect, i.e., the interactions between information structure and performance (to which we also refer as informational properties), for linear quadratic Gaussian (LQG) noncooperative games and, in particular, Nash and Stackelberg games.

In a previous paper (Ref. 10), the informational properties of a class of LQG Nash games were studied. A particular feature of the information structure of the game considered in Ref. 10 is that all the players have access to the same information source at all times, i.e., the information is *public.* This means that whenever a change is made in the information, each and every player's information is equally changed. In this paper, we consider two-person games, and each player has his own *private* information such that, whenever a change is made to one of the player's information, the other player's information remains the same.

There are several examples in the literature cited above which study the impact of changes in private information on the players' performance. Among these, Ref. 13 studies a two-person LQG static Nash game. It is shown for that example that, on the one hand, if one of the players improves his own information by acquiring his opponent's information (while his opponent's information does not change), then he ends up with a higher Nash cost (Case B of Ref. 13); on the other hand, if he improves his own information by getting an extra measurement not from his opponent, then he incurs lower Nash cost (Case D of Ref. 13). Such seemingly counterintuitive phenomena have not been fully analyzed, and a more important issue of what kind of private information is truly beneficial has not been clarified. In this paper, we partially answer this question by providing sufficient conditions; i.e., we prove that, in two-person LQG Nash and Stackelberg games, if one of the players acquires more information, then this extra information is beneficial to him provided that such information is orthogonal to both players' information. A special case is that, when one of the players is informationally stronger than the other (i.e., he knows all his opponent's information), then any new information is beneficial to him.

The structure of the paper is as follows. In Section 2, we study static Nash games and give sufficient conditions where more information is beneficial. In Section 3, we formulate an N-stage dynamic Nash game where one of the players' information is nested in the other's. At each stage k, player 1 is allowed to use a function of estimates $\hat{x}_1(k)$ and $\hat{x}_3(k)$ of the state $x(k)$, while player 2 is allowed to use a function of $\hat{x}_1(k)$ only, where $\hat{x}_1(k)$ and $\hat{x}_3(k)$ are generated through two Kalman filters that use linear, noise-corrupted measurements of $x(k)$, and $\hat{x}_3(k)$ is a refinement of $\hat{x}_1(k)$. In this setup, the Nash solution exists, is unique, and is linear in $\hat{x}_1(k)$ and $\hat{x}_3(k)$ under certain invertibility assumptions on some matrices. A nice feature of the solution is that a sort of separation principle (of estimation and control) holds and the estimation error is independent of the controls. In Section 4, we study the informational properties of the game formulated in Section 3. We prove that better information for player 1 alone is beneficial to himself. In Section 5, we extend the results obtained in Nash games to Stackelberg games. In Section 6, two examples are provided to illustrate the informational properties discussed in the previous sections. Finally, in Section 7, we present our concluding remarks.

2. Some Informational Properties of LQG Static Nash Games

Consider a two-person static Nash game. The cost functional of player i, $i=1,2$, is denoted by

$$J_i(\gamma_1, \gamma_2) = E[x^T P_i^T u_i + \tfrac{1}{2} u_i^T u_i + u_i^T Q_i u_j], \qquad u_i = \gamma_i(y_i), \qquad j \neq i,\ i, j = 1, 2, \tag{1}$$

where $x \in R^n$ is a Gaussian random vector, $x \sim N(0, \Omega)$, $u_i \in R^{l_i}$ is the control variable of player i, and P_i, Q_i are real matrices of appropriate dimensions. y_i, $i=1,2$, being the information available to player i, is given by

$$y_i = H_i x + w_i. \tag{2}$$

H_i is an $m_i \times n$ real matrix and $w_i \in R^{m_i}$ is a Gaussian random vector, which is independent of x. The control law γ_i is chosen from Γ_i, which consists of all the measurable functions from R^{m_i} to R^{l_i} such that $\gamma_i(y_i)$ is a second-order random vector. A pair (γ_1^*, γ_2^*) is called a Nash solution of the game if it satisfies the following two inequalities:

$$J_1(\gamma_1^*, \gamma_2^*) \leq J_1(\gamma_1, \gamma_2^*), \tag{3a}$$

$$J_2(\gamma_1^*, \gamma_2^*) \leq J_2(\gamma_1^*, \gamma_2), \tag{3b}$$

for every $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$. γ_i^* is called a Nash strategy of player i. A necessary and sufficient condition characterizing a Nash solution of the above game was given in Theorem 1 of Ref. 2, which we state below as a lemma.

Lemma 2.1. A pair (γ_1^*, γ_2^*) is a Nash solution of the game described above if and only if the following two equalities hold:

$$\gamma_i^*(y_i) = -P_i E[x \mid y_i] - Q_i E[\gamma_j^*(y_j) \mid y_i], \qquad j \neq i,\ i, j = 1, 2. \tag{4}$$

The question of existence and uniqueness of the Nash solution has also been studied in Ref. 3, where it was shown that almost always there exists a unique solution which has to be an affine function of the information.

By using Lemma 2.1, we will show, for certain information structures, how the Nash solution is affected by the information available to the players, and hence how the Nash performance is affected. We need the following definition of orthogonality and a lemma which consists of several well-known facts in estimation theory (see, e.g., Ref. 14).

Definition 2.1. Two zero-mean Gaussian random vectors z_1 and z_2 are said to be orthogonal (denoted by $z_1 \perp z_2$) if $E[z_1 z_2^T] = 0$.

Lemma 2.2. Let z_i, $i = 1, 2, 3$, be zero-mean Gaussian random vectors. Then,

(i) $\{z_1 - E[z_1 \mid z_2]\} \perp z_2$;
(ii) $E[z_1 \mid z_2] = C z_2$, where C is a real matrix.

If, in addition, $z_2 \perp z_3$, then

(iii) $E[z_2 \mid z_3] = 0$;
(iv) $E[z_1 \mid z_2, z_3] = E[z_1 \mid z_2] + E[z_1 \mid z_3]$.

Denote an extra measurement by y_e,

$$y_e = H_e x + w_e, \tag{5}$$

where H_e is an $m_e \times n$ real matrix and w_e is a Gaussian random vector; $w_e \sim N(0, \Sigma_e)$ and is independent of x. We consider the following two conditions:

(C1) $y_e \perp y_i$, $i = 1, 2$;
(C2) $y_2 = M y_1$, where M is an $m_2 \times m_1$ matrix.

Notice that Condition C2 has nothing to do with y_e; it simply means that the information provided by y_2 is contained in that provided by y_1.

Lemma 2.3. Under either Condition C1 or C2,

$$\{E[x \mid y_1, y_e] - E[x \mid y_1]\} \perp y_i, \qquad i = 1, 2.$$

Proof. The following equation holds:

$$E[x|y_1, y_e] - E[x|y_1] = E[x|y_1, y_e] - E[E[x|y_1, y_e]|y_1]. \tag{6}$$

Thus, by Lemma 2.2(i),

$$\{E[x|y_1, y_e] - E[x|y_1]\} \perp y_1.$$

Under Condition C2,

$$\{E[x|y_1, y_e] - E[x|y_1]\} \perp y_2.$$

Under Condition C1, Lemma 2.2(iv) and (ii) imply that

$$\begin{aligned} E[x|y_1, y_e] - E[x|y_1] &= E[x|y_1] + E[x|y_e] - E[x|y_1] \\ &= E[x|y_e] = Cy_e. \end{aligned} \tag{7}$$

The result holds, since $y_e \perp y_i$, $i = 1, 2$. □

Remark 2.1. C1 and C2 are, in general, different conditions; however, by acquiring y_e, the increment of player 1's information is $\{E[x|y_1, y_e] - E[x|y_1]\}$, which is always orthogonal to y_1. It is in this sense that C2 satisfies C1.

Theorem 2.1. Let condition C1 or C2 hold. Then, the following is true: If there exists a Nash solution under the information pattern where player 1 knows y_1 and player 2 knows y_2, then there exists a Nash solution under the information pattern where player 1 knows $\{y_1, y_e\}$ and player 2 knows y_2, and vice versa. Furthermore, the Nash strategy γ_2 remains the same under both information patterns. For the case where Condition C2 holds, a Nash solution exists and is unique if and only if the matrix $I - Q_2Q_1$ is invertible.

Proof. Lemma 2.1 implies that, when player 1 knows y_1 and player 2 knows y_2, a Nash solution $(\gamma_1(y_1), \gamma_2(y_2))$ exists if and only if

$$\begin{aligned} \gamma_2(y_2) = {} & Q_2Q_1E[E[\gamma_2(y_2)|y_1]|y_2] \\ & + Q_2P_1E[E[x|y_1]|y_2] - P_2E[x|y_2]. \end{aligned} \tag{8}$$

When player 1 knows $\{y_1, y_e\}$ and player 2 knows y_2, a Nash solution $(\gamma_1(y_1, y_e), \gamma_2(y_2))$ exists if and only if

$$\begin{aligned} \gamma_2(y_2) = {} & Q_2Q_1E[E[\gamma_2(y_2)|y_1, y_e]|y_2] \\ & + Q_2P_1E[E[x|y_1, y_e]|y_2] - P_2E[x|y_2]. \end{aligned} \tag{9}$$

Let Condition C1 hold. Then, Eq. (9) reduces to (8), by Lemma 2.2(iii)-(iv) and the fact that $\gamma_2(y_2)$ is affine in y_2. Let Condition C2 hold. Then, both Eqs. (8) and (9) reduce to

$$\gamma_2(y_2) = Q_2 Q_1 \gamma_2(y_2) + Q_2 P_1 E[x \mid y_2] - P_2 E[x \mid y_2]. \tag{10}$$

Hence, if a Nash solution exists in one of the information patterns, it exists in the other, and γ_2 is the same in both information patterns. Furthermore, when Condition C2 holds, a unique Nash solution exists if and only if $I - Q_2 Q_1$ is invertible. □

Theorem 2.2. Let Condition C1 or C2 hold. Then, the Nash cost incurred to player 1, when the information available to player 1 is $\{y_1, y_e\}$ and to player 2 is y_2, is less than or equal to the Nash cost incurred to him when the information available to player 1 is y_1 and to player 2 is y_2.

Proof. Let (γ_1^*, γ_2^*) denote the Nash solution when player 1 knows y_1 and player 2 knows y_2; and let $(\gamma_1^\circ, \gamma_2^\circ)$ denote the Nash solution when player 1 knows $\{y_1, y_e\}$ and player 2 knows y_2. Then, by Theorem 2.1,

$$\begin{aligned} J_1(\gamma_1^\circ, \gamma_2^\circ) &= \min_{\gamma_1(y_1, y_e) \in \Gamma_1'} J_1(\gamma_1, \gamma_2^\circ) = \min_{\gamma_1(y_1, y_e) \in \Gamma_1'} J_1(\gamma_1, \gamma_2^*) \\ &\leq \min_{\gamma_1(y_1) \in \Gamma_1} J_1(\gamma_1, \gamma_2^*) = J_1(\gamma_1^*, \gamma_2^*), \end{aligned} \tag{11}$$

where Γ_1' consists of all the measurable functions from $R^{m_1 + m_e}$ to R^{l_1}. □

Remark 2.2. Although we proved Theorem 2.1 (and hence Theorem 2.2) for Conditions C1 and C2 separately, recall from Remark 2.1 that C2 satisfies C1 in the sense of Lemma 2.3.

Remark 2.3. Notice that Theorems 2.1 and 2.2 hold regardless of the functional form of the costs (1), as long as they are quadratic, and all the results obtained go through even if we do not assume that x is of zero mean. This is easy to verify.

3. Formulation of an LQG Dynamic Nash Game and Its Solution

Consider a two-person, N-stage Nash game where the state of the system $x(\cdot)$ evolves according to

$$x(k+1) = Ax(k) + B_1 u_1(k) + B_2 u_2(k) + w(k), \qquad x(0) = x_0, \tag{12}$$

where $k \in \theta = \{0, 1, \dots, N-1\}$, $x(k) \in R^n$, and $u_i(k) \in R^{l_i}$ denotes the control variable of player i, $i = 1, 2$, at stage k. x_0 and $\{w(k), k \in \theta\}$ are independent Gaussian random vectors, $x_0 \sim N(\bar{x}_0, \Omega_0)$, $w(\cdot) \sim N(0, R)$.

At each stage k, the measurements $y_i(k) \in R^{m_i}$, $i = 1, 2$, are given by

$$y_i(k) = H_i x(k) + \nu_i(k), \tag{13}$$

where $\{\nu_i(k), k \in \theta, i = 1, 2\}$ are independent Gaussian random vectors, $\nu_i(\cdot) \sim N(0, \Sigma_i)$. ν_i's are also independent of x_0 and $\{w(k), k \in \theta_1\}$. The information available to the players is not $y_i(k)$'s, but $\hat{x}_1(k)$, $\hat{x}_3(k)$, the estimates of $x(k)$ given by two Kalman filters:

$$\hat{x}_i(k) = \hat{x}_i(k/k-1) + G_i(k)[y_i(k) - H_i \hat{x}_i(k/k-1)], \tag{14a}$$

$$\hat{x}_i(k+1/k) = A\hat{x}_i(k) + B_1 u_1(k) + B_2 u_2(k), \; \hat{x}_i(0/-1) = \bar{x}_0, \tag{14b}$$

$$G_i(k) = \Sigma_i(k/k-1) H_i^T (H_i \Sigma_i(k/k-1) H_i^T + \Sigma_i)^{-1}, \tag{14c}$$

$$\Sigma_i(k+1/k) = A[I - G_i(k) H_i] \Sigma_i(k/k-1) A^T + R, \quad \Sigma_i(0/-1) = \Omega_0, \tag{14d}$$

$$\Sigma_i(k) = [I - G_i(k) H_i] \Sigma_i(k/k-1), \tag{14e}$$

where $i = 1, 3$ and

$$H_3 \triangleq [H_1^T, H_2^T]^T, \tag{15}$$

$$y_3(\cdot) \triangleq [y_1^T(\cdot), y_2^T(\cdot)]^T, \tag{16}$$

$$\Sigma_3 \triangleq \operatorname{diag}[\Sigma_1, \Sigma_2]. \tag{17}$$

$\hat{x}_i(k+1/k)$ is the one-step prediction estimate, and $\Sigma_i(k)$ and $\Sigma_i(k+1/k)$ are the error covariance matrices associated with $\hat{x}_i(k)$ and $\hat{x}_i(k+1/k)$, respectively,

$$\Sigma_i(k) = E\{[x(k) - \hat{x}_i(k)][x(k) - \hat{x}_i(k)]^T\}, \tag{18}$$

$$\Sigma_i(k+1/k) = E\{[x(k+1) - \hat{x}_i(k+1/k)][x(k+1) - \hat{x}_i(k+1/k)]^T\}. \tag{19}$$

The information structure is defined as follows: At each stage k, player 1 knows

$$I_1(k) \triangleq \{\hat{x}_1(k), \hat{x}_3(k)\},$$

while player 2 knows

$$I_2(k) \triangleq \{\hat{x}_1(k)\}.$$

This information structure can be justified by considering that there are two referees 1 and 3, who compute respectively $\hat{x}_1(k)$ and $\hat{x}_3(k)$; referee 1 gives $\hat{x}_1(k)$ to both players, and referee 3 gives $\hat{x}_3(k)$ to player 1 only.

The cost of player i is $J_i \triangleq J_i(0)$, where $J_i(k)$ denotes the cost to go of player i at stage k and is defined by

$$J_i(k) = E\left\{\sum_{n=k}^{N-1} [x^T(n)P_i x(n) + u_i^T(n)u_i(n) + u_j^T(n)Q_i u_j(n)] \right.$$

$$\left. + x^T(N)P_i x(N)\right\}, \qquad j \neq i, \; i, j = 1, 2, \tag{20}$$

where P_i, $Q_i \geq 0$. $u_i(k)$ is chosen as $\gamma_i^k(I_i(k))$ and the γ_i^k's are measurable functions, $\gamma_1^k: R^n \times R^n \to R^{l_1}$ and $\gamma_2^k: R^n \to R^{l_2}$, with the property that $\gamma_i^k(I_i(k))$ is a second-order random vector. Let

$$g_i \triangleq \{\gamma_i^0, \gamma_i^1, \ldots, \gamma_i^{N-1}\}, \qquad i = 1, 2. \tag{21}$$

A pair $\{g_1^*, g_2^*\}$ is called a Nash solution of the game if

$$J_1(g_1^*, g_2^*) \leq J_1(g_1, g_2^*), \qquad \text{for every admissible } g_1, \tag{22a}$$

$$J_2(g_1^*, g_2^*) \leq J_2(g_1^*, g_2), \qquad \text{for every admissible } g_2. \tag{22b}$$

Before we give the Nash solution of the game, we need the following lemma which shows orthogonality in the information structure; the proof is given in Appendix A, Section 8.

Lemma 3.1. (i) $E[\hat{x}_3(k) | \hat{x}_1(k)] = \hat{x}_1(k)$. Let $\hat{x}_4(k) = \hat{x}_3(k) - \hat{x}_1(k)$. Then, (ii) $\hat{x}_4(k) \perp \hat{x}_1(k)$.

Notice that, by Lemma 3.1, the information structure $I_1(k)$ can equivalently be considered as $I_1(k) = \{\hat{x}_1(k), \hat{x}_4(k)\}$, which consists of two orthogonal elements.

The Nash solution of the game described above is provided in the following theorem, the proof of which is given in Appendix B, Section 9.

Theorem 3.1. Consider the equations

$$\begin{aligned} L_i(k) = P_i &+ A^T[(I + B_1 B_1^T L_1(k+1) + B_2 B_2^T L_2(k+1))^{-1}]^T \\ &\times [L_i(k+1) + L_i(k+1)B_i B_i^T L_i(k+1) \\ &+ L_j(k+1)B_j Q_i B_j^T L_j(k+1)] \\ &\times [I + B_1 B_1^T L_1(k+1) + B_2 B_2^T L_2(k+1)]^{-1} A, \end{aligned} \tag{23a}$$

$$L_i(N) = P_i, \qquad j \neq i, \; i, j = 1, 2, \tag{23b}$$

which evolve backward in time. We assume that the inverse of $(I+B_1B_1^TL_1(k)+B_2B_2^TL_2(k))$ exists for every $k\in\theta$. Then:

(i) There exists a unique Nash solution to the game, which is the following:

$$u_1^*(k)=*\gamma_1^k(I_1(k))=F_{11}(k)\hat{x}_1(k)+F_{14}(k)\hat{x}_4(k), \tag{24}$$

$$u_2^*(k)=*\gamma_2^k(I_2(k))=F_2(k)\hat{x}_1(k), \tag{25}$$

where

$$F_{11}(k)=-B_1^TL_1(k+1)[I+B_1B_1^TL_1(k+1)+B_2B_2^TL_2(k+1)]^{-1}A, \tag{26}$$

$$F_{14}(k)=-B_1^TL_1(k+1)[I+B_1B_1^TL_1(k+1)]^{-1}A, \tag{27}$$

$$F_2(k)=-B_2^TL_2(k+1)[I+B_1B_1^TL_1(k+1)+B_2B_2^TL_2(k+1)]^{-1}A. \tag{28}$$

(ii) The cost to go of player i at stage k is

$$J_i(k)=E[\hat{x}_3^T(k)L_i(k)\hat{x}_3(k)]+K_i(k), \tag{29}$$

where

$$\begin{aligned}K_i(k)=\operatorname{tr}\{&A^TL_i(k+1)A-L_{i4}(k)+P_i]\Sigma_3(k)\\&+L_i(k+1)[R-\Sigma_3(k+1)]+L_{i4}(k)\Sigma_1(k)\}+K_i(k+1),\end{aligned} \tag{30a}$$

$$K_i(N)=\operatorname{tr}\{P_i\Sigma_3(N)\}, \tag{30b}$$

$$\begin{aligned}L_{14}(k)=&A^T[(I+B_1B_1^TL_1(k+1))^{-1}]^T\\&\times[L_1(k+1)+L_1(k+1)B_1B_1^TL_1(k+1)]\\&\times[I+B_1B_1^TL_1(k+1)]^{-1}A-L_1(k)+P_1,\end{aligned} \tag{31}$$

$$\begin{aligned}L_{24}(k)=&A^T[(I+B_1B_1^TL_1(k+1))^{-1}]^T\\&\times[L_2(k+1)+L_1(k+1)B_1Q_2B_1^TL_1(k+1)]\\&\times[I+B_1B_1^TL_1(k+1)]^{-1}A-L_2(k)+P_2.\end{aligned} \tag{32}$$

Remark 3.1. Notice that the control laws $F_{11}(k)$, $F_{14}(k)$, and $F_2(k)$ in the above theorem are independent of the observation noise in the measurements (13); i.e., a sort of separation principle holds under such information structure. Also, we can see from the Kalman filter equations (14) that the estimation error $\Sigma_i(k)$, $i=1,3$, is independent of the controls.

Remark 3.2. The nonsingularity condition of the matrix $I+B_1B_1^TL_1(k)+B_2B_2B_2^TL_2(k)$ and the boundedness condition of $L_i(k)$, the solution of the coupled Riccati equations (23), were discussed in Theorem 2.2 and Remark 2 of Ref. 10.

4. Some Informational Properties of LQG Dynamic Nash Games

In this section, we first give the definition of "better information for player 1 alone," and then derive sufficient conditions that such better information would benefit each player.

Consider Information I and II. In Information I, the estimates $\hat{x}_1^{\mathrm{I}}(k)$ and $\hat{x}_3^{\mathrm{I}}(k)$ are generated through the past controls and the measurements

$$y_i^{\mathrm{I}}(\cdot)=H_i^{\mathrm{I}}x(\cdot)+\nu_i^{\mathrm{I}}(\cdot), \qquad \nu_i^{\mathrm{I}}\sim N(0,\Sigma_i^{\mathrm{I}}), \qquad i=1,2, \tag{33}$$

with corresponding estimation error $\Sigma_1^{\mathrm{I}}(k)$ and $\Sigma_3^{\mathrm{I}}(k)$. In Information II, the estimates $\hat{x}_1^{\mathrm{II}}(k)$ and $\hat{x}_3^{\mathrm{II}}(k)$ are generated through the past controls and the measurements

$$y_i^{\mathrm{II}}(\cdot)=H_i^{\mathrm{II}}x(\cdot)+\nu_i^{\mathrm{II}}(\cdot), \qquad \nu_i^{\mathrm{II}}\sim N(0,\Sigma_i^{\mathrm{II}}), \qquad i=1,2, \tag{34}$$

with corresponding estimation error $\Sigma_1^{\mathrm{II}}(k)$ and $\Sigma_3^{\mathrm{II}}(k)$.

Definition 4.1. We say that Information I provides better information for player 1 alone than Information II if

$$\Sigma_1^{\mathrm{I}}(k)=\Sigma_1^{\mathrm{II}}(k), \qquad \Sigma_3^{\mathrm{I}}(k)\le\Sigma_3^{\mathrm{II}}(k), \qquad \text{for every } k\in\theta,$$
$$\Sigma_3^{\mathrm{I}}(k)\ne\Sigma_3^{\mathrm{II}}(k), \qquad \text{for at least one } k\in\theta.$$

An obvious fact about the definition given above is that all the improvement is in the part of $\hat{x}_4(\cdot)$, player 1's private information, while there is no improvement in the part of $\hat{x}_1(\cdot)$, the public information of both players.

Let $J_i^{\mathrm{I}}(k)$ and $K_i^{\mathrm{I}}(k)$, $i=1,2$, be defined as in (29) and (30), corresponding to Information I, and

$$\Omega_j^{\mathrm{I}}(k)=E[\hat{x}_j^{\mathrm{I}}(k)\hat{x}_j^{\mathrm{I}T}(k)], \qquad j=1,3.$$

Similarly, we define $J_i^{\mathrm{II}}(k)$ and $K_i^{\mathrm{II}}(k)$, $i=1,2$, and $\hat{\Omega}_j^{\mathrm{II}}(k)$, $j=1,3$, for Information II. We need the following lemma to prove the next theorem, the proof of which can be found in Ref. 10.

Lemma 4.1. Let X, $Y\in R^{n\times n}$, $X\ge 0$, and Y be a nonzero, nonnegative definite matrix. Then, $\mathrm{tr}\{XY\}\ge 0$.

Theorem 4.1. The Nash solution given by Theorem 3.1 has the property that better information for player 1 alone lowers player i's cost if

$$P_i+A^TL_i(k+1)A-L_i(k)-L_{i4}(k)\ge 0, \qquad \text{for every } k\in\theta. \tag{35}$$

Proof. From part (ii) of Theorem 3.1,

$$J_i^{\mathrm{I}}(0)=E[\hat{x}_3^{\mathrm{I}T}(0)L_i(0)\hat{x}_3^{\mathrm{I}}(0)]+K_i^{\mathrm{I}}(0)=\mathrm{tr}\{L_i(0)\hat{\Omega}_3^{\mathrm{I}}(0)\}+K_i^{\mathrm{I}}(0). \tag{36}$$

From the recursive expression of $K_i(\cdot)$ in (30), we obtain

$$\begin{aligned}J_i^{\mathrm{I}}(0)=\mathrm{tr}\{&L_i(0)\hat{\Omega}_3^{\mathrm{I}}(0)+[P_i+A^TL_i(1)A-L_{i4}(0)]\Sigma_3^{\mathrm{I}}(0)\\&+L_i(1)R+L_{i4}(0)\Sigma_1^{\mathrm{I}}(0)\\&+\sum_{k=1}^{N-1}[[P_i+A^TL_i(k+1)A-L_i(k)-L_{i4}(k)]\Sigma_3^{\mathrm{I}}(k)\\&+L_i(k+1)R+L_{i4}(k)\Sigma_1^{\mathrm{I}}(k)]\bigg\}.\end{aligned} \tag{37}$$

Similarly,

$$\begin{aligned}J_i^{\mathrm{II}}(0)=\mathrm{tr}\bigg\{&L_i(0)\hat{\Omega}_3^{\mathrm{II}}(0)+[P_i+A^TL_i(1)A-L_{i4}(0)]\Sigma_3^{\mathrm{II}}(0)\\&+L_i(1)R+L_{i4}(0)\Sigma_1^{\mathrm{II}}(0)\\&+\sum_{k=1}^{N-1}[[P_i+A^TL_i(k+1)A-L_i(k)-L_{i4}(k)]\Sigma_3^{\mathrm{II}}(k)\\&+L_i(k+1)R+L_{i4}(k)\Sigma_1^{\mathrm{II}}(k)]\bigg\}.\end{aligned} \tag{38}$$

By using the fact that

$$\hat{\Omega}_3^{\mathrm{II}}(0)-\hat{\Omega}_3^{\mathrm{I}}(0)=-(\Sigma_3^{\mathrm{II}}(0)-\Sigma_3^{\mathrm{I}}((0)), \tag{39}$$

we obtain

$$\begin{aligned}J_i^{\mathrm{II}}(0)-J_i^{\mathrm{I}}(0)=\sum_{k=0}^{N-1}&\mathrm{tr}\{[P_i+A^TL_i(k+1)A-L_i(k)-L_{i4}(k)]\\&\times[\Sigma_3^{\mathrm{II}}(k)-\Sigma_3^{\mathrm{I}}(k)]+L_{i4}(k)[\Sigma_1^{\mathrm{II}}(k)-\Sigma_1^{\mathrm{I}}(k)]\}.\end{aligned} \tag{40}$$

Suppose now that Information I provides better information for player 1 alone than Information II. Then, Lemma 4.1 implies that $J_i^{\mathrm{II}}(0)\geq J_i^{\mathrm{I}}(0)$, if (35) holds. □

Corollary 4.1. Better information for player 1 alone lowers player 1's Nash cost.

Proof. Substituting (23) and (31) into (35), and letting $i=1$, we obtain

$$\begin{aligned}&P_1+A^TL_1(k+1)A-L_1(k)-L_{14}(k)\\&=A^TL_1(k+1)A-A^T[(I+B_1B_1L_1(k+1))^{-1}]^T\\&\times[L_1(k+1)+L_1(k+1)B_1B_1^TL_1(k+1)][I+B_1B_1^TL_1(k+1)]^{-1}A\\&=U^TVU\geq 0,\end{aligned} \tag{41}$$

where

$$U = [I + B_1 B_1^T L_1(k+1)]^{-1} A, \tag{42}$$

$$\begin{aligned} V = {} & L_1(k+1) B_1 B_1^T L_1(k+1) \\ & + L_1(k+1) B_1 B_1^T L_1(k+1) B_1 B_1^T L_1(k+1) \geq 0. \end{aligned} \tag{43}$$

Then, Theorem 4.1 implies the desired result. □

Remark 4.1. In Corollary 4.1, we see that better information for player 1 alone is beneficial to him, and this fact is independent of the number of stages N and it is not necessary for the better information to be dynamically better (Ref. 10). In contrast with Theorem 5.1 of Ref. 10, the above two features reveal the essential difference between improving the players' private information and public information in a dynamic Nash game.

5. Related Properties of Static and Feedback Stackelberg Games

In this section, we extend the results obtained in Nash games to static and feedback Stackelberg games. The difference between a Stackelberg game and a Nash game lies partially in that the roles of the players are asymmetric in Stackelberg games, while they are symmetric in a Nash game. However, the Stackelberg solution of a static game is also a Nash solution of the same problem under explicit control sharing (Ref. 11) and a feedback Stackelberg solution of an N-stage dynamic game is also a Nash solution of a $2N$-stage game (Ref. 15). Hence, we expect some different, as well as some similar, properties between Stackelberg and Nash games.

Consider a two-person Stackelberg game. Let player 1 be the leader and player 2 be the follower. Their cost functionals are given by $J_1(\gamma_1, \gamma_2)$ and $J_2(\gamma_1, \gamma_2)$, respectively, where

$$J_i(\gamma_1, \gamma_2) = E[\tfrac{1}{2} u_i^T u_i + \tfrac{1}{2} u_j^T P_i u_j + u_i^T Q_i u_j + u_i^T S_{ii} x + u_j^T S_{ij} x], \tag{44a}$$

$$u_i = \gamma_i(y_i), \qquad j \neq i,\ i, j = 1, 2, \tag{44b}$$

where $x \in R^n$ is a Gaussian random vector, $x \sim N(0, \Omega)$, $u_i \in R^{l_i}$ is the control variable of player i and P_i, Q_i, S_{ii}, S_{ij} are real constant matrices of appropriate dimensions. The linear measurement of player i is given by

$$y_i = H_i x + w_i. \tag{45}$$

H_i is an $m_i \times n$ real matrix and w_i is a Gaussian random vector, $w_i \sim N(0, \Sigma_i)$, which is independent of x. The control law γ_i is chosen from Γ_i, which consists of all the measurable functions mapping from $R^{m_i} \times R^{l_i}$ such that

$\gamma_i(y_i)$ is a second-order random vector. A pair (γ_1^*, γ_2^*) is called a Stackelberg solution, with player 1 as the leader, if γ_1^* satisfies the following inequality:

$$\sup_{\gamma_2 \in R_2(\gamma_1^*)} J_1(\gamma_1^*, \gamma_2) \leq \sup_{\gamma_2 \in R_2(\gamma_1)} J_1(\gamma_1, \gamma_2), \tag{46}$$

for every $\gamma_1 \in \Gamma_1$ and $\gamma_2^* \in R_2(\gamma_1^*)$, where $R_2(\gamma_1)$ is called the rational reaction set of the follower to the strategy γ_1 announced by the leader and is defined by

$$R_2(\gamma_1) = \{\bar{\gamma}_2 \in \Gamma_2 \,|\, J_2(\gamma_1, \bar{\gamma}_2) \leq J_2(\gamma_1, \gamma_2), \text{ for every } \gamma_2 \in \Gamma_2\}. \tag{47}$$

Notice that, if $R_2(\gamma_1)$ is a singleton for each $\gamma_1 \in \Gamma_1$, then (46) can equivalently be written as

$$J_1(\gamma_1^*, \bar{\gamma}_2(\gamma_1^*)) \leq J_1(\gamma_1, \bar{\gamma}_2(\gamma_1)). \tag{48}$$

It turns out that $R_2(\gamma_1)$ is a singleton indeed (Ref. 16) and is given by

$$\bar{\gamma}_2(y_2) = -S_{22}E[x \,|\, y_2] - Q_2 E[\gamma_1(y_1) \,|\, y_2]. \tag{49}$$

A sufficient condition that a unique linear Stackelberg solution exists was given in Ref. 16, which condition is determined by the matrices P_i and Q_i, $i = 1, 2$, and has nothing to do with the information available to the players. We assume in the following derivations that a unique linear Stackelberg solution exists under every information structure that we consider. The result of the following lemma is known, but we include a short proof for reasons of completeness.

Lemma 5.1. The leader's cost decreases if he acquires an extra measurement y_e.

Proof. Let (γ_1^*, γ_2^*) and $(\gamma_1^\circ, \gamma_2^\circ)$ denote respectively the Stackelberg solution before and after the leader acquires y_e. After the leader acquires y_e, he can choose a suboptimal strategy $\gamma_1^S(y_1, y_e) = \gamma_1^*(y_1)$. Then, the follower will react by choosing $\gamma_2^S(y_2) = \gamma_2^*(y_2)$, and hence

$$\begin{aligned} J_1(\gamma_1^\circ(y_1, y_e), \gamma_2^\circ(y_2)) &\leq J_1(\gamma_1^S(y_1, y_e), \gamma_2^S(y_2)) \\ &= J_1(\gamma_1^*(y_1), \gamma_2^*(y_2)). \end{aligned} \tag{50}$$

□

The follower, who is in the lower level of a hierarchy, sees things differently from the leader, and knowing more is not necessarily beneficial to him. As in the Nash case, we prove in the following theorems that, if the follower acquires an extra measurement y_e which satisfies a certain orthogonality condition or if the follower knows all that the leader knows, then the leader's strategy does not change, and such y_e is beneficial to the follower. As in Section 2, we consider the following conditions:

(C1) $y_e \perp y_i$, $i=1,2$;

($\overline{\text{C2}}$) $y_1 = My_2$, where M is an $m_1 \times m_2$ matrix.

Theorem 5.1. If the follower acquires extra measurement y_e such that either one of Conditions C1, $\overline{\text{C2}}$ holds, then the leader's strategy does not change.

Proof. Let $\gamma_1^*(y_1)$, $\gamma_1^\circ(y_1)$ denote the leader's strategy before and after the follower acquires y_e, and let $\gamma_2^*(y_2)$, $\gamma_2^\circ(y_2, y_e)$ denote respectively the follower's reaction before and after he acquires y_e. Then, by (49),

$$\gamma_2^*(y_2) = -S_{22}E[x \mid y_2] - Q_2 E[\gamma_1(y_1) \mid y_2], \tag{51}$$

$$\gamma_2^\circ(y_2, y_e) = -S_{22}E[x \mid y_2, y_e] - Q_2 E[\gamma_1(y_1) \mid y_2, y_e]. \tag{52}$$

Under either one of Conditions C1, $\overline{\text{C2}}$, the following is true:

$$E[\gamma_1(y_1) \mid y_2, y_e] = E[\gamma_1(y_1) \mid y_2]. \tag{53}$$

Hence, (52) can be written as

$$\begin{aligned}\gamma_2^\circ(y_2, y_e) &= \gamma_2^*(y_2) - S_{22}\{E[x \mid y_2, y_e] - E[x \mid y_2]\}\\ &= \gamma_2^*(\gamma_1, y_2) - S_{22}\hat{y},\end{aligned} \tag{54}$$

where

$$\hat{y} \triangleq E[x \mid y_2, y_e] - E[x \mid y_2],$$

which by Lemma 2.3 is orthogonal to y_1 and y_2. The leader's strategy after the follower acquires y_e is the following [we omit the arguments in the strategies $y_i^*(\cdot)$ and $\gamma_i^\circ(\cdot)$ for a while to avoid the tedious expressions]:

$$\begin{aligned}\gamma_1^\circ = \arg\min_{\gamma_1 \in \Gamma_1} & E\{\tfrac{1}{2}\gamma_1^T\gamma_1 + \tfrac{1}{2}\gamma_2^{\circ T}P_1\gamma_2^\circ + \gamma_1^T Q_1 \gamma_2^\circ + \gamma_1^T S_{11}x\\ &\qquad + \gamma_2^{\circ T} S_{12} x\}\\ = \arg\min_{\gamma_1 \in \Gamma_1} & E\{\tfrac{1}{2}\gamma_1^T\gamma_1 + \tfrac{1}{2}\gamma_2^{*T}P_1\gamma_2^* - \gamma_2^{*T}P_1 S_{22}\hat{y}\\ & + \tfrac{1}{2}(S_{22}\hat{y})^T P_1 (S_{22}\hat{y})\\ & + \gamma_1^T Q_1 \gamma_2^* - \gamma_1^T Q_1 S_{22}\hat{y} + \gamma_1^T S_{11} x + \gamma_2^{*T} S_{12} x - (S_{22}\hat{y})^T S_{12} x\}\\ = \arg\min_{\gamma_1 \in \Gamma_1} & E\{\tfrac{1}{2}\gamma_1^T\gamma_1 + \tfrac{1}{2}\gamma_2^{*T}P_1\gamma_2 + \gamma_1^T Q_1 \gamma_2^* + \gamma_1^T S_{11} x + \gamma_2^{*T} S_{12} x\}\\ = \gamma_1^*, &\end{aligned} \tag{55}$$

where we use the fact that $\hat{y} \perp \{y_1, y_2\}$ to get rid of the terms $\gamma_2^{*T}P_1 S_{22}\hat{y}$ and $\gamma_1^T Q_1 S_{22}\hat{y}$ in taking the expectation operations. □

Theorem 5.2. If the follower acquires extra measurement y_e such that either one of the Conditions C1, $\overline{\text{C2}}$ holds, then the follower can do better by incurring lower cost.

Proof. The proof is similar to Theorem 2.2, and hence is omitted. □

Now, consider a feedback Stackelberg game with the same formulation as in the feedback Nash game of Section 3, except that we consider two cases which correspond to two different information structures. Let $I_i(k)$ denote the information available to player i at stage k. Then, we consider two cases:

Case A, $I_1^A(k)=\{\hat{x}_1(k), \hat{x}_3(k)\}, I_2^A(k)=\{\hat{x}_1(k)\}$;
Case B, $I_1^B(k)=\{\hat{x}_1(k)\}, I_2^B(k)=\{\hat{x}_1(k), \hat{x}_3(k)\}$.

Let us call player 1 the leader and player 2 the follower. A pair (g_1^*, g_2^*) is a feedback Stackelberg solution to the game if

$$\sup_{\gamma_2^k\in R_k(*\gamma_1^k)} J_1(g_1^*, g_{2k}^*, \gamma_2^k)$$
$$\leq \sup_{\gamma_2^k\in R_k 0\gamma_1^k)} J_1(g_{1k}^*, \gamma_1^k, g_{2k}^*, \gamma_2^k), \qquad \text{for every admissible } \gamma_1^k,$$

where

$$g_{ik} \triangleq \{\gamma_i^0, \gamma_i^1, \ldots, \gamma_i^{k-1}, \gamma_i^{k+1}, \ldots, \gamma_i^{N-1}\}.$$

$R_k(\gamma_1^k)$ is called the rational reaction set of the follower at stage k to the strategy γ_1^k announced by the leader and is defined by

$$R_k(\gamma_1^k)=\{\bar{\gamma}_2^k \,|\, J_2(g_{1k}^*, \gamma_1^k, g_{2k}^*, \bar{\gamma}_2^k)$$
$$\leq J_2(g_{1k}^*, \gamma_1^k, g_{2k}^*, \gamma_2^k), \text{for every admissible } \gamma_2^k\}.$$

The feedback Stackelberg solution for Cases A and B are provided in Appendix C, Section 10.

Let Informations I and II be defined as in Section 3 and satisfy the conditions in Definition 4.1. Then, in Case A, Information I provides better information for the leader alone than Information II; in Case B, Information I provides better information for the follower alone than Information II. We have the following theorem.

Theorem 5.3. Under the information structure of Cases A and B, the feedback Stackelberg solution has respectively the following properties:

(i) better information for the leader alone is beneficial to the leader;

(ii) better information for the follower alone is beneficial to the follower.

Proof. One way of proving this theorem is by using the connection of the feedback Stackelberg solution to the feedback Nash solution according to the procedure of Ref. 15, where it was proved that a feedback Stackelberg solution of an N-stage dynamic game is also a feedback Nash solution of a $2N$-stage dynamic game and the result is then implied by Corollary 4.1. An independent proof of this theorem is provided in Appendix D, Section 11. □

Remark 5.1. A similar feedback Stackelberg game was studied in Ref. 17, where the expressions of the solution obtained were so complicated that it was not possible to investigate its informational properties. The expressions of the solution could have been simplified if the authors of Ref. 17 had observed the orthogonality in the information structure, i.e., Lemma 3.1(ii).

6. Examples

Example 6.1. This example illustrates Theorem 2.1 and 2.2 under Condition C1. Consider a static Nash game where all the notations follow those defined in Section 2. The cost functionals are

$$J_1(\gamma_1, \gamma_2) = E[(x+u_1+u_2)^2+u_1^2],$$
$$J_2(\gamma_1, \gamma_2) = E[(x+u_2+u_2)^2+u_2^2];$$

player i has measurement y_i, $y_i = x + w_i$; x, w_1, w_2 are independent random variables with zero mean and unit variance.

This example was previously considered in Ref. 13, and the Nash solution was given by

$$\gamma_1^*(y_1) = -\tfrac{1}{5}y_1, \qquad \gamma_2^*(y_2) = -\tfrac{1}{5}y_2,$$

with corresponding Nash costs

$$J_1(\gamma_1^*, \gamma_2^*) = J_2(\gamma_1^*, \gamma_2^*) = 468/900.$$

Now, if in addition to y_2, player 2 acquires an extra measurement y_e, what is the impact to his Nash cost? It was shown (Case B of Ref. 13) that, if $y_e = y_1$, then player 2 incurs higher Nash cost. In the following, we will find an y_e such that $y_e \perp \{y_1, y_2\}$ and demonstrate that this y_e lowers player 2's Nash cost.

Let $y_e = x - w_1 - w_2$. Then, it is easy to check that $y_e \perp \{y_1, y_2\}$. Denote the Nash solution after player 2 acquires this y_e by $(\gamma_1^\circ, \gamma_2^\circ)$. Then, by direct calculation, we obtain

$$\gamma_1^\circ(y_1) = -\tfrac{1}{5}y_1,$$
$$\gamma_2^\circ(y_2, y_e) = -\tfrac{1}{5}y_2 - \tfrac{1}{6}y_e.$$

The corresponding Nash solution of player 2 is $J_2(\gamma_1^\circ, \gamma_2^\circ)$ and

$$J_2(\gamma_1^\circ, \gamma_2^\circ) = 318/900 < 468/900 = J_2(\gamma_1^*, \gamma_2^*).$$

Example 6.2. This example illustrates Corollary 4.1. Consider a dynamic Nash game with the general formulation given in Sections 3 and 4. We choose

$$A = 0.5, \qquad \bar{x}_0 = 0, \qquad \Omega_0 = 10,$$

$$B_i = P_i = R = 1, \qquad Q_i = 20, \qquad i = 1, 2.$$

Two kinds of information are described below.

Information I, $\hat{x}_1^{\mathrm{I}}(\cdot)$, $\hat{x}_3^{\mathrm{I}}(\cdot)$ correspond to

$$y_1^{\mathrm{I}}(\cdot) = x(\cdot) + \nu_1^{\mathrm{I}}(\cdot),$$

$$y_2^{\mathrm{I}}(\cdot) = x(\cdot) + \nu_2^{\mathrm{I}}(\cdot),$$

$$\nu_i^{\mathrm{I}}(\cdot) \sim N(0, 1), \qquad i = 1, 2;$$

Table 1. Costs of player 1 in Example 6.2 under different information versus different number of stages.

N	Information I	Information II	Benefit of player 1 due to better information for him alone
1	16.72872	16.98826	0.259544
2	19.79963	20.12271	0.323073
3	21.68059	22.06824	0.387644
4	23.31423	23.76004	0.445805
5	24.90147	25.40363	0.502162
6	26.48017	27.03831	0.558140
7	28.05730	28.67135	0.614047
8	29.63415	30.30409	0.669940
9	31.21094	31.93677	0.725830
10	32.78773	33.56945	0.781720
11	34.36451	35.20212	0.837610
12	35.94123	36.83479	0.893500
13	37.51808	38.46747	0.949390
14	39.09486	40.10014	1.005280
15	40.67164	41.73281	1.061170
16	42.24843	43.36549	1.117060
17	43.82521	44.99816	1.172950
18	45.40199	46.63083	1.228840
19	46.97877	48.26350	1.284730

Information II, $\hat{x}_1^{\text{II}}(\cdot)$, $\hat{x}_3^{\text{II}}(\cdot)$ correspond to

$$y_1^{\text{II}}(\cdot) = x(\cdot) + \nu_1^{\text{II}}(\cdot),$$

$$y_2^{\text{II}}(\cdot) = 0 \cdot x(\cdot) + \nu_2^{\text{II}}(\cdot),$$

$$\nu_i^{\text{II}}(\cdot) \sim N(0, 1), \qquad i = 1, 2.$$

It is easy to see that, for Information II, $\hat{x}_1^{\text{II}}(k) = \hat{x}_3^{\text{II}}(k)$ at every stage k, and Information I provides better information for player 1 alone than Information II. We compute the Nash cost of player 1 for different number of stages, i.e., N from 1 to 19. Information I is more beneficial to player 1 than Information II. The resulting costs are shown in Table 1.

7. Conclusions

In a general two-person LQG Nash game (static or dynamic), we proved that more or better information for one of the players alone is beneficial to him if he is informationally stronger than his opponent or if such information is orthogonal to both players' information. Such results are quite understandable. Since the Nash solution is an equilibrium solution with consistency constraints (Ref. 12), any unilateral improvement of information does not guarantee benefit to either party. A unilateral improvement of information does guarantee benefit to the one who has the improvement, however, if his opponent's strategy does not change by such improvement, such that he who has the improved information can use it to optimize his strategy without further constraints. In order that his opponent's strategy does not change, his opponent should be totally ignorant of this improved information, as implied by the orthogonality condition given by Lemma 2.3.[4] Similar results hold in static and feedback Stackelberg games for both the leader and the follower. The leader in a static Stackelberg game can use any extra information to his benefit, however.

As we noted before, the investigation of the informational property of the dynamic games is greatly simplified by the formulation where a sort of separation principle holds and the estimation error is independent of the controls. Without these nice properties, it would be difficult either in defining

[4] As a referee pointed out, another view to see the ideas behind the results is this. Consider two Nash games with two players. Player 2 in both games has the same information, while player 1 has different information. Assume that the Nash strategy of player 2 happens to be the same in both games. In this case, player 1's optimization problem in the two different information structures can be considered as team problems. In a team problem, a player can never be worse off with more information. Therefore, the problem boils down to how to find conditions to make the above assumption true.

"better information for one player alone" or in solving for the solutions. Either one of the difficulties makes the problem extremely hard. An extension of the results obtained in this paper to N-person, $N \geq 3$, Nash games can be carried out using similar arguments, and such results should constitute a fundamental step in designing information structures (Refs. 18-20) for large-scale systems.

8. Appendix A: Proof of Lemma 3.1

Consider the following state equation and measurements:

$$\tilde{x}(k+1) = A\tilde{x}(k) + w(k), \qquad \tilde{x}(0) = x_0, \tag{56}$$

$$\tilde{y}_i(k) = H_i\tilde{x}(k) + \nu_i(k), \qquad i = 1, 2, \tag{57}$$

where x_0, $\{w(k)\}$ and $\{\nu_i(k)\}$ are defined as in Section 3. By comparing (56) with (12), we immediately have

$$x(k) = \tilde{x}(k) + \sum_{n=0}^{k-1} A^{k-n-1}[B_1u_1(n) + B_2u_2(n)]. \tag{58}$$

Let

$$\hat{\hat{x}} \triangleq E[\tilde{x}(k) \mid \tilde{y}_1(0), \ldots, \tilde{y}_1(k)], \tag{59}$$

$$\hat{\hat{x}}_3(k) \triangleq E[\tilde{x}(k) \mid \tilde{y}_1(0), \ldots, \tilde{y}_1(k), \tilde{y}_2(0), \ldots, \tilde{y}_2(k)]. \tag{60}$$

Then, $\hat{\hat{x}}_i(k)$, $i = 1, 3$, are given exactly by the Kalman filter equations (14), except that (14b) is replaced by

$$\hat{\hat{x}}_i(k+1/k) = A\hat{\hat{x}}_i(k). \tag{61}$$

By the construction of $\hat{\hat{x}}_i(k)$ and $\hat{x}_i(k)$, $i = 1, 3$, it is easy to see that

$$\hat{x}_i(k) = \hat{\hat{x}}_i(k) + \phi_k, \tag{62}$$

where

$$\phi_k = \sum_{n=0}^{k-1} A^{k-n-1}[B_1u_1(n) + B_2u_2(n)]. \tag{63}$$

Since $\hat{\hat{x}}_3(k)$ is a refinement of $\hat{\hat{x}}_1(k)$, we obtain

$$\begin{aligned}\hat{\hat{x}}_1(k) &= E[\tilde{x}(k) \mid \tilde{y}_1(0), \ldots, \tilde{y}_1(k)]\\ &= E[E[\tilde{x}(k) \mid \tilde{y}_1(0), \ldots, \tilde{y}_1(k), \tilde{y}_2(0),\\ &\quad \ldots, \tilde{y}_2(k)] \mid \tilde{y}_1(0), \ldots, \tilde{y}_1(k)]\\ &= E[\hat{\hat{x}}_3(k) \mid \hat{y}_1(0), \ldots, \hat{y}_1(k)].\end{aligned} \tag{64}$$

Hence,

$$\begin{aligned} E[\hat{\hat{x}}_3(k) \,|\, \hat{\hat{x}}_1(k)] &= E[E[\hat{\hat{x}}_3(k)\hat{y}_1(0), \ldots, \tilde{y}_1(k)] \,|\, \hat{\hat{x}}_1(k)] \\ &= E[\hat{\hat{x}}_1(k) \,|\, \hat{\hat{x}}_1(k)] = \hat{\hat{x}}_1(k). \end{aligned} \tag{65}$$

Eq. (62) indicates that

$$\begin{aligned} E[\hat{x}_3(k) \,|\, \hat{x}_1(k)] &= E[\hat{\hat{x}}_3(k) + \phi_k \,|\, \hat{\hat{x}}_1(k) + \phi_k] \\ &= E[\hat{\hat{x}}_3(k) \,|\, \hat{\hat{x}}_1(k)] + \phi_k = \hat{\hat{x}}_1(k) + \phi_k \\ &= \hat{x}_1(k). \end{aligned} \tag{66}$$

By Lemma 2.2(i), $\hat{x}_3(k) - E[\hat{x}_3(k) \,|\, \hat{x}_1(k)]$ is orthogonal to $\hat{x}_1(k)$, i.e., $\hat{x}_4(k) \perp \hat{x}_1(k)$. □

9. Appendix B: Proof of Theorem 3.1

The proof is similar to that of Theorem 2.1 of Ref. 10. The difference is that here $I_1(k)$ has an additional element $\hat{x}_4(k)$ which, being orthogonal to both players' public information $\hat{x}_1(k)$, does not give rise to major changes in the Nash solution. The problem will be solved by a dynamic programming approach.

At stage N,

$$\begin{aligned} J_i(N) &= E[x^T(N)P_i x(N)] = E[\hat{x}_3^T(N)P_i\hat{x}_3(N)] + \mathrm{tr}[P_i\Sigma_3(N)] \\ &= E[\hat{x}_3^T(N)L_i(N)\hat{x}_3(N)] + K_i(N), \qquad i = 1, 2, \end{aligned} \tag{67}$$

where

$$L_i(N) \triangleq P_i, \qquad K_i(N) \triangleq \mathrm{tr}[P_i\Sigma_3(N)].$$

At stage $N-1$,

$$\begin{aligned} J_i(N-1) = E[&x^T(N-1)P_i x(N-1) \\ &+ u_i^T(N-1)u_i(N-1) \\ &+ u_j^T(N-1)Q_i u_j(N-1) \\ &+ x^T(N)P_i x(N)], \qquad j \neq i,\ i, j = 1, 2. \end{aligned} \tag{68}$$

After receiving $I_i(N-1)$, player i's objective is to minimize $\bar{J}_i(N-1)$ given by

$$\bar{J}_i(N-1) = E[J_i(N-1) \mid I_i(N-1)]. \tag{69}$$

Since $\bar{J}_i(N-1)$ is convex in $u_i(N-1)$, the Nash pair at stage $N-1$, $({}^*\gamma_1^{N-1}, {}^*\gamma_2^{N-1})$ is chosen such that

$$\left.\frac{\partial \bar{J}_i(N-1)}{\partial \gamma_i(N-1)}\right|_{{}^*\gamma_1^{N-1}, {}^*\gamma_2^{N-1}} = 0, \qquad i = 1, 2. \tag{70}$$

We then have

$${}^*\gamma_1^{N-1}(I_1(N-1)) = F_{11}(N-1)\hat{x}_1(N-1) + F_{14}(N-1)\hat{x}_4(N-1), \tag{71a}$$

$${}^*\gamma_2^{N-1}(I_2(N-1)) = F_2(N-1)\hat{x}_1(N-1), \tag{71b}$$

where $F_{11}(N-1)$, $F_{14}(N-1)$, $F_2(N-1)$ are given by (26) through (28). Notice that $({}^*\gamma_1^{N-1}, {}^*\gamma_2^{N-1})$ given by (71) exists and is unique, if $[I + B_1 B_1^T L_1(N) + B_2 B_2^T(N)]$ is nonsingular. Substituting (71) into (68), we obtain

$$J_i(N-1) = E[\hat{x}_3^T(N-1) L_i(N-1) \hat{x}_3(N-1)] + K_i(N-1), \tag{72}$$

where $L_i(N-1)$ and $K_i(N-1)$ are given by (23) and (30), respectively. As we can see, (72) and (67) are of the same form. In deriving the Nash pair $({}^*\gamma_1^{N-2}, {}^*\gamma_2^{N-2})$ at stage $N-2$, we will repeat what we did at stage $N-1$. An inductive argument then proves the theorem. □

10. Appendix C: Feedback Stackelberg Solution

In this appendix, we derive the feedback Stackelberg solution. The problem was stated in Section 5.

Theorem 10.1. There exists a unique solution to the feedback Stackelberg game.

(i) The solution for Case A is

$$u_{1A}^*(k) = {}^*\gamma_{1A}^k(I_1^A(k)) = F_{11A}(k)\hat{x}_1(k) + F_{14A}(k)\hat{x}_4(k), \tag{73}$$

$$u_{2A}^*(k) = {}^*\gamma_{2A}^k(I_2^A(k)) = F_{21A}(k)\hat{x}_1(k), \tag{74}$$

where

$$F_{11A}(k)=-B_1^T Z_A(k+1)[I+B_1B_1^T Z_A(k+1)]^{-1}A, \tag{75}$$

$$F_{14A}(k)=-B_1^T L_{1A}(k+1)[I+B_1B_1^T L_{1A}(k+1)]^{-1}A, \tag{76}$$

$$F_{21A}(k)=-B_2^T L_{2A}(k+1) \times[I+B_2B_2^T L_{2A}(k+1)]^{-1}[I+B_1B_1^T Z_A(k+1)]^{-1}A, \tag{77}$$

$$Z_A(k)=[I+B_2B_2^T L_{2A}(k)]^{-1^T}[L_{2A}(k)B_2Q_1B_2^T L_{2A}(k)+L_{1A}(k)] \times[I+B_2B_2^1 L_{2A}(k)]^{-1}, \tag{78}$$

$$L_{1A}(k)=P_1+F_{11A}^T(k)F_{11A}(k)+F_{21A}^T(k)Q_1F_{21A}(k) +(A+B_1F_{11A}(k)+B_2F_{21A}(k))^T \times L_{1A}(k+1)(A+B_1F_{11A}(k)+B_2F_{21A}(k)), \tag{79a}$$

$$L_{1A}(N)=P_1, \tag{79b}$$

$$L_{2A}(k)=P_2+F_{11A}^T(k)Q_2F_{11A}(k)+F_{21A}^T(k)F_{21A}(k) +(A+B_1F_{11A}(k)+B_2F_{21A}(k))^T \times L_{2A}(k+1)(A+B_1F_{11A}(k)+B_2F_{21A}(k)), \tag{80a}$$

$$\times L_{2A}(N)=P_2. \tag{80b}$$

Their costs to go at stage k are respectively

$$J_{1A}(k)=E\{\hat{x}_3^T(k)L_{1A}(k)\hat{x}_3(k)\}+K_{1A}(k), \tag{81}$$

$$J_{2A}(k)=E\{\hat{x}_4^T(k)L_{2A}(k)\hat{x}_3(k)\}+K_{2A}(k), \tag{82}$$

where

$$K_{1A}(k)=\mathrm{tr}\{[P_1+A^T L_{1A}(k+1)A-L_{14A}(k)]\Sigma_3(k) -L_{1A}(k+1)\Sigma_3(k+1)+L_{14A}(k)\Sigma_1(k)+L_{1A}(k+1)R\} +K_{1A}(k+1),\ K_{1A}(N)=\mathrm{tr}\{P_1\Sigma_3(N)\}, \tag{83}$$

$$K_{2A}(k)=\mathrm{tr}\{[P_2+A^T L_{2A}(k+1)A-L_{24A}(k)]\Sigma_3(k) -L_{2A}(k+1)\Sigma_3(k+1)+L_{24A}(k)\Sigma_1(k)+L_{2A}(k+1)R\} +K_{2A}(k+1),\ K_{2A}(N)=\mathrm{tr}\{P_2\Sigma_3(N)\}, \tag{84}$$

$$L_{14A}(k)=P_1+F_{14A}^T(k)F_{14A}(k) +(A+B_1F_{14A}(k))^T L_{1A}(k+1)(A+B_1F_{14A}(k))-L_{1A}(k), \tag{85}$$

$$L_{24A}(k)=P_2+F_{14A}^T(k)Q_2F_{14A}(k) +(A+B_1F_{14A}(k))^T L_{2A}(k+1)(A+B_1F_{14A}(k))-L_{2A}(k). \tag{86}$$

(ii) The solution for Case B is

$$u_{1B}^*(k) = {}^*\gamma_{1B}^k(I_1^B(k)) = F_{11B}(k)\hat{x}_1(k), \tag{87}$$

$$u_{2B}^*(k) = {}^*\gamma_{2B}^k(I_2^B(k)) = F_{21B}(k)\hat{x}_1(k) + F_{24B}(k)\hat{x}_4(k), \tag{88}$$

where

$$F_{11B}(k) = F_{11A}(k), \qquad F_{21B}(k) = F_{21A}(k), \tag{89a}$$

$$F_{24B}(k) = -B_2^T L_{2B}(k+1)[I + B_2 B_2^T L_{2B}(k+1)]^{-1} A, \tag{89b}$$

$$L_{2B}(k) = L_{2A}(k), \qquad L_{1B}(k) = L_{1A}(k). \tag{89c}$$

Their costs to go at stage k are respectively

$$J_{1B}(k) = E\{\hat{x}_3^T(k) L_{1B}(k)\hat{x}_3(k)\} + K_{1B}(k), \tag{90}$$

$$J_{2B}(k) = E\{\hat{x}_3^T(k) L_{2B}(k)\hat{x}_3(k)\} + K_{2B}(k), \tag{91}$$

where

$$\begin{aligned} K_{1B}(k) = \operatorname{tr}\{&[P_1 + A^T L_{1B}(k+1)A - L_{14B}(k)]\Sigma_3(k) \\ &- L_{1B}(k+1)\Sigma_3(k+1) + L_{14B}(k)\Sigma_1(k) + L_{1B}(k+1)R\} \\ &+ K_{1B}(k+1), \; K_{1B}(N) = \operatorname{tr}\{P_1\Sigma_3(N)\}, \end{aligned} \tag{92}$$

$$\begin{aligned} K_{2B}(k) = \operatorname{tr}\{&[P_2 + A^T L_{2B}(k+1)A - L_{24B}(k)]\Sigma_3(k) \\ &- L_{2B}(k+1)\Sigma_3(k+1) + L_{24B}(k)\Sigma_1(k) + L_{2B}(k+1)R\} \\ &+ K_{2B}(k+1), \; K_{2B}(N) = \operatorname{tr}\{P_2\Sigma_3(N)\}, \end{aligned} \tag{93}$$

$$\begin{aligned} L_{14B}(k) = {}&P_1 + F_{24B}^T(k) Q_1 F_{24}(k) \\ &+ (A + B_2 F_{24B}(k))^T L_{1B}(k+1)(A + B_2 F_{24B}(k)) - L_{1B}(k), \end{aligned} \tag{94}$$

$$\begin{aligned} L_{24B}(k) = {}&P_2 + F_{24B}^T(k) F_{24B}(k) \\ &+ (A + B_2 F_{24B})^T L_{2B}(k+1)(A + B_2 F_{24B}(k)) - L_{2B}(k). \end{aligned} \tag{95}$$

Proof. We will prove part (i) only. The proof for part (ii) is similar.

Feedback Stackelberg strategies have the property that they are in static Stackelberg equilibrium at every stage of the problem. This property can be observed from its definition, and hence we can solve the problem by going backward (a dynamic programming type of approach).

At stage N (no more decisions to be made), the cost to go of player i is

$$\begin{aligned} J_i(N) &= E[x^T(N)P_ix(N)] \\ &= E[\hat{x}_3^T(N)P_i\hat{x}_3(N)] + \mathrm{tr}\{P_i\Sigma_3(N)\} \\ &= E[\hat{x}_3^T(N)L_{iA}(N)\hat{x}_3(N)] + K_{iA}(N), \end{aligned} \tag{96}$$

where

$$L_{iA}(N) = P_i, \qquad K_{iA}(N) = \mathrm{tr}\{P_i\Sigma_3(N)\}.$$

At stage $N-1[I_i^A(N-1)$ is available], player i's objective is to minimize $\bar{J}_i(N-1)$ given by

$$\begin{aligned} \bar{J}_i(N-1) = E[&x^T(N-1)P_ix(N-1) + u_i^T(N-1)u_i(N-1) \\ &+ u_j^T(N-1)Q_iu_j(N-1) + x^T(N)P_ix(N)\,|\,I_i(N-1)]. \end{aligned} \tag{97}$$

By applying the Kalman filter equations (19) and Lemma 3.1, we obtain

$$\begin{aligned} \bar{J}_1(N-1) = {} & u_1^T(N-1)u_1(N-1) + u_2^T(N-1)Q_1u_2(N-1) \\ & + (A\hat{x}_3(N-1) + B_1u_1(N-1) + B_2u_2(N-1))^T \\ & \times L_{1A}(N)(A\hat{x}_3(N-1) + B_1u_1(N-1) + B_2u_2(N-1)) \\ & + \hat{x}_3^T(N-1)P_1\hat{x}_3(N-1) + \mathrm{tr}\{P_1\Sigma_3(N-1) \\ & + L_{1A}(N)[\Sigma_3(N/N-1) - \Sigma_3(N)]\} + K_{1A}(N), \end{aligned} \tag{98}$$

$$\begin{aligned} \bar{J}_2(N-1) = {} & E[(A\hat{x}_3(N-1) + B_1u_1(N-1) + B_2u_2(N-1))^T \\ & \times L_{2A}(N)(A\hat{x}_3(N-1) + B_1u_1(N-1) \\ & + B_2u_2(N-1))\,|\,I_2(N-1)] \\ & + E[u_1^T(N-1)Q_2u_1(N-1)\,|\,I_2^A(N-1)] \\ & + \hat{x}_1(N-1)P_2\hat{x}_1(N-1) \\ & + \mathrm{tr}\{P_1\Sigma_1(N-1) + L_{2A}(N-1)[\Sigma_3(N/N-1) - \Sigma_3(N)]\} \\ & + K_{2A}(N). \end{aligned} \tag{99}$$

To any strategy $\gamma_{1A}^{N-1}[I_1^A(N-1)]$ announced by the leader, the follower's rational reaction set is a singleton, i.e.,

$$\gamma_{2A}^{N-1}(I_2^A(N-1)) = -B_2^T L_{2A}[I + B_2 B_2^T L_{2A}(N)]^{-1} A\hat{x}_1(N-1) + B_1 E(\gamma_{1A}^{N-1}(I_1^A(N-1) \mid I_2^A(N-1))]. \tag{100}$$

Substituting $u_2(N-1)$ given by (100) into (98) and optimizing $\bar{J}_1(N-1)$ with respect to $u_1(N-1)$, we obtain

$$u_1^*(N-1) = F_{11A}(N-1)\hat{x}_1(N-1) + F_{14A}(N-1)\hat{x}_4(N-1), \tag{101}$$

where $F_{11A}(N-1)$ and $F_{14A}(N-1)$ are given respectively by (75) and (76). Substituting (101) into (100), we obtain $u_2^*(N-1)$ given by (74). Substituting $u_1^*(N-1)$ and $u_2^*(N-1)$ into $J_i(N-1)$, we obtain (81) and (82) for $k = N-1$. The proof of this feedback Stackelberg solution can then be concluded by an inductive argument. □

11. Appendix D: Proof of Theorem 5.3

We will prove part (i) only. The proof for part (ii) is similar.

From Eqs. (81) and (83) of Appendix C, we obtain the cost for the leader in Case A, which is

$$\begin{aligned} J_{1A}(0) = \operatorname{tr}\Big\{ & L_{1A}(0)\hat{\Omega}_3^{\mathrm{I}}(0) + [P_i + A^T L_{1A}(1)A - L_{14A}(0)]\Sigma_3(0) \\ & + L_{1A}(1)R + L_{14A}(0)\Sigma_1(0) \\ & + \sum_{k=1}^{N-1} [[P_1 + A^T L_{1A}(k+1)A - L_{1A}(k) - L_{14A}(k)]\Sigma_3(k) \\ & + L_{1A}(k+1)R + L_{14A}(k)\Sigma_1(k)] \Big\}. \end{aligned} \tag{102}$$

Let $J_{1A}^{\mathrm{I}}(0)$ and $J_{1A}^{\mathrm{II}}(0)$ correspond to Information I and II, respectively. Then,

$$\begin{aligned} J_{1A}^{\mathrm{II}}(0) - J_{1A}^{\mathrm{I}}(0) = \sum_{k=0}^{N-1} \operatorname{tr}\{ & [P_1 + A^T L_{1A}(k+1)A - L_{1A}(k) - L_{14A}(k)] \\ & \times [\Sigma_3^{\mathrm{II}}(k) - \Sigma_3^{\mathrm{I}}(k)] + L_{14A}(k)[\Sigma_1^{\mathrm{II}}(k) - \Sigma_1^{\mathrm{I}}(k)]\}. \end{aligned} \tag{103}$$

If Information I provides better information for the leader alone than Information II, then Lemma 4.1 implies $J_{1A}^{\mathrm{II}}(0) \geq J_{1A}^{\mathrm{I}}(0)$ if

$$P_1 + A^T L_{1A}(k+1)A - L_{1A}(k) - L_{14A}(k) \geq 0, \qquad \text{for every } k \in \theta.$$

Substituting Eqs. (79) and (85) of Appendix C into the left-hand side of the above equation, we obtain

$$\begin{aligned} & P_1 + A^T L_{1A}(k+1)A - L_{1A}(k) - L_{14}(k) \\ & = \{B_1^T L_{1A}(k+1)[I + B_1 B_1^T L_{1A}(k+1)]^{-1} A\}^T \\ & \times [I + B_1^T L_{1A}(k+1) B_1] B_1^T L_{1A}(k+1)[I + B_1 B_1^T L_{1A}(k+1)]^{-1} A \\ & \geq 0. \end{aligned} \tag{104}$$

□

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