SIAM J. CONTROL AND OPTIMIZATION
Vol. 24, No. 4, July 1986

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ITERATIVE TECHNIQUES FOR THE NASH SOLUTION IN QUADRATIC GAMES WITH UNKNOWN PARAMETERS*

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Abstract. We study adaptive schemes for repeated quadratic Nash games in a deterministic and a stochastic framework. The convergence of the schemes is demonstrated under certain conditions.

Key words. Nash equilibrium, adaptive Games, stochastic approximation

AMS(MOS) subject classifications. 60G42, 62L20, 90D05, 90D15

1. Introduction. The object of this paper is the study of a static quadratic Nash game where the players do not have knowledge of the parameters involved in the description of the cost of their opponents and of their opponent's information. The game is played repeatedly and at each stage the players know the past actions of their opponents. The only dynamics involved are in the accumulation of the information on their opponent's previous actions; apart from this dynamic aspect, the problem considered is a repeated static game. We examine both the deterministic and stochastic case, consider some adaptive schemes for updating the players decisions, and we show convergence to the optimal decisions (in the mean square sense and with probability one for the stochastic case), under some conditions. The scheme for the stochastic case is actually a stochastic approximation algorithm of the Robbins-Monro type.

The underlying motivation for the present paper is to study situations of conflict where the players do not know some of the parameters involved in the description of the others' cost functionals, or in the state equation. Such situations have been and are being studied for the single player—i.e., control problem—case and come under the name of Adaptive Control; the corresponding problems for situations of conflict, i.e., Adaptive Games, has received very little attention up to now. The problem studied here can be considered as a very simple type of adaptive game where the players adapt their decisions as to converge in the limit to the solution of a static Nash game. It should be noted that the strategies exhibited in this paper do not constitute a Nash equilibrium pair for the construed dynamic—dynamic due to the dynamic information—game; but similarly, the adaptive control strategy in the self-tuning regulator problem [5], converges in the limit to the optimal solution without being necessarily optimal at each stage. Adaptive games are important for several reasons. For example, when two players are involved in a situation of conflict, it is reasonable to assume that each player knows his own objective, but not that of his opponent; in addition, he might not know several of the parameters of the dynamic system which couples him with the other. In decentralized control, we think of decentralization as a scheme according to which each controller knows his own objective and information but not those of the others. If each controller knew the objectives of the others—as is implicitly assumed in many existing decentralized schemes—then the notion of decentralization is weakened. Although considerable progress has been achieved for the centralized controller, single objective adaptive control [4]-[6], the area of adaptive games is in

* Received by the editors June 26, 1984, and in revised form April 1, 1985. This research was supported in part by the U.S. Air Force Office of Scientific Research under grant AFOSR-82-0174 and by the University of Southern California Faculty Research and Innovation Fund.

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its infancy. The only work that the author is familiar with in this area is [7] and [8]. In [7], adaptive schemes based on self-tuning for stochastic Nash and Stackelberg games are considered, where the players have the same information. (In the present paper the information of the players is different.) In [8] two adaptive schemes are studied for repeated Stackelberg games in a determinisitic framework.

The structure of the paper is as follows. In § 2 we consider the determinisitic case and study three simple adaptive schemes. In § 3 we consider an adaptive scheme for the stochastic case. The stochastic scheme is a Robbins-Monro type of stochastic approximation algorithm. Although several results exist for such algorithms, many of which can be used to provide convergence for the scheme considered here, the conditions of convergence that they would obtain for our scheme are more stringent than those that we prove here. In each section we provide several comments relating the results with previous work, expand on their meaning and provide appropriate motivation. Finally, we have a conclusions section.

2. Deterministic case. Let $J_1, J_2: R^{m_1}\times R^{m_2}\to R$ be two functions defined by:

$$J_i(u_1, u_2)=\tfrac{1}{2}u_i'u_i+u_i'R_iu_j+u_i'c_i, \qquad i\neq j, \quad i,j=1,2 \tag{1}$$

where $u_i\in R^{m_i}$, R_1, R_2 are real constant matrices and c_1, c_2 are real constant vectors of appropriate dimensions. A pair (u_1^*, u_2^*) is a Nash equilibrium if it satisfies ([1], [2]):

$$J_1(u_1^*, u_2^*)\leqq J_1(u_1, u_2^*) \quad \forall u_1\in R^{m_1}, \tag{2}$$

$$J_2(u_1^*, u_2^*)\leqq J_2(u_1^*, u_2) \quad \forall u_2\in R^{m_2}, \tag{3}$$

or equivalently if

$$R\begin{bmatrix}u_1^*\\ u_2^*\end{bmatrix}+c=0, \quad R=\begin{bmatrix}1 & R_1\\ R_2 & 1\end{bmatrix}, \quad c=\begin{bmatrix}c_1\\ c_2\end{bmatrix}. \tag{4}$$

J_i and u_i are the cost and the decision of player i.

Let us assume that player i knows R_i and c_i, but not R_j and c_j $(j\neq i)$; then he cannot solve (4) for u_i^*. Consider also that this game is played repeatedly at times $t=1,2,3,\cdots$, that at time t, player i knows $I_t^i=\{u_{1,1},\cdots,u_{1,t-1},u_{2,1},\cdots,u_{2,t-1}\}$ and plays u_{it} which is chosen as a function of I_t^i, i.e.,

$$u_{it}=F_i(I_t^i, t), \qquad i=1,2, \quad t=2,3,\cdots. \tag{5}$$

The question is: For what F_1, F_2 the recursion (5) will converge to a solution of (4). Let us now examine three possible choices of F_1, F_2.

CASE 1.

$$F_i(I_t^i, t)=-R_iu_{j,t-1}-c_i, \qquad i=1,2, \quad i\neq j. \tag{6}$$

The meaning of (6) is that player 1 minimizes $J_1(u_1, u_{2,t-1})$, i.e., he reacts only to the last announced decision of player 2. Recursion (5) assumes the form:

$$\begin{bmatrix}u_{1t}\\ u_{2t}\end{bmatrix}=\begin{bmatrix}u_{1,t-1}\\ u_{2,t-1}\end{bmatrix}-\left(R\begin{bmatrix}u_{1,t-1}\\ u_{2,t-1}\end{bmatrix}+c\right), \qquad t\geqq 2. \tag{7}$$

Recursion (7) will converge to a solution of (4) for any initial condition $(u_{1,1}, u_{2,1})$ if and only if all the eigenvalues of the matrix R lie within the open disc of radius 1 centered at the point 1 in the complex plane, i.e.,

$$|\lambda(R)-1|<1 \tag{8}$$

((8) is equivalent to $|\lambda(R_1R_2)|<1$). Condition (8) also guarantees that (4) has a unique solution.

CASE 2.

$$F_i(I_t^i, t) = -R_i[u_{j,t-1} + \theta u_{j,t-2} + \cdots + \theta^{t-2} u_{j,1}] \frac{1-\theta}{1-\theta^{t-1}} - c_i, \tag{9}$$

$$1 > \theta \geqq 0, \quad i = 1, 2, \quad i \neq j.$$

The meaning of (9) is that player 1 minimizes J_1 with respect to u_1, with u_2 fixed to a value that is a weighted average of $u_{2,t-1}, \cdots, u_{2,1}$ where more weight is put on the recent values of u_2. We assume that both players use the same θ. Recursion (9) can be written equivalently:

$$\begin{bmatrix} u_{1t} \\ u_{2t} \end{bmatrix} = \begin{bmatrix} u_{1,t-1} \\ u_{2,t-1} \end{bmatrix} - \frac{1-\theta}{1-\theta^{t-1}} \left(R \begin{bmatrix} u_{1,t-1} \\ u_{2,t-1} \end{bmatrix} + c \right), \qquad t \geqq 2. \tag{10}$$

Recursion (10) will converge to a solution of (4) for any initial condition $(u_{1,1}, u_{2,1})$ if and only if all the eigenvalues of the matrix R lie within the open disc of radius $(1-\theta)^{-1}$ centered at the point $(1-\theta)^{-1}$ in the complex plane, i.e.,

$$\left| \lambda(R) - \frac{1}{1-\theta} \right| < \frac{1}{1-\theta}. \tag{11}$$

Condition (11) also guarantees that (4) has a unique solution. (Notice that as $t \to +\infty$, $\theta^{t-1} \to 0$ and thus $(1-\theta)R$ in (10) assumes the role of R in (7).) Obviously, for $\theta = 0$, (11) reduces to (8) and (10) to (7).

CASE 3.

$$F_i(I_t^i, t) = -R_i[u_{j,t-1} + u_{j,t-2} + \cdots + u_{j,1}] \frac{1}{t-1} - c_i, \qquad i = 1, 2, \quad i \neq j. \tag{12}$$

The meaning of (12) is that player 1 minimizes J_1 with respect to u_1, with u_2 fixed to the arithmetic mean of $u_{2,t-1}, \cdots, u_{2,1}$. Recursion (12) can be written equivalently:

$$\begin{bmatrix} u_{1,t} \\ u_{2,t} \end{bmatrix} = \begin{bmatrix} u_{1,t-1} \\ u_{2,t-1} \end{bmatrix} - \frac{1}{t-1} \left(R \begin{bmatrix} u_{1,t-1} \\ u_{2,t-1} \end{bmatrix} + c \right), \qquad t \geqq 2. \tag{13}$$

Recursion (13) will converge to a solution of (4), for any initial condition $(u_{1,1}, u_{2,1})$ if and only if all the eigenvalues of R has positive real parts, i.e.,

$$\operatorname{Re} \lambda(R) > 0 \tag{14}$$

(for proof see Appendix A, Lemma A3). Condition (14) also guarantees that (4) has a unique solution. Notice that as $\theta \to 1$, (11) reduces to (14).

Remark 1. Obviously (8)$\Rightarrow$(11)$\Rightarrow$(14). If (8) holds, (7) converges faster than (10) and if (11) holds, (10) converges faster than (13).

Remark 2. In all three cases we assumed that both players use the same scheme. Nonetheless, it might happen that they use different ones. It is easy to verify that if player 1 uses scheme 1 and player 2 uses scheme 2, the region of convergence is larger than if both were using scheme 1 and worse than if both were using scheme 2. Similar results holds for the other combinations.

Remark 3. If we consider (10) with $\theta > 1$, i.e., more weight is assigned to the old measurements, the scheme will not converge. This can be easily verified by considering the scalar version of (10) with $c = 0$:

$$u_t = u_{t-1} \left(1 - r \frac{1-\mu}{\mu} \frac{\mu^{t-1}}{1-\mu^{t-1}} \right), \qquad \mu = \frac{1}{\theta}$$

which for $t \to +\infty$ behaves like

$$y_t = y_{t-1}\left(1 - r\frac{1-\mu}{\mu}\mu^{t-1}\right)$$

(since $0<\mu<1$) and is easily seen to fail to converge.

Remark 4. Conditions (8), (11) and (14) can be expressed equivalently in terms of the eigenvalues of R_1R_2.

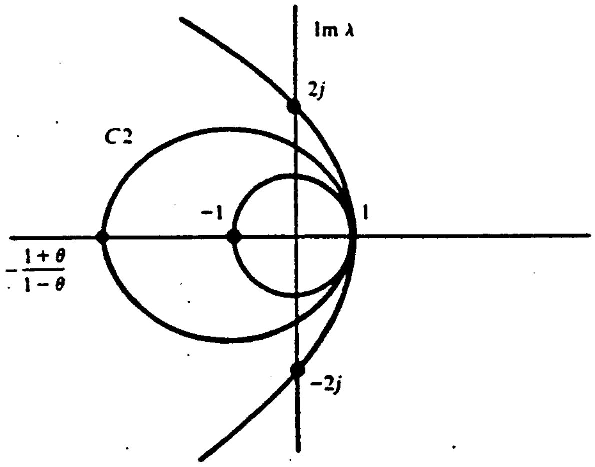

FIG. 1

Condition (8) corresponds to $|\lambda(R_1R_2)|<1$, i.e., inside the unit disc, (11) corresponds to

$$(1-\theta)|\lambda| \pm 2\theta\cos\frac{\varphi}{2}|\lambda|^{1/2} - (1+\theta) < 0,$$

$$\lambda(R_1R_2) = |\lambda|e^{j\varphi},$$

i.e., inside the curve $C2$ of Fig. 1. Condition (14) corresponds to eigenvalues of R_1R_2 being inside the parabola defined by

$$\text{Re}\,\lambda + \tfrac{1}{4}(\text{Im}\,\lambda)^2 < 1, \qquad \lambda = \lambda(R_1R_2).$$

Remark 5. If (8) (or equivalently $|\lambda(R_1R_2)|<1$) holds, the solution of (4) is called in game theory a stable equilibrium, and the game is called stable [1]. The reason is that if player i deviates from u_i^*, then player $j(j\neq i)$ responds according to scheme (6) and to that player i responds according to scheme (6) and so on and eventually they both converge back to (u_1^*, u_2^*). Obviously the notion of stable equilibrium depends on the reaction scheme that the players employ. If schemes (9) or (12) are used as reaction schemes, we have an enlarged class of stable games.

Remark 6. Since the scheme of Case 3 (12) has the best convergence region out of the three schemes, in the next section we will deal with the stochastic analogue of (12).

Remark 7. All three schemes considered can actually be viewed as schemes for solving $Ru+c=0$ (see (4)), by using an iteration of the form:

(15) $$u_{n+1} = u_n - D_n[Ru_n + c]$$

where D_n has to have the structure

$$D_n = \begin{bmatrix} D_n^1 & 0 \\ 0 & D_n^2 \end{bmatrix}.$$

(Iterative solutions of linear equations is a vast subject, see for e.g. [16].) Scheme (13) employed: $D_n^i = (1/n)I$. We can create new schemes which converge under weaker conditions than (14) by allowing $D_n^i = (1/n)D^i$ where D^1, D^2 are properly chosen constant matrices. For example, if R_1, R_2 are scalars, (14) is equivalent to $1 > r_1 r_2$; but if we use $D_n^i = (1/n)d_i$ in (15), the convergence condition becomes

$$\operatorname{Re} \lambda \left(\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \begin{bmatrix} 1 & r_1 \\ r_2 & 1 \end{bmatrix} \right) > 0$$

which is equivalent to:

$$d_1 + d_2 > 0, \qquad d_1 d_2 (1 - r_1 r_2) > 0,$$

and can always be satisfied for some d_1, d_2 as long as $1 \neq r_1 r_2$. Notice that $1 \neq r_1 r_2$ is the necessary and sufficient condition for solvability of (4) for any c.

Remark 8. Another way of going about the problem of this section is to consider that at each stage, each player uses a certain scheme to estimate the R and C of his opponent and then calculates his action by solving (4) wherein he employs the estimates of the R and c of his opponent. In such a scheme, each player should know at each stage not only the previous actions of his opponent—as in our scheme—but also the rationale according to which his opponent calculates his actions. This is necessary in order just to estimate his opponent's parameters at each stage. Nonetheless, such an additional knowledge can be permitted and the convergence of the resulting scheme studied. Finally, it should be noted that the problem considered here and the schemes proposed, besides having their own merit, provide a certain motivation for the scheme considered for the stochastic case of the next section.

3. The stochastic case. Let x be a Gaussian random vector in R^n with zero mean and unit covariance matrix. Let

$$y_i = C_i x, \qquad i = 1, 2 \tag{16}$$

represent the measurements of the two players, where C_1, C_2 are fixed real matrices of dimensions $n_1 \times n$, $n_2 \times n$ respectively. Let Γ_i be the set of all measurable $\gamma_i: R^{n_i} \to R^{m_i}$ functions with $E[\gamma_i(y_i)'\gamma_i(y_i)] < +\infty$. Set $u_i = \gamma_i(y_i)$ and let

$$J_i(\gamma_1, \gamma_2) = E[\tfrac{1}{2}u_i'u_i + u_i'R_iu_j + u_i'S_ix], \qquad i \neq j, \quad i, j = 1, 2 \tag{17}$$

represent the costs of the two players. R_1, R_2, S_1, S_2 are fixed real matrices of appropriate dimensions. A pair $(\gamma_1^*, \gamma_2^*) \in \Gamma_1 \times \Gamma_2$ is called a Nash equilibrium if it satisfies

$$\begin{aligned} &J_1(\gamma_1^*, \gamma_2^*) \leq J_1(\gamma_1, \gamma_2^*) \quad \forall \gamma_1 \in \Gamma_1, \\ &J_2(\gamma_1^*, \gamma_2^*) \leq J_2(\gamma_1^*, \gamma_2) \quad \forall \gamma_2 \in \Gamma_2. \end{aligned} \tag{18}$$

For background concerning the formulation of the stochastic Nash game see [18]. (18) is equivalent to (see [2], [3]):

$$\gamma_1^*(y_1) + R_1 E[\gamma_2^*(y_2)|y_1] + S_1 E[x|y_1] = 0, \tag{19a}$$

$$\gamma_2^*(y_2) + R_2 E[\gamma_1^*(y_1)|y_2] + S_2 E[x|y_2] = 0. \tag{19b}$$

It is known (see [3]) that if no eigenvalue of $R_1 R_2$ equals the inverse of any arbitrary but finite product of powers of the squares of the canonical correlation coefficients of

y_1, y_2 (i.e., of $\sigma_1, \sigma_2, \cdots$), then (19) has a unique solution which as to be linear in the information. The set of values where the eigenvalues of R_1R_2 should not lie is a countable isolated set of points in $[1, +\infty)$ and thus it is generically true that (19) admits a unique solution which has to be linear in the information. We can assume without loss of generality (see [3, Lemma 1]) that

$$n_1 \leqq n_2 C_1C_1' = I_{n_1\times n_1}, \quad C_2C_2' = I_{n_2\times n_2}, \quad C_1C_2' = \begin{bmatrix} \sigma_1 & & 0 & \vdots & \\ & \sigma_2 & & \vdots & 0 \\ 0 & & \sigma_{n_1} & \vdots & \end{bmatrix}_{n_1\times n_2} \tag{20}$$

$$1 \geqq \sigma_1 \geqq \sigma_2 \geqq \sigma_{n_1} \geqq 0$$

and then $\gamma_i^*(y_i) = L_i y_i$ where L_1, L_2 are the solutions to the system:

$$\begin{aligned} L_1 + R_1L_2C_2C_1' + S_1C_1' &= 0, \\ L_2 + R_2L_1C_1C_2' + S_2C_2' &= 0. \end{aligned} \tag{21}$$

Let us assume that player i knows R_i, S_i, C_i, but not R_j, S_j, C_j, $i \neq j$; then he cannot solve (21) for L_i. Consider also that this game is played repeatedly at times $t = 1, 2, 3, \cdots$, that at time t player i knows

$$I_t^i = \{u_{1,1}, \cdots, u_{1,t-1}, u_{2,1}, \cdots, u_{2,t-1}, y_{i,1}, \cdots, y_{i,t}\} \tag{22}$$

where y_{it} is the measurement of player i at time t. We assume that

$$y_{it} = C_i x_t \tag{23}$$

where the x_t's are independent Gaussian vectors with zero mean and unit covariance. At time t, player 1 employs the following scheme for finding u_{1t}:

$$u_{1t} + R_1\left(\frac{1}{t-1}\sum_{k=1}^{t-1} u_{2,k}y_{1,k}'\right) y_{1t} + S_1C_1'y_{1t} = 0. \tag{24}$$

A justification of this scheme is the following: at time t player 1 has to solve (19a) for u_{1t} and thus he has to calculate $E[u_{2,t}|y_{1t}]$, $E[x|y_{1t}]$. If u_{2t} is linear in y_{2t}, then u_{2t}, y_{1t} are jointly Gaussian and thus

$$E[u_{2,t}|y_{1t}] = E[u_{2t}y_{1t}'](E[y_{1t}y_{1t}'])^{-1}y_{1,t}. \tag{25}$$

Player 1 approximates $E[u_{2t}y_{1t}']$ by $1/(t-1)\sum_{k=1}^{t-1}(u_{2,k}y_{1k}^T)$; a motivation for this approximation is the following: If player 1 knew all the parameters of (16), (17), he would then solve equation (19) at state t, employing (23); due to the independence of the x_t's, $1/(t-1)\sum_{k=1}^{t-1}(u_{2k}y_{1k}^T)$ would provide a reasonable approximation of $E[u_{2t}|y_{1t}]$, since u_{2k} would be independent of u_{2l}, y_{1l}, $l \neq k$. By overlooking the lack of independence of u_{2k} on u_{2l}, y_{1l}, $l \neq k$, he still employs the above approximation, hoping that things will work out. The convergence results of Theorems 1′ and 2′ provide a posterior justification for the reasonableness of this approximation.

By our assumption (20) $E[y_{1t}y_{1t}'] = I$ and $E[x_t|y_{1t}] = S_1C_1'y_{1t}$. (24) yields that u_{1t} is linear in y_{1t}, i.e., $u_{1t} = L_{1t}y_{1t}$ where L_{1t} satisfies

$$L_{1t} + R_1\left[\frac{1}{t-1}\sum_{k=1}^{t-1} u_{2k}y_{1k}'\right] + S_1C_1' = 0. \tag{26}$$

A similar equation is satisfied by L_{2t}, if we consider that u_{2t} is calculated by an equation corresponding to (24) and $u_{2t} = L_{2t}y_{2t}$. The equations for L_{1t}, L_{2t} can be written recursively as:

(27a) $$L_{1t}=L_{1,t-1}-\frac{1}{t-1}[L_{1,t-1}+R_1L_{2,t-1}y_{2,t-1}y'_{1,t-1}+S_1C'_1],$$

(27b) $$L_{2t}=L_{2,t-1}-\frac{1}{t-1}[L_{2,t-1}+R_2L_{1,t-1}y_{1,t-1}y'_{2,t-1}+S_2C'_2].$$

Recursion (27) is the recursion that we intend to study and show that under some conditions converges to the solution of (21) in the q.m. sense and w.p.1. The initial condition L_{11}, L_{12} of (27) is taken to be an arbitrary pair of real constant matrices and we are interested in convergence for any initial condition. The recursion (27) defines a Markovian stochastic process (L_{1t}, L_{2t}) and is obviously a stochastic approximation algorithm of the Robbins-Monro type [9] for solving (21). Recursion (27) is a stochastic analogue of the scheme of Case 3 of the deterministic case.

Let us now study the convergence of (27). Let us call l_{it}, m_{it}, c_i, d_i the ith columns of L_{1t}, L_{2t}, $S_1C'_1$, $S_2C'_2$ respectively, i.e.,

(28) $$L_{1t}=[l_{1t},\cdots,l_{n_1t}],\qquad L_{2t}=[m_{1t},\cdots,m_{n_2t}],$$
$$S_1C'_1=[c_1,\cdots,c_{n_1}],\qquad S_2C'_2=[d_2,\cdots,d_{n_2}].$$

Let

(29) $$\bar{l}_{it}=E[l_{it}],\qquad \bar{m}_{it}=E[m_{it}].$$

Using (20) and the fact that L_{1t} depends on $y_{11},\cdots,y_{1t-1},y_{21},\cdots,y_{2,t-1}$, we obtain from (27):

(30a) $$\bar{l}_{it}=\bar{l}_{it-1}-\frac{1}{t-1}[\bar{l}_{i,t-1}+\sigma_iR_1\bar{m}_{i,t-1}+c_i],\qquad i=1,\cdots,n_1$$

(30b) $$\bar{m}_{it}=\bar{m}_{it-1}-\frac{1}{t-1}[\bar{m}_{i,t-1}+\sigma_iR_2\bar{l}_{i,t-1}+d_i],\qquad i=1,\cdots,n_1$$

and

(30c) $$\bar{m}_{it}=\bar{m}_{i,t-1}-\frac{1}{t-1}[\bar{m}_{i,t-1}+d_i],\qquad i=n_1+1,\cdots,n_2.$$

Recursion (30c) converges for any initial condition (see Lemma A3). Recursions (30a) and (30b) can be written as

(31) $$\begin{bmatrix}\bar{l}_{it}\\ \bar{m}_{it}\end{bmatrix}=\begin{bmatrix}\bar{l}_{i,t-1}\\ \bar{m}_{i,t-1}\end{bmatrix}-\frac{1}{t-1}\left(\begin{bmatrix}I & \sigma_iR_1\\ \sigma_iR_2 & I\end{bmatrix}\begin{bmatrix}\bar{l}_{i,t-1}\\ \bar{m}_{i,t-1}\end{bmatrix}+\begin{bmatrix}c_i\\ d_i\end{bmatrix}\right)$$

and using Lemma A3 yields that (31) converges for any initial condition if and only if

(32) $$\operatorname{Re}\lambda\left(\begin{bmatrix}I & \sigma_iR_1\\ \sigma_iR_2 & I\end{bmatrix}\right)>0.$$

It is easy to see that if (32) holds for σ_1 then it holds for any σ_i, $0\leqq\sigma_i\leqq\sigma_1$. We thus have proven the following theorem.

THEOREM 1′. *The means of L_{1t}, L_{2t} as defined by the recursion* (27) *converge to a solution of* (21) *for any initial condition, if and only if*

(33) $$\operatorname{Re}\lambda\left(\begin{bmatrix}I & \sigma_1R_1\\ \sigma_1R_2 & I\end{bmatrix}\right)>0.$$

It is easy to see that if (33) holds then (21) has a unique solution. If we want (27) to converge to a solution of (21) not only for any initial condition, but also for any pair of measurements, i.e., any C_1, C_2, we have to consider $\sigma_1 = 1$ in (33) which is exactly the condition for convergence of Case 3 of the deterministic case.

Next we will show that L_{1t}, L_{2t} converge to a solution of (21) in the mean square sense, under condition (33). For simplicity and w.l.o.g. we will assume $S_1C_1' = 0$, $S_2C_2' = 0$. We can write (27) component-wise in tems of l_{it}, m_{it} and then form the products $l_{it}l_{jt}'$, $i, j = 1, \cdots, n_1$, $m_{it}m_{jt}'$, $i, j = 1, \cdots, n_2$ and $l_{it}m_{mt}'$, $i = 1, \cdots, n_1$, $j = 1, \cdots, n_2$. These products satisfy recursions that can be easily calculated, and taking expectations of which result in a recursion which gives the evolution of $E(l_{it}l_{jt}')$, $E(m_{it}m_{jt}')$, $E(l_{it}m_{jt}')$ in terms of $E(l_{i,t-1}l_{j,t-1}')$, $E(m_{i,t-1}, m_{j,t-1}')$ $E(l_{i,t-1}m_{j,t-1}')$. Before writing down this recursion we introduce some notation:

(34a) $$\Lambda_{ij}^t = E[l_{it}l_{jt}'], \qquad i, j = 1, \cdots, n_1,$$

(34b) $$M_{ij}^t = E[m_{it}m_{jt}'], \qquad i, j = 1, \cdots, n_2,$$

(34c) $$K_{ij}^t = E[l_{it}m_{jt}'], \quad i = 1, \cdots, n_1, \quad j = 1, \cdots, n_2,$$

(35) $$N_t = \left[\begin{array}{ccc|ccc} \Lambda_{11}^t & \cdots & \Lambda_{1n_1}^t & K_{1,1}^t & \cdots & K_{1,n_2}^t \\ \vdots & & \vdots & \vdots & & \vdots \\ \Lambda_{n_11}^t & \cdots & \Lambda_{n_1n_1}^t & K_{n_1,1} & \cdots & K_{n_1,n_2}^t \\ \hline (K_{11}^t)' & \cdots & (K_{n_1,1}^t)' & M_{11}^t & \cdots & M_{1,n_2}^t \\ \vdots & & \vdots & \vdots & & \vdots \\ (K_{1,n_2}^t)' & \cdots & (K_{n_1,n_2}^t)' & M_{n_2,1} & \cdots & M_{n_2,n_2}^t \end{array}\right],$$

(36) $$Q = \left[\begin{array}{ccc|ccccc} I & & 0 & \sigma_1 R_1 & & 0 & 0 & 0 \\ & \ddots & & & \sigma_2 R_1 & & \vdots & \\ & & & & & \ddots & & \\ 0 & & I & 0 & & \sigma_{n_1} R_1 & 0 \cdots & 0 \\ \hline \sigma_1 R_2 & & 0 & I & & & & \\ & \ddots & & & & & & \\ & 0 & \sigma_{n_1} R_2 & & & & & 0 \\ 0 & \cdots & 0 & & 0 & & \ddots & \\ \vdots & & \vdots & & & & & \\ 0 & \cdots & 0 & & & & & I \end{array}\right].$$

Then N_t satisfies

(37) $$N_t = N_{t-1} - \frac{1}{t-1}[N_{t-1}Q' + QN_{t-1}] + \frac{1}{(t-1)^2}\mathscr{L}(N_{t-1}).$$

where $\mathscr{L}(\cdot)$ denotes a linear time invariant function of its argument. (For details of this derivation, see Appendix B.)

Using Lemma A4 we conclude that N_t goes to zero for any initial condition if and only if the matrix Q has eigenvalues with positive real parts which is easily seen to be equivalent to (33). We thus have proven.

THEOREM 2'. *L_{1t}, L_{2t} as defined by recursion* (27) *converge to a solution of* (21) *for any initial condition, in the mean square sense, if and only if* (33) *holds.*

Next, we will show that (L_{1t}, L_{2t}) converges under (33) for any initial condition to the solution of (21) with probability 1 (i.e., a.s. convergence). We again assume for simplicity and w.l.o.g. that $S_1C_1' = 0$, $S_2C_2' = 0$. We will use the theorem in paragraph 3 of [11] (or [13, Lemma 3.5]) which we restate here and which is an easy consequence of the martingale convergence theorem of Doob.

LEMMA 1. *Let $\{V_t\}$ be a sequence of random variables such that $E(V_1)$ exists. Let A be a real number and suppose $V_t \geqq A$. Furthermore, assume that $\sum_{t=1}^{\infty} E(E[V_{t+1} - V_t | V_1, \cdots, V_t]^+)$ converges. Then the sequence $\{V_t\}$ converges with probability 1.*

(Recall that if x is a random variable: $x^+ = \frac{1}{2}(|x| + x)$.) Let $x_t = (l'_{1t}, \cdots, l'_{n_1,t}, m'_{1,t}, \cdots, m'_{n_2,t})'$. We will prove that x_t converges to 0 w.p.l. or equivalently that $V_t = \|x_t\|^2$ does. Let $A = 0$. From (27) we can easily obtain (see Appendix C)

$$|E[V_{t+1} - V_t | V_1, \cdots, V_t]| \leqq \frac{\alpha}{t} V_t$$

for some positive number α and thus

$$E[V_{t+1} - V_t | V_1, \cdots, V_t]^+ \leqq \frac{\alpha}{t} V_t.$$

In order to fulfill the assumption of Lemma 1, it suffices to show that

$$\sum_{t=1}^{\infty} \frac{\alpha}{t} E(V_t) < +\infty. \tag{38}$$

$E[V_t] = \operatorname{tr} N_t$ holds, and thus it suffices to show that

$$\sum_{t=1}^{\infty} \frac{\operatorname{tr} N_t}{t} < +\infty. \tag{39}$$

From (37) we obtain

$$N_{t+1} = N_1 - Q\left[\sum_{t=1}^{t} \frac{N_k}{k}\right] - \left[\sum_{k=1}^{t} \frac{N_k}{k}\right] Q' + \mathcal{L}\left(\sum_{k=1}^{t} \frac{N_k}{k^2}\right). \tag{40}$$

If we assume that Q has eigenvalues with real parts (40) can be solved for $\sum_{k=1}^{t} (N_k/k)$ to yield

$$\sum_{k=1}^{t} \frac{N_k}{k} = \mathcal{L}'\left(N_{t+1}, N_1, \sum_{k=1}^{t} \frac{N_k}{k^2}\right).$$

Since N_k converges, it is bounded and so is $\sum_{k=1}^{t} (N_k/k^2)$. Thus $\sum_{k=1}^{t} (N_k/k)$ is uniformly bounded and thus (39) and (38) are bounded. We thus conclude that $\|x_t\|^2 = V_t$ converges with probability 1. $\|x_t\|^2$ converges to 0 in the mean square sense by Theorem 2′ and thus in probability and thus it has a subsequence converging to zero with probability one [17, Thms. 2, 5, 3, p. 93]. Since we just showed that $\|x_t\|^2$ converges with probability one, this limit has to be zero. Let us now summarize the results of this section in a theorem.

THEOREM. *L_{1t}, L_{2t} as defined by recursion (27) converge to a solution of (21) for any initial condition, in the mean square sense and with probability one if and only if*

$$\operatorname{Re} \lambda\left(\begin{bmatrix} I & \sigma_1 R_1 \\ \sigma_1 R_2 & I \end{bmatrix}\right) > 0$$

(*under this condition* (21) *admits a unique solution*).

Remark 1. N_n (37), goes to zero but it does not have to converge monotonically.

Remark 2. One can construct the stochastic analogues of the deterministic schemes of Cases 1 and 2, if a different—appropriate—approximation is used for $E[u_{2t}|y_{1t}]$ in (25). A little reflection, though, will persuade the reader that these schemes will converge under conditions more stringent than (33).

Remark 3. For a repeated Stackelberg game one can consider schemes similar to those considered here, if one assumes that the Leader does not know the parameters involved in the Follower's cost. An idea of this sort was recently studied in a deterministic framework in [8].

Remark 4. It should be clear from (30) and (37) that the rate of convergence of the means and the covariances of l_{in}, m_{it} depend on the eigenvalues of the matrices in (32) for $\sigma_i = 1, \sigma_1, \cdots, \sigma_{n_1}$, or equivalently of Q. Actually, a recursion of the form (A1) with $\bar{\lambda} = \text{Re}(\lambda) > 0$ goes to zero like $(n^{\bar{\lambda}})^{-1}$, (see [12]). Thus if λ_m denotes the real part of the eigenvalues of Q, $m = 1, \cdots, n_1 + n_2$ and $\bar{\lambda} = \min \text{Re}(\lambda_m)$ the mean converges no slower than $(t^{\bar{\lambda}})^{-1}$, the covariances no slower than $(t^{2\bar{\lambda}})^{-1}$, the third moments no slower than $(t^{3\bar{\lambda}})^{-1}$ and so on. Thus if one were to consider whether $t^{\theta}[L_{1n}, L_{2t}]$ converges weakly to a Gaussian random variable as $t \to \infty$, θ should be chosen equal to $\bar{\lambda}$ so that the second moments converge to a nonzero constant, but then automatically all the moments will also do so. Thus in general one cannot have asymptotic normality of $n^{\theta}[L_{1n}, L_{2t}]$ for some $\theta > 0$. As a matter of fact, Theorem (1) of [12] cannot be applied since its assumption (A4) fails for the stochastic approximation algorithm (27), considered here, as should be expected from the above remarks. Finally, it should be pointed out that the fact that the rate of convergence of the algorithm is given by $t^{-\bar{\lambda}}$ and $t^{-2\bar{\lambda}}$ for the first and second moments, is a useful fact when implementing it, in deciding when to stop, what is the probability of error when stopping in a finite number of iterations, etc.

Remark 5. Stochastic approximation has been an object of intensive study (see [9]–[15]). Several of the results available can be used to prove convergence of the iteration (27) but they demand conditions stronger than (33), or they are not applicable to it. For example, in [9] it is required that in the scheme $x_{n+1} = x_n - (1/n)y_n$, y_n is uniformly bounded. Assumptions III and IV of [10] do not hold for (27). In proving asymptotic normality [12], he uses Assumption (A4) which does not hold for (27). Assumptions A5, A5′ of [11] do not hold for our scheme. Lemma 3.1 and Theorem 4.3 of [13] can be applied to (27) but result in more stringent conditions than (33). The convergence analysis of [15] demands boundness of the second term in (27) which is not applicable to our case. Assumption iii in [14, Problem 1, p. 92] does not hold for (27).

4. Conclusions. There are several directions in which this research can be continued. One of them is the corresponding problem for the Stackelberg game (see Remark 3 in § 3). The dynamic case where the players are also coupled through the evolution of a discrete time equation is obviously important and useful. We hope that the analysis presented here will be helpful in such further research.

Appendix A.

LEMMA A1. *Consider the scalar recursion*

$$\text{(A1)} \qquad x_{n+1} = \left(1 - \frac{\lambda}{n}\right) x_n, \qquad n = 1, 2, 3, \cdots$$

where λ and x_1 are complex numbers. Then $x_n \to 0$ for any x_1 if and only if $\text{Re}(\lambda) > 0$.

(*If we set* $t_n = 1 + \cdots + 1/n$, *we see that* (A1) *is a discrete approximation of* $\dot{x} = -\lambda x$ *and thus* $\mathrm{Re}(\lambda) > 0$ *is expected in order to have asymptotic stability of* (A1).)

LEMMA A2. *Consider the scalar recursion*

$$x_{n+1} = \left(1 - \frac{\lambda}{n} + O\left(\frac{1}{n^2}\right)\right) x_n, \qquad n = 1, 2, 3, \cdots \tag{A2}$$

where x *and* x_1 *are complex numbers. Then* $x_n \to 0$ *for any* x_1 *if and only if* $\mathrm{Re}(\lambda) > 0$.

Proof. It is an immediate consequence of Lemma A1 since λ/n dominates $O(1/n^2)$. □

LEMMA A3. *Consider the recursion*

$$x_{n+1} = \left(I - \frac{1}{n} A + O\left(\frac{1}{n^2}\right)\right) x_n, \qquad n = 1, 2, 3, \cdots \tag{A3}$$

where A *is a real square matrix and* x_1 *is a vector. Then* $x_n \to 0$ *for any* x_1 *if and only if* $\mathrm{Re}\,\lambda(A) > 0$.

Proof. We bring A to its Jordan form and apply Lemma A2. It is helpful to notice that if P is a real symmetric matrix

$$x'_{n+1} P x_{n+1} = x'_n P x_n - \frac{1}{n} x'_n [PA + A'P] x_n + x'_n O\left(\frac{1}{n^2}\right) x_n$$

and thus if A has $\mathrm{Re}\,\lambda(A) > 0$, we can find a positive definite P so that $A'P + PA > 0$. Therefore if n is sufficiently large

$$\frac{1}{n} x'_n [PA + A'P] x_n > x'_n O\left(\frac{1}{n^2}\right) x_n$$

and thus $x'_{n+1} P x_n < x'_n P x_n$ and consequently x_n is bounded. This justifies the fact that the $1/n$ term dominates in (A3). □

LEMMA A4. *Consider the recursion*

$$N_{t+1} = N_t - \frac{1}{t} [N_t Q' + Q N_t] + \frac{1}{t_2} \mathcal{L}(N_t), \qquad t = 1, 2, \cdots \tag{A4}$$

where N_t, Q *are square matrices.* $N_t \to 0$ *for any initial condition if and only if* $\mathrm{Re}\,\lambda(Q) > 0$.

Proof. Let x_t be the vector composed of the columns of N_t. We can write the recursion equivalently as

$$x_{t+1} = x_t - \frac{1}{t} A x_t + \frac{1}{t^2} \mathcal{L}(x_t).$$

It can be checked that $\mathrm{Re}\,\lambda(A) > 0$ if and only if $\mathrm{Re}\,\lambda(Q) > 0$ and thus Lemma A3 can be applied. □

It should be pointed out that if x_n evolves as in (A1), and λ is real, x_n behaves like $n^{-\lambda}$ (see 12, (2.3)). If λ is complex, then (A2) implies that $|x_n|^2$ behaves like n^{-2a} and thus $|x_n|$ behaves like n^{-a}, i.e., $n^{-\mathrm{Re}\lambda}$. Consequently x_{n+1} in (A3) behaves like $n^{-\bar{\lambda}}$, where $\bar{\lambda} = \min \mathrm{Re}\,\lambda\,(A)$ and N_t in (A4) behaves like $t^{-2\bar{\lambda}}$ where $\bar{\lambda} = \min \mathrm{Re}\,\lambda(Q)$.

Appendix B. Let l_{in}, m_{in}, c_i, d_i be as in (28). For convenience, let

$$y_{1,t-1} = \begin{bmatrix} y_1 \\ \vdots \\ y_{n_1} \end{bmatrix}, \qquad y_{2,t-1} = \begin{bmatrix} z_1 \\ \vdots \\ z_{n_2} \end{bmatrix}. \tag{B1}$$

Equation (27) can be written as

$$\text{(B2)}\qquad l_{it}=l_{i,t-1}-\frac{1}{t-1}\left[l_{i,t-1}+y_iR_1\sum_{j=1}^{n_2}z_jm_{j,t-1}+c_i\right],\qquad i=1,\cdots,n_1,$$

$$\text{(B3)}\qquad m_{it}=m_{i,t-1}-\frac{1}{t-1}\left[m_{i,t-1}+z_iR_2\sum_{j=1}^{n_1}y_jl_{j,t-1}+d_i\right],\qquad i=1,\cdots,n_2.$$

For convenience, let us drop the subscript $t-1$ from $l_{i,t-1}$, $m_{i,t-1}$. From (B2), (B3), we obtain:

$$\text{(B4)}\qquad \begin{aligned} l_{it}l'_{jt}=l_il'_j&-\frac{1}{t-1}\left[2l_il'_j+y_j\sum_{l=1}^{n_2}z_il_im'_lR'_1+y_iR_1\sum_{k=1}^{n_2}z_km_kl'_j+l_ic'_j+c_il'_j\right]\\ &+\frac{1}{(t-1)^2}\left[l_il'_j+y_j\sum_{l=1}^{n_2}z_il_im'_lR'_1+y_iR_1\sum_{k=1}^{n_2}z_km_kl'_j+y_iy_jR_1\sum_{k,l=1}^{n_2}z_kz_lm_km'_lR'_1\right.\\ &\qquad\left.+y_iR_i\sum_{k=1}^{n_2}z_km_kc'_j+y_j\sum_{l=1}^{n_2}z_lc_im'_lR'_1+l_ic'_j+c_il'_j+c_ic'_j\right],\\ &\qquad i,j=1,\cdots,n_1,\end{aligned}$$

$$\text{(B5)}\qquad \begin{aligned} m_{it}m'_{jt}=m_im'_j&-\frac{1}{t-1}\left[2m_im'_j+z_j\sum_{l=1}^{n_1}y_lm_il'_lR'_2+z_iR_2\sum_{k=1}^{n_1}y_kl_km'_j+m_id'_j+d_im'_j\right]\\ &+\frac{1}{(t-1)^2}\left[m_im'_j+z_j\sum_{l=1}^{n_1}y_lm_il'_lR'_2+z_iR_2\sum_{k=1}^{n_1}y_kl_km'_j+z_iz_jR_2\sum_{k,l=1}^{n_1}y_ky_ll_kl'_lR'_2\right.\\ &\qquad\left.+z_iR_2\sum_{k=1}^{n_1}y_kl_kd'_j+z_j\sum_{l=1}^{n_1}y_ld_il'_lR'_2+m_id'_j+d_im'_j+d_id'_j\right],\\ &\qquad i,j=1,\cdots,n_2,\end{aligned}$$

$$\text{(B6)}\qquad \begin{aligned} l_{it}m'_{jt}=l_im'_j&-\frac{1}{t-1}\left[2l_im_j+z_j\sum_{l=1}^{n_1}y_ll_il'_lR'_2+y_iR_1\sum_{k=1}^{n_2}z_km_km'_j+l_id'_j+c_im'_j\right]\\ &+\frac{1}{(t-1)^2}\left[l_im'_j+z_j\sum_{l=1}^{n_1}y_ll_il'_lR'_2+y_iR_1\sum_{k=1}^{n_2}z_km_km'_j+y_iz_jR_1\sum_{k=1}^{n_2}\sum_{l=1}^{n_1}z_ky_lm_kl'_lR'_2\right.\\ &\qquad\left.+y_iR_1\sum_{k=1}^{n_2}z_km_kd'_j+z_j\sum_{l=1}^{n_1}y_lc_il'_lR'_2+l_id'_j+c_im'_j+c_id'_j\right],\\ &\qquad i=1,\cdots,n_1,\quad j=1,\cdots,n_2.\end{aligned}$$

Let Λ^t_{ij}, M^t_{ij}, K^t_{ij} be defined as in (34), let c_i, $d_i=0$ for simplicity and w.l.o.g.. We take expectation in (B4)–(B6) and drop for convenience the superscript $t-1$ from Λ^{t-1}_{ij}, M^{t-1}_{ij}, K^{t-1}_{ij} in the right-hand side. (When taking expectations, we use the fact that l^{t-1}_i, m^{t-1}_i are independent of $y_{1,t-1}$, $y_{2,t-1}$.) We obtain:

$$\text{(B7)}\qquad \begin{aligned} \Lambda^t_{ij}=\Lambda_{ij}&-\frac{1}{t-1}[2\Lambda_{ij}+\sigma_jK_{ij}R'_1+\sigma_iR_1(K_{ij})']\\ &+\frac{1}{(t-1)^2}\left[\Lambda_{ij}+\sigma_jK_{ij}R'_1+\sigma_iR_1(K_{ij})'\right.\\ &\qquad\left.+\begin{Bmatrix}\sigma_i\sigma_jR_1(M_{ij}+M_{ji})R'_1, & \text{if } i\neq j\\ \hat{R}_1\left(\sum_{\substack{k=1\\k\neq i}}^{n_2}M_{kk}+E(y_i^2z_i^2)M_{ii}\right)R'_1, & \text{if } i\neq j\end{Bmatrix}\right],\\ &\qquad i,j=1,\cdots,n_1,\end{aligned}$$

$$M'_{ij}=M_{ij}-\frac{1}{t-1}[2M_{ij}+\sigma_j K_{ij}R'_2+\sigma_i R_1(K_{ij})']+\frac{1}{(t-1)^2}\mathscr{L}_2(\Lambda_{ij}\text{'s})$$

(B8)
$$i,j=1,\cdots,n_2 \text{ and } \sigma_i=0 \quad \text{if } i>n_1,$$
$$\sigma_j=0 \quad \text{if } j>n_1,$$

$$K'_{ij}=K_{ij}-\frac{1}{t-1}[2K_{ij}+\sigma_j\Lambda_{ij}R'_2+\sigma_i R_1 M_{ij}]+\frac{1}{(t-1)^2}\mathscr{L}_3(\Lambda_{ij}, M_{ij}, K_{ij}\text{'s}),$$

(B9)
$$i=1,\cdots,n_1,\quad \sigma_i=0 \text{ if } i>n_1,$$
$$j=1,\cdots,n_2,\quad \sigma_j=0 \text{ if } j>n_2.$$

Defining N_t and Q as in (35), (36) we see that (B7)-(B9) can be written in compact form as in (37).

Appendix C. Let $x_t=(l'_{it},\cdots,l'_{n_1,t},m'_{1,t},\cdots,m'_{n_2,t})$. Using (27) or the equivalent (B2), (B3) we have

(C1)
$$x_{t+1}=x_t-\frac{1}{t}[R(y_{1t},y_{2t})x_t]$$

where the definition of $R(y_{1t},y_{2t})=\bar{R}_t$ is obvious from (B2), (B3). From (C1) we obtain

(C2)
$$\|x_{t+1}\|^2=\|x_t\|^2-\frac{2}{t}x'_t\bar{R}_tx_t+\frac{1}{t^2}x'_t\bar{R}'_t\bar{R}_tx_t.$$

It holds

(C3)
$$\begin{aligned}&E[\|x_{t+1}\|^2-\|x_t\|^2|\|x_1\|^2,\cdots,\|x_t\|^2]\\&\quad=E[E[\|x_{t+1}\|^2-\|x_t\|^2|\|x_1\|^2,\cdots,\|x_t\|^2]|x_1,\cdots,x_t],\end{aligned}$$

(C4a)
$$\begin{aligned}E[x'_t\bar{R}_tx_t|\|x_1\|^2,\cdots,\|x_t\|^2]&=E[E[x'_t\bar{R}_tx_t|\|x_1\|^2,\cdots,\|x_t\|^2]|x_1,\cdots,x_t]\\&=E[E[E[x'_t\bar{R}_tx_t|x_1,\cdots,x_t]|x_1,\cdots,x_t]|\|x_1\|^2,\cdots,\|x_t\|^2]\\&=E[x'_tE[\bar{R}_t|x_1,\cdots,x_t]x_t|\|x_1\|^2,\cdots,\|x_t\|^2]\\&=E[x'_tR_1x_t|\|x_1\|^2,\cdots,\|x_t\|^2],\end{aligned}$$

since $\bar{R}_t$ depends only on y_{1t}, y_{2t} which are independent of $x_1,\cdots,x_t$ and where R_1 is a constant matrix defined by

(C4b)
$$E[R(y_{1t},y_{2t})]=R_1.$$

Similarly

(C4c)
$$E[x'_t\bar{R}_t\bar{R}_tx_t|\|x_1\|^2,\cdots,\|x_t\|^2]=E[x'_tR_2x_t|\|x_1\|^2,\cdots,\|x_t\|^2]$$

where R_2 is a constant matrix defined by

(C5)
$$E[R'(y_{1,t},y_{2,t})R(y_{1t},y_{2t})]=R_2.$$

From (C3)-(C5) we obtain

(C6)
$$\begin{aligned}&E[\|x_{t+1}\|^2-\|x_t\|^2|\|x_1\|^2,\cdots,\|x_t\|^2]\\&\quad=E\left[x'_t\left(-\frac{2}{t}R_1+\frac{1}{t^2}R_2\right)x_t|\|x_1\|^2,\cdots,\|x_t\|^2\right].\end{aligned}$$

It holds

$$-\frac{\alpha}{t} I \leq -\frac{2}{t} R_1 + \frac{1}{t^2} R_2 \leq \frac{\alpha}{t} I \tag{C7}$$

for some positive constant α and thus

$$\left| \left[E \left[x_t' \left(-\frac{2}{t} R_1 + \frac{1}{t^2} R_2 \right) x_t \,\middle|\, \|x_1\|^2, \cdots, \|x_t\|^2 \right] \right| \right. \tag{C8}$$
$$\leq \frac{\alpha}{t} E[\|x_t\|^2 | \|x_1\|^2, \cdots, \|x_t\|^2] = \frac{\alpha}{t} \|x_t\|^2.$$

Let $V_t = \|x_t\|^2$; then from (C6) and (C7) we obtain

$$|E[V_{t+1} - V_t | V_1, \cdots, V_t]| \leq \frac{\alpha}{t}. \tag{C9}$$

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