On the Informational Properties of the Nash Solution of LQG Dynamic Games

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Abstract—The *M*-person, *N*-stage discrete-time LQG Nash game is considered. The players use strategies that are linear functions of the current estimate of the state, generated by a Kalman filter. We study the impact of improvements of the information on the costs of the players. For certain classes of such problems, we show that better information is beneficial to all the players if the number of stages, or the number of players, is larger than some bounds, and which bounds are given explicitly in terms of the coefficient matrices. Related properties of the two-person zero-sum game are also investigated. It is shown that under certain conditions, better information is beneficial to the player who has better maneuverability while the saddle-point cost is independent of the information if both players have the same maneuverability. Conditions guaranteeing the uniform boundedness of the solutions of the coupled Riccati equations which arise in such games are also given.

I. INTRODUCTION

NONCOOPERATIVE multiobjective control problems have received much attention, due to their usefulness in economics, engineering, etc. In particular, the Nash equilibrium concept has been studied by several researchers (see [1]-[3], [9], [10]). There are several features which are characteristic of the Nash game and differentiate it quite sharply from the single objective control problem. The particular feature which we consider here is the impact that changes in the information have on the optimal costs in a Nash game. It is known that better information is beneficial in a single objective problem, in the sense that it results in smaller cost, but as it has been observed in a Nash setup, better information is not necessarily beneficial to the players. The purpose of this paper is to study the impact on the optimal costs of an LQG Nash game, of improvements in the information available to the players.

Several examples concerning informativeness and performance (i.e., the impact of changes in the information on the costs) have been considered. Ho and Blau [4] gave an example of a two-person Nash game, where better information for both players results in higher costs for both of them. Basar and Ho [5] gave two examples which show that better information for either player might result in lower average cost for both players in the LQG model, whereas for the dupoly model, better information for one player helps him alone while it hurts the other player. Walsh and Cruz [6] had another example of a two person Nash game in which worse information for one of the players could help him and hurt the other. Informational competitiveness and consistency constraints are used to explain such phenomena in the quoted work. Still, the above examples consider special cases which are too specific for any concrete general conclusions to be drawn. In this paper we study the impact of informativeness on performance for a more general dynamic problem and relate it to the number of stages or players involved. We consider the case where the information available to the players is the same, i.e., it is public information. It should be noted that several of the examples in [4]-[6] consider the case where the information of only one player is improved, i.e., they examine the impact of changes of the private information of the players.[1]

The structure of the paper is as follows. In Section II we state the LQG Nash game. At each stage k, the players are allowed to use a function of an estimate $\hat{x}(k)$ of the state $x(k)$, which is generated through a Kalman filter that uses linear, noise corrupted measurements of $x(k)$. In this setup the solution exists, is unique and linear in $\hat{x}(k)$, under invertibility assumptions on some matrices. This solution can also be obtained by solving the deterministic game where the players use linear strategies in the current state $x(k)$ and using $\hat{x}(k)$ in place of $x(k)$, i.e., by imposing separation. In this section, we also derive a bound on the norms of the solutions of the coupled Riccati equations involved, which guarantees the uniform boundedness—for any k—of these solutions. In Sections III-V several sufficient conditions that better information (to be defined later) will lower the players' costs are derived in terms of the coefficients of the game. An interesting result that turns out in our analysis is that, for a certain class of games, if the number of players is large enough, then better information is beneficial to all the players. An intuitive interpretation of this result is that if the number of players is very large, then each player can think of all the others as being very little affected by him, and thus improvement of his information is not used by the others in order to increase his cost. Another result that we prove, for a certain class of games, is that if the number of stages is large enough, then dynamically better information (to be defined later) is beneficial to all the players. An interpretation of this result is that if the number of stages is very large, then the players become more coupled and tend to cooperate, bringing about the feature of the team problem where more information is beneficial. In Section VI we consider the two-player zero-sum game and we investigate the impact of changes in the information on the saddle-point cost. Under certain conditions, we show that if $B_iB_i^T > B_jB_j^T$, then better information is beneficial to player i and if $B_1 = B_2$, then the saddle-point cost is independent of the information available to the players. A similar result for the continuous-time LQG zero-sum game was given in [11] where $B_iB_i^T$ is viewed as the maneuverability of player i. The above results could be interpreted as follows. If $B_1B_1^T > B_2B_2^T$ (player 1 has better maneuverability which implies higher controllability [11]), then player 1 dominates the system, and thus he can use the better information to his benefit. When $B_1 = B_2$, both players can equally influence the system and the controls exerted by the players cancel each other, so that the system is free of control, and hence the cost is free of the information available to the controllers. In Section VII we present three numerical examples to illustrate the informational properties discussed in the previous sections. Finally, in Section VIII we present our conclusions and delineate further directions of research.

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[1] In a sequel paper we present an analysis parallel to the one of the present paper studying the impact of changes of the private information. In regard to this, see also [15].

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II. Description of the Game and Its Solution

Consider an M-person, N-stage Nash game where $x(k)$ evolves according to

$$x(k+1)=Ax(k)+\sum_{j\in\theta_2} B_j u_j(k)+w(k), \qquad x(0)=x_0 \tag{1}$$

where

$$k\in\theta_1=\{0, 1, \cdots, N-1\}$$

$$\theta_2=\{1, 2, \cdots, M\}$$

$x(k) \in R^n$ and $u_j(k) \in R^{l_j}$ denotes the action of player j at stage k. x_0 and $\{w(k), k \in \theta_1\}$ are Gaussian independent random vectors and $E[w(k)w^T(k)] = R$, $E[w(k)] = 0$, cov $[x_0, x_0] = \Omega_0$, $E[x_0] = \bar{x}_0$. Consider the measurements

$$y(k)=Hx(k)+v(k) \tag{2}$$

where the $v(k)$'s are zero-mean Gaussian, independent of x_0, $\{w(k), k \in \theta_1\}$ and $E[v(n)v^T(k)] = \Sigma\, \delta_{nk}$, $\Sigma > 0$. Let

$$\hat{x}(k)=E[x(k)\,|\,y(0), y(1), \cdots, y(k)] \tag{3}$$

and given explicitly by the Kalman filter equations:

$$\hat{x}(k)=\hat{x}(k/k-1)+D(k)[y(k)-H\hat{x}(k/k-1)] \tag{4a}$$

$$\hat{x}(k+1/k)=A\hat{x}(k)+\sum_{j\in\theta_2} B_j u_j(k), \qquad \hat{x}(0/-1)=\bar{x}_0 \tag{4b}$$

$$D(k)=\Sigma(k/k-1)H'(H\Sigma(k/k-1)H'+\Sigma)^{-1} \tag{4c}$$

$$\Sigma(k+1/k)=A[I-D(k)H]\Sigma(k/k-1)A'+R, \qquad \Sigma(0/-1)=\Omega_0 \tag{4d}$$

$$\Sigma(k)=[I-D(k)H]\Sigma(k/k-1) \tag{4e}$$

where

$$\hat{x}(k+1/k)=E[x(k+1)|y(0), y(1), \cdots, y(k)] \tag{5}$$

is the one-step prediction of the state and

$$\Sigma(k)=E[x(k)-\hat{x}(k)][x(k)-\hat{x}(k)]' \tag{6}$$

$$\Sigma(k+1/k)=E[x(k+1)-\hat{x}(k+1/k)][x(k+1)-\hat{x}(k+1/k)]' \tag{7}$$

are, respectively, the current estimation error covariance and the one-step prediction error covariance.

The cost of player i, $i \in \theta_2$ is

$$J_i=E\left\{\sum_{n=0}^{N-1}\left[x^T(n)P_i x(n)+\sum_{j=1}^{M} u_j^T(n)Q_{ij}u_j(n)\right] + x^T(N)P_i x(N)\right\} \tag{8}$$

where $P_i \geqq 0$, $Q_{ij} \geqq 0$, $Q_{ii} = I$, and E denotes total expectation. $u_j(k)$ is chosen as $\gamma_j^k(\hat{x}(k))$ and the γ_j^k's are measurable functions, $\gamma_j^k: R^n \to R^{l_j}$ with the property that $\gamma_j^k(\hat{x}(k))$ is a second-order random vector. The employment of such strategies can be justified by considering that there is an impartial referee who computes $\hat{x}(k)$ and communicates it to the players (see also Remark 1).

Let

$$g_i=\{\gamma_i^0, \gamma_i^1, \cdots, \gamma_i^{N-1}\}, \qquad i\in\theta_2.$$

A set $\{g_1^*, \cdots, g_M^*\}$ is called a Nash equilibrium solution to the game if for every $i \in \theta_2$

$$J_i(g_1^*, \cdots, g_M^*) \leqq J_i(g_1^*, \cdots, g_{i-1}^*, g_i, g_{i+1}^*, \cdots, g_M^*) \tag{9}$$

for all admissible g_i's.

The Nash equilibrium solution to the game under consideration is provided in the following theorem, the proof of which is given in Appendix A.

Theorem 2.1: Consider the equations

$$L_i(k)=P_i+A^T\left[\left(I+\sum_{j\in\theta_2} B_jB_j^TL_j(k+1)\right)^{-1}\right]^T \cdot\left[L_i(k+1)+\sum_{j\in\theta_2} L_j(k+1)B_jQ_{ij}B_j^TL_j(k+1)\right] \cdot\left[I+\sum_{j\in\theta_2} B_jB_j^TL_j(k+1)\right]^{-1}A, \qquad L_i(N)=P_i, \; i\in\theta_2 \tag{10}$$

which evolve backwards in time. We assume the inverse of $[I + \sum_{j\in\theta_2} B_jB_j^TL_j(k + 1)]$ exists for every $k \in \theta_1$, then the following holds.

i) There exists a unique Nash solution to the game which is the following:

$$u_i^*(k)={}^*\gamma_i^k(\hat{x}(k))=F_i(k)\hat{x}(k), \qquad k\in\theta_1, \; i\in\theta_2 \tag{11}$$

where

$$F_i(k)=-B_i^TL_i(k+1)\left[I+\sum_{j\in\theta_2} B_jB_j^TL_j(k+1)\right]^{-1}A. \tag{12}$$

ii) The cost to go of player i at stage k (defined as $J_i(k)$ equal to the expression in (8) where the summation is from k up to $N - 1$) is

$$J_i(k)=E[\hat{x}^T(k)L_i(k)\hat{x}(k)]+K_i(k) \tag{13}$$

where

$$K_i(k)=\text{tr}\;\{P_i\Sigma(k)+A^TL_i(k+1)A\Sigma(k) + L_i(k+1)(R-\Sigma(k+1))\}+K_i(k+1),$$
$$K_i(N)=\text{tr}\;\{P_i\Sigma(N)\}. \tag{14}$$

Remark 1: From (12) and (10) we see that the $F_i(k)$ is independent of the observation noise. As a matter of fact, the $F_i(k)$'s are the same as those that will be used in this Nash game if all the players have perfect measurements of the state and use feedback (closed-loop no memory) strategies. This means that the separation principle of estimation and control holds for the strategies considered. This fact provides another motivation for considering such strategies.

Remark 2: The nonsingularity conditions required in Theorem 2.1 can be satisfied for several cases as, for example, when $B_j = b_jB$, b_j, a real number, since then the eigenvalues of $I + BB^T \sum_{j\in\theta_2} b_j^2L_j(k + 1)$ are $\geqq 1$.

Remark 3: If the players know the past history $\{y(0), \cdots, y(k)\}$ and use it explicitly when they calculate their decisions at stage k, rather than just $\hat{x}(k)$, then the Nash solution is different (see [18], [1]). In [17], the same strategies as those considered here are derived, after restricting them to be affine in $\hat{x}(k)$; notice that we do not impose *a priori* the affine character of the strategies.

In our later derivations, it is sometimes crucial to have $L_i(k)$ uniformly bounded as k goes to infinity. Before giving the next theorem which provides a sufficient condition to that effect, we state the following lemma, the proof of which is given in Appendix B (the norm of any matrix considered here is the sup norm, and the norm of any vector is the Euclidean one).

Lemma 2.1: a) If $X > Y \geq 0$, then $X^{-1} > X^{-1}YX^{-1}$. b) If $X \geq Y > 0$, then $Y^{-1} \geq X^{-1}$.

Theorem 2.2: If $B_j = b_jB$, b_j a real number, $j \in \theta_2$ and $B_jB_j^T$ is nonsingular, then $L_i(k)$ given by (8) is bounded by a constant which is independent of k provided that

$$\|A\|^2 \cdot \sigma_{\max}(W) < \lambda_{\min}\left(BB^T + BB^T\sum_{j\in\theta_2} b_j^2 P_j BB^T\right) \quad (15)$$

where

$$W = \begin{bmatrix} \|BB^T\| & \|BQ_{12}B^T\| & \cdots & \|BQ_{1M}B^T\| \\ \|BQ_{21}B^T\| & \|BB^T\| & \cdots & \|BQ_{2M}B^T\| \\ \vdots & & & \\ \|BQ_{M1}B^T\| & \|BQ_{M2}B^T\| & \cdots & \|BB^T\| \end{bmatrix} \quad (16)$$

$\sigma_{\max}(W)$ denotes the largest singular value of W and $\lambda_{\min}(\cdot)$ denotes the smallest eigenvalue of $(\cdot)$.

Proof: First notice that since $B_j = b_jB$, the nonsingularity condition of Theorem 2.1 holds, and thus the $L_i(k)$'s are well defined $\forall i, k$. From (10) we have that $L_i(k) = P_i$ + a nonnegative definite matrix, and thus

$$L_i(k) \geq P_i. \quad (17)$$

We will now show that $\|b_iL_i^{1/2}(k)(I + BB^T\sum_{j\in\theta_2} b_j^2L_j(k))^{-1}\|$ is bounded by a constant c. It holds that

$$\begin{aligned}
&\left\| b_iL_i^{1/2}(k)\left(I + BB^T\sum_{j\in\theta_2} b_j^2L_j(k)\right)^{-1}\right\| \\
&= \sup_{\|x\|=1}\left\| b_iL_i^{1/2}(k)\left((BB^T)^{-1} + \sum_{j\in\theta_2} b_j^2L_i(k)\right)^{-1}(BB^T)^{-1}x\right\| \\
&= \left\{\lambda\max\left[(BB^T)^{-1}\left((BB^T)^{-1} + \sum_{j\in\theta_2} b_j^2L_j(k)\right)^{-1} b_i^2L_i(k)\right.\right. \\
&\quad \left.\left.\cdot\left((BB^T)^{-1} + \sum_{j\in\theta_2} b_j^2L_j(k)\right)^{-1}(BB^T)^{-1}\right]\right\}^{1/2} \\
&\leq \left\{\lambda\max\left[(BB^T)^{-1}\left((BB^T)^{-1} + \sum_{j\in\theta_2} b_j^2L_j(k)\right)^{-1}\right.\right. \\
&\quad \left.\left.\cdot (BB^T)^{-1}\right]\right\}^{1/2} \quad \text{[by Lemma 2.1 a)]} \\
&\leq \left\{\lambda\max\left[BB^T + BB^T\sum_{j\in\theta_2} b_j^2P_jBB^T\right]^{-1}\right\}^{1/2} \triangleq c^{1/2} \\
&\text{[by Lemma 2.1 b)]}
\end{aligned} \quad (18)$$

where $\lambda_{\max}(\cdot)$ denotes the largest eigenvalue of $(\cdot)$. From (10) we have

$$\begin{aligned}
\|L_i(k)\| &\leq \|P_i\| + \|A\|^2c \\
&\quad \cdot\left[1/b_i^2 + \sum_{j\in\theta_2}\|L_j^{1/2}(k+1)BQ_{ij}B^TL_j^{1/2}(k+1)\|\right] \\
&\leq \|P_i\| + \|A\|^2c\left[1/b_i^2 + \sum_{j\in\theta_2}\|BQ_{ij}B^T\|\ \|L_j(k+1)\|\right]
\end{aligned} \quad (19)$$

for every $i \in \theta_2$, i.e.,

$$\begin{bmatrix} \|L_1(k)\| \\ \vdots \\ \|L_M(k)\| \end{bmatrix} \leq \begin{bmatrix} \|P_1\| \\ \vdots \\ \|P_M\| \end{bmatrix} + \|A\|^2c \cdot \left(\begin{bmatrix} 1/b_1^2 \\ \vdots \\ 1/b_M^2 \end{bmatrix} + W\begin{bmatrix} \|L_1(k+1)\| \\ \vdots \\ \|L_M(k+1)\| \end{bmatrix}\right). \quad (20)$$

Hence, $\|L_i(k)\|$ will be uniformly bounded if

$$\|A\|^2c\sigma_{\max}(W) < 1, \quad (21)$$

i.e.,

$$\|A\|^2 \cdot \sigma_{\max}(W) < \lambda_{\min}\left(BB^T + BB^T\sum_{j\in\theta_2} b_j^2P_jBB^T\right) \quad (22)$$

where W is given by (16). □

III. An Informational Property of the Nash Solution

In this section we define the concept of better information and dynamically better information and we investigate how the Nash costs are affected by the quality of the information. A theorem is then established which gives a sufficient condition that better information will lower player i's cost. In the next two sections, based on the work done in this section, we will delineate how the number of players and the number of stages affect the informational properties of the Nash solution.

Let

$$y^I(\cdot) = H^Ix(\cdot) + v^I(\cdot) \quad (23a)$$

$$y^{II}(\cdot) = H^{II}x(\cdot) + v^{II}(\cdot) \quad (23b)$$

be two different observations of $x(\cdot)$ and let $\hat{x}^I(k)$ and $\hat{x}^I(k+1/k)$ be defined as in (3) and (5) based on $y^I(\cdot)$ with associated error covariance $\Sigma^I(k)$ and $\Sigma^I(k+1/k)$. $\hat{x}^{II}(k)$, $\hat{x}^{II}(k+1/k)$, $\Sigma^{II}(k)$ and $\Sigma^{II}(k+1/k)$ are similarly defined.

Definition 3.1: We say that the estimate $\hat{x}^I(\cdot)$ provides *better information*[2] about the random vector $x(\cdot)$ than $\hat{x}^{II}(\cdot)$ does, if and only if $\Sigma^{II}(k) - \Sigma^I(k) \geq 0$ for all k and $\Sigma^{II}(k) \neq \Sigma^I(k)$ for at least one k. If, in addition, we have

$$\Sigma^{II}(k/k-1) - \Sigma^I(k/k-1) \geq \Sigma^{II}(k) - \Sigma^I(k) \quad (24)$$

then we say that $\hat{x}^I(\cdot)$ provides *dynamically better information* about $x(\cdot)$ than $\hat{x}^{II}(\cdot)$ does.

Better information simply means that the mean-square estimation error $\Sigma(k)$ is reduced and dynamically better information means that the one-step prediction error $\Sigma(k/k-1)$ is reduced no less than the reduction of the estimation error $\Sigma(k)$. For convenience, we denote

$$\hat{\Omega}^I(k) = E[\hat{x}^I(k)\hat{x}^{IT}(k)]. \quad (25)$$

J_i^I and $K_i^I(k)$ are defined as in (13) and (14) corresponding to information $\hat{x}^I(\cdot)$. Similarly, we define $\hat{\Omega}^{II}(k)$, J_i^{II}, and $K_i^{II}(k)$. Before giving a sufficient condition that better information is

[2] Notice that "an increase of information" of [6], [13] or "more observable system" of [11], or "better information" of [5] implies "better information" by our definition.

beneficial to the players, we state the following lemma, the proof of which is given in Appendix C.

Lemma 3.1: If $X > 0$ and Y is a nonzero, nonnegative definite matrix, then tr $\{XY\} > 0$.

Theorem 3.1: The Nash solution given by Theorem 2.1 has the property that better information lowers the cost of player i if $P_i + A^TL_i(k+1)A - L_i(k) > 0$ for all $k \in \theta_1$.

Proof: From part ii) of Theorem 2.1,

$$J_i^I = E[\hat{x}^{IT}(0)L_i(0)\hat{x}^I(0)] + K_i^I(0) = \text{tr}\ \{L_i(0)\hat{\Omega}^I(0) + K_i^I(0)\}. \quad (26)$$

From the recursive expression of K_i in (14), we have

$$K_i^I(0) = \text{tr}\ \Big\{P_i\Sigma^I(0) + A^TL_i(1)A\Sigma^I(0) + \sum_{k=1}^{N-1}[P_i + A^TL_i(k+1)A - L_i(k)]\Sigma^I(k) + \sum_{k=0}^{N-1} RL_i(k+1)\Big\}. \quad (27)$$

Thus,

$$J_i^I = \text{tr}\ \Big\{L_i(0)\hat{\Omega}^I(0) + [P_i + A^TL_i(1)A]\Sigma^I(0) + \sum_{k=1}^{N-1}[P_i + A^TL_i(k+1)A - L_i(k)]\Sigma^I(k) + \sum_{k=0}^{N-1} RL_i(k+1)\Big\}. \quad (28)$$

Similarly,

$$J_i^{II} = \text{tr}\ \Big\{L_i(0)\hat{\Omega}^{II}(0) + [P_i + A^TL_i(1)A]\Sigma^{II}(0) + \sum_{k=1}^{N-1}[P_i + A^TL_i(k+1)A - L_i(k)]\Sigma^{II}(k) + \sum_{k=0}^{N-1} RL_i(k+1)\Big\}. \quad (29)$$

By using the fact that

$$\hat{\Omega}^{II}(0) - \hat{\Omega}^I(0) = -(\Sigma^{II}(0) - \Sigma^I(0)) \quad (30)$$

we obtain

$$J_i^{II} - J_i^I = \sum_{k\in\theta_1} \text{tr}\ \{[P_i + A^TL_i(k+1)A - L_i(k)][\Sigma^{II}(k) - \Sigma^I(k)]\}. \quad (31)$$

Suppose now that $\hat{x}^I(\cdot)$ provides better information than $\hat{x}^{II}(\cdot)$ does, then Lemma 3.1 implies $J_i^{II} - J_i^I > 0$ if

$$P_i + A^TL_i(k+1)A - L_i(k) > 0 \quad \text{for all } k\in\theta_1. \quad (32)$$

□

IV. Effect of M on the Informational Property of the Nash Solution

We will demonstrate in this section that under some conditions, as the number of players increases, better information is beneficial to all the players. We will need the following lemma which we prove in Appendix D.

Lemma 4.1: If $Y \geq 0$, then $\lambda_{\min}(X^TYX) \geq \lambda_{\min}(Y)\cdot\lambda_{\min}(X^TX)$.

Theorem 4.1: If $B_j = b_jB$, b_j a real number, $P_j \geq P > 0, j \in \theta_2$, A, $B_jB_j^T$ are nonsingular, and $L_j(k)$ is uniformly bounded for all $j \in \theta_2$, then better information lowers all the players' costs if M, the number of players, is large enough.

Proof: From Theorem 3.1, a sufficient condition that better information lowers the cost of player i is (32). Substituting (10) into (32), we obtain

$$P_i + A^TL_i(k+1)A - L_i(k) = U^TVU \quad (33)$$

where

$$U \triangleq \Big[I + BB^T\sum_{j\in\theta_2} b_j^2L_j(k+1)\Big]^{-1}A \quad (34)$$

$$V \triangleq \Big[I + BB^T\sum_{j\in\theta_2} b_j^2L_j(k+1)\Big]^T L_i(k+1)\cdot\Big[I + BB^T\sum_{j\in\theta_2} b_j^2L_j(k+1)\Big] - L_i(k+1) - \sum_{j\in\theta_2} b_j^2L_j(k+1)BQ_{ij}B^TL_j(k+1). \quad (35)$$

Since U is nonsingular, the condition for (32) to hold is the same one as $V > 0$. By writing

$$I + BB^T\sum_{j\in\theta_2} b_j^2L_j(k+1) = BB^T\Big[(BB^T)^{-1} + \sum_{j\in\theta_2} b_j^2L_j(k+1)\Big] \quad (36)$$

and let $\lambda_{\min}(P) = \delta$, $\lambda_{\min}(BB') = \beta$, $\max_{i,j}\|Q_{ij}\| = q$, $L_j(k) \leq \bar{L}$, $\bar{b} \geq |b_j| \geq b > 0$, then by Lemma 4.1 we have

$$V \geq [(Mb^2\beta\delta)^2\delta - \|\bar{L}\| - M\bar{b}^2q\|BB^T\|\ \|\bar{L}\|^2]I. \quad (37)$$

Thus, $V > 0$ if

$$M > \frac{\|\bar{L}\| + \bar{b}^2q\|BB^T\|\ \|\bar{L}\|^2}{b^4\delta^3\beta^2}. \quad (38)$$

□

Remark 4: If all the players have the same cost functional to minimize, i.e., $P_j = P$, $Q_{ij} = I$ for all i, j and release the condition that $B_j = b_jB$, then the decision problem is known as an LQG team problem and a Nash solution leads to person-by-person optimality which coincides with team optimality. In this case, $L_j(k) = L(k)$ for all $j \in \theta_2$ and

$$\begin{aligned} V &= \Big[I + \sum_{j\in\theta_2} B_jB_j^TL(k+1)\Big]^T L(k+1)\Big[I + \sum_{j\in\theta_2} B_jB_j^TL(k+1)\Big] - L(k+1) - L(k+1)\sum_{j\in\theta_2} B_jB_j^TL(k+1) \\ &= L(k+1)\sum_{j\in\theta_2} B_jB_j^TL(k+1) + L(k+1)\sum_{j\in\theta_2} B_jB_j^TL(k+1)\sum_{j\in\theta_2} B_jB_j^TL(k+1) \\ &> 0 \text{ for } M \geq 1. \end{aligned} \quad (39)$$

Hence, better information always lowers the team cost no matter how many players are in a team problem.

Remark 5: Conditions for boundedness of $L_j(k)$ were derived in Theorem 2.2, and thus the related assumption of Theorem 4.1 can be made to hold.

Remark 6: Notice that the bound on M, given in (33) can be ad hoc calculated explicitly.

Remark 7: An interesting effect of M on the asymptotic value of communication in a team situation was studied in [16].

V. Effect of $\bar{N}$ on the Informational Property of the Nash Solution

We shall demonstrate that under certain conditions, as the number of stages increases, dynamically better information is beneficial to all the players.

Assumption 5.1: Let (A, C) be observable and (A, H^T) controllable where $R = C^TC$ [recall $E[w(k)w^T(k)] = R$ and $w(k)$ is the random vector in (1)].

We need the following lemma, which actually states two well-known results from the control and filtering theorem (see [8], [9]).

Lemma 5.1: a) $\Sigma(k+1/k) = A\Sigma(k/k)A^T + R$. b) Under Assumption 5.1, $\Sigma^I(k+1/k)$ and $\Sigma^{II}(k+1/k)$ converge as k goes to infinity, and hence $\Delta\Sigma \triangleq \lim_{k\to\infty} \Delta\Sigma(k) \triangleq (\Sigma^{II}(k) - \Sigma^I(k))$ exists.

Note that from Definition 3.1 and Lemma 5.1 a), if $\hat{x}^I(\cdot)$ is a dynamically better information than $\hat{x}^{II}(\cdot)$, then

$$A\Delta\Sigma A^T \geq \Delta\Sigma. \tag{40}$$

Theorem 5.1: Under Assumption 5.1 and if, in addition, i) $P_i > 0$ and ii) $L_i(k) \leq \bar{L}\ \forall\ k$, then dynamically better information lowers the cost of player i if N, the number of stages, is large enough.

Proof: Let $\hat{x}^I(\cdot)$ provide dynamically better information than $\hat{x}^{II}(\cdot)$. By Lemma 5.1 b), for $\xi > 0$, there exists s such that

$$(1+\xi)\Delta\Sigma \geq \Delta\Sigma(k) \geq (1-\xi)\Delta\Sigma, \qquad \text{for } k \geq s-1. \tag{41}$$

From (31)

$$\begin{aligned} J_i^{II} - J_i^I &= \operatorname{tr}\left\{\sum_{k=0}^{N-1}(P_i + A^TL_i(k+1)A - L_i(k))\Delta\Sigma(k)\right\} \\ &= \operatorname{tr}\left\{\sum_{k=0}^{s-1}(P_i + A^TL_i(k+1)A - L_i(k))\Delta\Sigma(k)\right. \\ &\quad \left. + \sum_{k=s}^{N-1}(A^TL_i(k+1)A - L_i(k))\Delta\Sigma(k) + \sum_{k=s}^{N-1} P_i\Delta\Sigma(k)\right\}. \end{aligned} \tag{42}$$

Let $\lambda_{\min}(P_i) = \delta$, then

$$\sum_{k=s}^{N-1} P_i\Delta\Sigma(k) \geq (N-s)P_i(1-\xi)\Delta\Sigma \geq (N-s)(1-\xi)\delta\Delta\Sigma \tag{43}$$

and

$$\begin{aligned} &\operatorname{tr}\left\{\sum_{k=s}^{N-1}(A^TL_i(k+1)A - L_i(k))\Delta\Sigma(k)\right\} \\ &\geq \sum_{k=s}^{N-1}[\operatorname{tr}\{(A^TL_i(k+1)A - L_i(k))\Delta\Sigma\} \\ &\quad - \operatorname{tr}\{(1+\|A\|^2)\|\bar{L}\|\xi\Delta\Sigma\}] \\ &\geq \operatorname{tr}\left\{(A^TL_i(N)A - L_i(s))\Delta\Sigma + \sum_{k=s}^{N-2} L_i(k+1)(A\Delta\Sigma A^T - \Delta\Sigma)\right\} \\ &\quad - \xi(N-s)(1+\|A\|^2)\|\bar{L}\| \operatorname{tr}\Delta\Sigma \\ &\geq -[1+\xi(N-s)(1+\|A\|^2)]\|\bar{L}\| \operatorname{tr}\Delta\Sigma \end{aligned} \tag{44}$$

where we use the fact that

$$\begin{aligned} &\operatorname{tr}\{(A^TL_i(k+1)A - L_i(k+1))\Delta\Sigma\} \\ &= \operatorname{tr}\{L_i(k+1)(A\Delta\Sigma A^T - \Delta\Sigma)\} \geq 0. \end{aligned}$$

Substituting (43), (44) into (42) we obtain

$$\begin{aligned} J_i^{II} - J_i^I \geq \operatorname{tr}&\left\{\sum_{k=0}^{s-1}(P_i + A^TL_i(k+1)A - L_i(k))\Delta\Sigma(k)\right. \\ &\left. + (N-s)(1-\xi)\delta\Delta\Sigma - [1+\xi(N-s)(1+\|A\|^2)]\|\bar{L}\|\Delta\Sigma\right\} \\ \geq -|\operatorname{tr}&\{\bar{B}\}| + (N-s)[\delta - \xi(\delta + \|\bar{L}\|(1+\|A\|^2))] \operatorname{tr}\Delta\Sigma \end{aligned} \tag{45}$$

where

$$\bar{B} = \sum_{k=0}^{s-1}(P_i + A^TL_i(k+1)A - L_i(k))\Delta\Sigma(k) - \|\bar{L}\|\Delta\Sigma. \tag{46}$$

If we choose

$$\xi = (1/2)\frac{\delta}{\delta + \|\bar{L}\|(1+\|A\|^2)} \tag{47}$$

then $J_i^{II} - J_i^I > 0$ provided that

$$N > s + \frac{2\,|\operatorname{tr}\{\bar{B}\}|}{\delta \operatorname{tr}\{\Delta\Sigma\}}. \tag{48}$$

□

Remark 8: The third example in Section VII shows that Theorem 5.1 might not hold if dynamically better information is replaced by better information.

Remark 9: The "dynamically better information" concept used in the above theorem could also be explained in the following manner. Since (22) is equivalent to

$$\Sigma^{II}(k/k-1) - \Sigma^{II}(k) \geq \Sigma^I(k/k-1) - \Sigma^I(k) \geq 0 \tag{49}$$

the advantage of dynamically better information not only resides in smaller estimation error, but also in that the difference of the one-step prediction error and the estimation error is reduced. The best that one-step prediction error could be is equal to the estimation error, so it is something that really helps in a long-period dynamical multiobjective control process when both the estimation error and its difference with the prediction error are reduced.

Remark 10: The condition (40) used in Theorem 5.1 is true in the scalar case if and only if $|A| \geq 1$, i.e., a better information is also a dynamically better information if and only if $|A| \geq 1$. $|A| \geq 1$ in turn implies that the players are stronger coupled than when $|A| < 1$. In vector case, it is necessary that $\|A\| \geq 1$ and for different A, (40) suggests the right direction to improve the estimation such that better information will be beneficial to the players in the dynamic process. Another interpretation for the dynamically better information condition in Theorem 5.1 is that under dynamically better information (which implies $\|A\| \geq 1$) the open-loop system is unstable; to the players' benefit, when N is large, they had better stabilize the system first and that makes them cooperate.

Remark 11: Notice that the bound on N, given in (48) can be ad hoc calculated explicitly.

Remark 12: Notice that when the condition (38) holds, it is independent of N. Also when (48) holds, it is independent of M. In general, we could think of a joint bound for (M, N) which guarantees that dynamically better information implies lower costs to all the players.

VI. Related Properties of the Zero-Sum Game

Although the two-person zero-sum game [9] is a special case of the Nash problem, not all the theorems that we derived so far apply to it, and this is due to the specific structure of the zero-sum games. For example, the nonnegative definiteness of the matrices that we imposed in the cost functional (8) do not hold for the zero-sum case.

Let us now describe the zero-sum problem that we will consider. Player 1 (the minimizer) wants to minimize while player 2 (the maximizer) wants to maximize the following cost:

$$J=E\left\{\sum_{k=0}^{N-1}[x^T(k)PX(k)+u_1^T(k)u_1(k) - u_2^T(k)u_2(k)]+x^T(N)Px(N)\right\}. \quad (50)$$

The evolution of $x(k)$ is described by

$$x(k+1)=Ax(k)+B_1x_1(k)+B_2x_2(k)+w(k). \quad (51)$$

The counterpart of Theorem 2.1 in the zero-sum game defined above is Theorem 2.1′ given below. The proof of this theorem is similar to that of Theorem 2.1 but here we need $J(k)$ to be convex in $u_1(k)$ and concave in $u_2(k)$ at each stage k while the corresponding convexity conditions in Theorem 2.1 were implied by the nonnegative definiteness of the matrices we imposed in (8).

Theorem 2.1′: Consider the equation

$$\begin{aligned}L(k)=P+A^T[(I+(B_1B_1^T-B_2B_2^T)L(k+1))^{-1}]^T[L(k+1)\\ +L(k+1)(B_1B_1^T-B_2B_2^T)L(k+1)]\\ \cdot[I+(B_1B_1^T-B_2B_2^T)L(k+1)]^{-1}A, \quad L(N)=P \quad (52)\end{aligned}$$

which evolves backwards in time. We assume the inverse of $[I+(B_1B_1^T-B_2B_2^T)L(k+1)]$ exists for every $k\in\theta_1$ and

$$I+B_1^TL(k+1)B_1>0 \quad \text{and} \quad I-B_2^TL(k+1)B_2>0 \quad (53)$$

then the following holds.

i) There exists a unique saddle-point solution to the zero-sum game which is the following:

$$u_i^*(k)={}^*\gamma_i^k(\hat{x}(k))=F_i(k)\hat{x}(k) \qquad k\in\theta_1, \quad i=1,2 \quad (54)$$

where

$$F_i(k)=-B_i^TL(k+1)[I+B_1B_1^T-B_2B_2^T)L(k+1)]^{-1}A. \quad (55)$$

ii) The cost to go of player 1 (player 2) at stage k is $J(k)(-J(k))$ where

$$J(k)=E[\hat{x}^T(k)L(k)\hat{x}(k)]+K(k) \quad (56)$$

$$\begin{aligned}K(k)=\text{tr }\{P\Sigma(k)+A^TL(k+1)A\Sigma(k)\\ +L(k+1)(R-\Sigma(k+1)\}+K(k+1)\\ K(N)=\text{tr }\{P\Sigma(N)\}. \quad (57)\end{aligned}$$

□

Notice that the convexity and concavity conditions which are necessary for the above solution to exist results in (53) which is a kind of boundedness condition for $L(k)$ to hold. The following theorem gives sufficient conditions that $L(k)$ will be bounded and the bound is given explicitly. We omit the proof since it is similar to the proof of Theorem 2.2.

Theorem 2.2′: i) If $P>0$ and $B_1B_1^T>B_2B_2^T$, then $L(k)\geq P>0$. ii) If $P<0$ and $B_1B_1^T<B_2B_2^T$, then $L(k)\leq P<0$. If in addition to either 1) or 2), $\|A\|^2c\|B_1B_1^T-B_2B_2^T\|<1$, then $\|L(k)\|$ is uniformly bounded by $\|\bar{L}\|$ where

$$\|\bar{L}\|\triangleq\frac{\|P\|+\|A\|^2c}{1-\|A\|^2c\|B_1B_1^T-B_2B_2^T\|}+\|A\|^2c\|B_1B_1^T-B_2B_2^T\|\,\|P\| \quad (58)$$

$$c\triangleq\|[(B_1B_1^T-B_2B_2^T)+(B_1B_1^T-B_2B_2^T)P(B_1B_1^T-B_2B_2^T)]^{-1}\|. \quad (59)$$

iii) If $B_1=B_2$, then $L(k)=P+A^TL(k+1)A$. If, in addition, $\|A\|<1$, then

$$\|L(k)\|\leq\frac{\|P\|}{1-\|A\|^2}+\|A\|^2\|P\|. \quad (60)$$

By Theorem 2.1′ ii) and by carrying out similar derivation as in Theorem 3.1, we have the following theorem which is the counterpart of Theorem 3.1 in the zero-sum game.

Theorem 3.1′: The saddle-point solution given by Theorem 2.1′ has the property that better information lowers the minimizer's cost if, for all $k\in\theta_1$, the matrix

$$P+A^TL(k+1)A-L(k) \quad (61)$$

is positive definite. Better information lowers the maximizer's cost if (61) is negative definite. Both costs are independent of the information if (61) is a zero matrix.

□

Since $M=2$, the idea of Theorem 4.1 does not apply to the two-person zero-sum game. We can use, however, a similar procedure as that used in proving Theorem 4.1 to obtain a sufficient condition that better information lowers one of the player's cost in the zero-sum game. For this reason, the following theorem is a counterpart of Theorem 4.1.

Theorem 4.1′: i) Better information lowers the minimizer's cost if A is nonsingular, $P>0$, and $B_1B_1^T>B_2B_2^T$. ii) Better information lowers the maximizer's cost if A is nonsingular, $P<0$, and $B_1B_1^T<B_2B_2^T$. iii) The costs are independent of the information if $B_1=B_2$.

Proof: i) Substituting (52) into (61) we obtain

$$P+A^TL(k+1)A-L(k)=Y^T[(B_1B_1^T-B_2B_2^T)+L(k+1)]Y \quad (62)$$

where

$$Y\triangleq(B_1B_1^T-B_2B_2^T)L(k+1)[I+(B_1B_1^T-B_2B_2^T)L(k+1)]^{-1}A. \quad (63)$$

Under the conditions given, and by Theorem 2.2′i) we have $L(k)>0$ and Y is nonsingular and, hence, the right-hand side of (62) is positive definite. Theorem 3.1′ then implies the desired result:

ii) is similar to (i) with the corresponding matrices negative definite

iii) immediately from Theorem 2.2′iii) and Theorem 3.1′.

□

Remark 13: In Theorem 3.1′ and 4.1′, a sufficient condition for one of the players to benefit by better information is also a sufficient condition for the other player to suffer by that information.

Remark 14: The zero-sum game was extensively studied in the past decade (e.g., [9], [11], [12], [14]). Reference [9] considers the deterministic zero-sum games while [11], [12], and [14] consider the stochastic zero-sum games. The informational properties of the zero-sum LQG game were studied in [11] in the continuous-time setup. Our results which concern the discrete-time case are analogous to those of [11]. The concept of maneuverability was introduced in [11] in order to provide an intuitive interpretation of the results. As it turns out, this concept is not sufficient to provide an intuitive interpretation of the informational properties of the nonzero-sum game, which as our results of the previous sections indicate, depend on other factors as well.

VII. Examples

In this section we give three examples which illustrate the theorems we obtained in the previous sections. All the examples are scalar nonzero-sum cases with the general formulation given in Section II. For simplicity we choose $\bar{x}_0 = 0$, $\Omega_0 = 10$, $B_j = P_j = Q_{jj} = R = H = 1 \; \forall j \in \theta_2$ and $Q_{ij} = 20$, $i \neq j$. We consider two different measurements which correspond to two different information available to the players.

Information I: $\hat{x}^I$, based on measurements $y^I(\cdot) = x(\cdot) + v^I(\cdot)$, $v^I(\cdot) = N(0, 1)$.

Information II: $\hat{x}^{II}$, based on measurements $y^{II}(\cdot) = x(\cdot) + v^{II}(\cdot)$, $v^{II}(\cdot) = N(0, 2)$.

Note that Information I provides better information than II, according to Definition 3.1, and the fact that the larger the variance of observation noise is, the larger is the estimation error.

Example 1: This example demonstrates Theorems 3.1 and 4.1. Consider an M-person, one-stage ($N = 1$) game where $A = 1$. Since the parameters of the game are symmetric (it is not a team situation, however), the costs incurred to the players are of the same value. Before we use (38) to estimate the number of players under which better information is guaranteed to lower all the players' costs, we have to obtain a bound for $L(k)$ from (10). For this single-stage problem, it is easy to see that $L_i(k) \leq 1 + (20M - 18)/(1 + M)^2$. Without loss of generality we can assume that there are more than 40 players so that $L_i(k) < 1.5$. Equation (38) then gives $M > 46$, i.e., if $M > 46$, then better information lowers all the players' costs. We can see from Table I that if M is larger than 16 (which is within the bound of 46), then better information gives positive benefit. Also, if M is larger than 16, then $P_i + A^T L_i(1) A - L_i(0) > 0$, a sufficient condition given in Theorem 3.1 holds.

Example 2: This example demonstrates Theorem 5.1. Consider a two-person ($M = 2$), N-stage game where $A = 1$. Notice that here by Remark 8, Information I is also a dynamically better information relative to II. Again, the costs incurred to the players are of the same value. As in Example 1, we first obtain a bound for $L(k)$ from (10). It turns out that $L(k) < 7$. Choose $\xi = 0.03$ from (47). Then we can use the Kalman filter equations to obtain $\Delta \Sigma(k)$ and $\Delta \Sigma$. Values of s from (41) and $\hat{B}$ from (44) can be obtained accordingly. Equation (48) then gives a bound of $N > 18$, i.e., if $N > 18$, then dynamically better information lowers their Nash costs. We can see from Table II that if N is larger than 4 (which is within the bound of 18), then dynamically better information gives positive benefit. Moreover, the benefit per stage due to dynamically better information is increasing with N.

Example 3: This example demonstrates that dynamically better information is sometimes crucial for Theorem 5.1 to hold. Consider the same game as in Example 2 except $A = 0.5$. Notice that by Remark 8, in the scalar system, a better information is a dynamically better information if and only if $|A| \geq 1$. Hence, in this example, although Information I is better than II, it is no longer a dynamically better information relative to II. The result is that as N increases, better information does not lower the costs. The costs are shown in Table III.

TABLE I
COSTS OF PLAYERS IN EXAMPLE 1 UNDER DIFFERENT INFORMATION VERSUS DIFFERENT NUMBER OF PLAYERS

	Information I	Information II	Benefit due to Better Information	$P_i+A^T L_i(1)A-L_i(0)$
M= 2	34.1313	33.0370	-1.0948	-1.4444
M= 3	35.7727	34.5417	-1.2311	-1.6250
M= 4	34.4545	33.3333	-1.1212	-1.4800
M= 5	32.6162	31.6481	-0.9680	-1.2778
M= 6	30.8330	30.0136	-0.8194	-1.0816
M= 7	29.2386	28.5521	-0.6865	-0.9063
M= 8	27.8462	27.2757	-0.5705	-0.7531
M= 9	26.6364	26.1667	-0.4697	-0.6200
M=10	25.5830	25.2011	-0.3819	-0.5041
M=11	24.6616	24.3565	-0.3051	-0.4028
M=12	23.8510	23.6134	-0.2376	-0.3136
M=13	23.1336	22.9558	-0.1778	-0.2347
M=14	22.4949	22.3704	-0.1246	-0.1644
M=15	21.9233	21.8464	-0.0769	-0.1016
M=16	21.4089	21.3749	-0.0341	-0.0450
M=17	20.9439	20.9486	0.0047	0.0062
M=18	20.5215	20.5614	0.0399	0.0526
M=19	20.1364	20.2083	0.0720	0.0950
M=20	19.7838	19.8851	0.1014	0.1338

TABLE II
COSTS OF PLAYERS IN EXAMPLE 2 UNDER DIFFERENT INFORMATION VERSUS DIFFERENT NUMBER OF STAGES

	Information I	Information II	Benefit due to Dynamically Better Information	Benefit per Stage due to Dynamically Better Information
N= 1	34.1313	33.0370	-1.0943	-0.54714
N= 2	53.5253	52.3570	-1.1682	-0.38941
N= 3	62.6977	62.3126	-0.3851	-0.09628
N= 4	69.2737	69.5068	0.2331	0.04662
N= 5	75.4141	76.1124	0.6983	0.11638
N= 6	81.4866	82.5919	1.1052	0.15789
N= 7	87.5487	89.0429	1.4942	0.18677
N= 8	93.6092	95.4872	1.8780	0.20867
N= 9	99.6695	101.9299	2.2605	0.22605
N=10	105.7297	108.3723	2.6425	0.24023
N=11	111.7900	114.8145	3.0245	0.25205
N=12	117.8502	121.2567	3.4065	0.26204
N=13	123.9104	127.6989	3.7885	0.27061
N=14	129.9706	134.1411	4.1704	0.27803
N=15	136.0309	140.5833	4.5524	0.28453
N=16	142.0911	147.0255	4.9344	0.29026
N=17	148.1513	153.4676	5.3163	0.29535
N=18	154.2115	159.9098	5.6983	0.29991
N=19	160.2713	166.3520	6.0803	0.30401

TABLE III
COSTS OF PLAYERS IN EXAMPLE 3

	Information I	Information II	Benefit due to Better Information	Benefit per Stage due to Better Information
N= 1	16.7828	16.5093	-0.2736	-0.13678
N= 2	19.9329	19.5413	-0.3916	-0.13052
N= 3	21.8794	21.4080	-0.4714	-0.11785
N= 4	23.5712	23.0214	-0.5498	-0.10996
N= 5	25.2148	24.5857	-0.6291	-0.10485
N= 6	26.8495	26.1409	-0.7086	-0.10123
N= 7	28.4825	27.6943	-0.7883	-0.09853
N= 8	30.1153	29.2474	-0.8679	-0.09643
N= 9	31.7480	30.8004	-9.9475	-0.09475
N=10	33.3806	32.3535	-1.0272	-0.09338
N=11	35.0133	33.9065	-1.1068	-0.09224
N=12	36.6460	35.4595	-1.1865	-0.09127
N=13	38.2787	37.0126	-1.2661	-0.09044
N=14	39.9113	38.5656	-1.3458	-0.08972
N=15	41.5440	40.1186	-1.4254	-0.08909
N=16	43.1767	41.6716	-1.5050	-0.08853
N=17	44.8094	43.2247	-1.5847	-0.08804
N=18	46.4420	44.7777	-1.6643	-0.08760
N=19	48.0747	46.3307	-1.7440	-0.08720

VIII. Conclusion

It is not surprising that better information might be harmful in a situation of conflict, but it still remains a cause of discomfort for the players. Besides this discomfort, there are situations where one would expect better information to be beneficial, as for example would be the case in a Nash game which is used to model the decentralized decision making in a large system, which is too large to be solved in centralized fashion. The conditions of Theorems 4.1 and 5.1 single out classes of problems where more information is beneficial to all the players and delineate the importance of the number of stages or players for such a result to hold. A salient feature of two-person zero-sum games that distinguishes them from other types of games is that they do not allow for any cooperation between the players. This feature is also revealed in their informational properties. In Theorem 4.1′ we showed that in the zero-sum games, better information is beneficial to the dominant player only. The dominancy is determined by the maneuverability and the P matrix. This result in discrete-time system is consistent with the previous result in continuous-time differential games.

In this paper we studied the impact of changes of the information quality on the optimal costs under a specific information structure defined in Section II. It is also interesting to investigate the impact of changes of the "information structure" on the optimal costs. For example, if in the formulation of the problem, instead of $\hat{x}(k)$, all of the players know $\{y(0), \cdots, y(k)\}$ at stage k, then it can be shown that there exists a unique Nash solution under certain nonsingularity conditions and the solution is different from the one given by Theorem 2.1. We can then compare the Nash costs resulting from these two different information structures and ask under what conditions one of the information structure is more beneficial to player i than the other information structure. Another problem related to this research is to investigate the informational properties of the games with players in different hierarchical levels, e.g., the Stackelberg games. These are still open questions in this area of research.

Finally, notice that Theorem 2.2, although used as a stepping stone in our analysis, is of interest in its own, since it provides conditions guaranteeing uniform boundedness of the solutions of the coupled Riccati equations.

Appendix A

In this Appendix we prove Theorem 2.1

Notice that $\hat{x}(k)$ given by (4) is such that the error term

$$\tilde{x}(k) \triangleq x(k) - \hat{x}(k)$$

is independent of the controls and J_i can be written solely in the terms of $\hat{x}(k)$, $k \geqq 0$, with some remainder terms depending only on $\tilde{x}(k)$. This construction makes the problem similar to a deterministic problem considered in [10, Theorem 3] with $\hat{x}(k)$ replacing $x(k)$ and M players instead of two. The idea behind the proof of the two theorems is, however, the same. In the following we sketch the decision making at stage $N-1$, $u_j^*(N-1)$, $j \in \theta_2$.

At stage N (no more decisions to be made)

$$\begin{aligned} J_i(N) &= E[x^T(N)P_i x(N)] = E[E[x^T(N)P_i x(N) \mid \hat{x}(N)]] \\ &= E[\hat{x}^T(N)P_i\hat{x}(N)] + \operatorname{tr}\,[P_i\Sigma(N)] \\ &= E[\hat{x}^T(N)L_i(N)\hat{x}(N)] + K_i(N) \end{aligned} \tag{A-1}$$

where

$$L_i(N) = P_i,\quad K_i(N) = \operatorname{tr}\,[P_i\Sigma(N)]$$

at stage $N-1$,

$$J_i(N-1) = E\left[x^T(N-1)P_i x(N-1) + \sum_{j\in\theta_2} u_j^T(N-1)Q_{ij}u_j(N-1) + x^T(N)P_i x(N)\right] \tag{A-2}$$

after receiving $\hat{x}(N-1)$, player i's objective is to minimize $\bar{J}_i(N-1)$ given by

$$\begin{aligned} \bar{J}_i(N-1) &= E\Bigg[x^T(N-1)P_i x(N-1) + \sum_{j\in\theta_2} u_j^T(N-1)Q_{ij}u_j(N-1) \\ &\qquad + x^T(N)P_i x(N) \mid \hat{x}(N-1)\Bigg] \\ &= \hat{x}^T(N-1)P_i\hat{x}(N-1) + \sum_{j\in\theta_2} u_j^T(N-1)Q_{ij}u_j(N-1) \\ &\quad + \operatorname{tr}\,[P_i\Sigma(N-1)] \\ &\quad + E[\hat{x}^T(N)L_i(N)\hat{x}(N) \mid \hat{x}(N-1)] + K_i(N) \end{aligned} \tag{A-3}$$

$$\begin{aligned} &E[\hat{x}^T(N)L_i(N)\hat{x}(N) \mid \hat{x}(N-1)] \\ &= \hat{x}^T(N/N-1)L_i(N)\hat{x}(N/N-1) \\ &\quad + \operatorname{tr}\,\{L_i(N)(\Sigma(N/N-1) - \Sigma(N))\} \\ &= \Big(A\hat{x}(N-1) + \sum_{j\in\theta_2} B_j u_j(N-1)\Big)^T L_i(N)\Big(A\hat{x}(N-1) \\ &\quad + \sum_{j\in\theta_2} B_j u_j(N-1)\Big) \\ &\quad + \operatorname{tr}\,\{L_i(N)(\Sigma(N/N-1) - \Sigma(N))\}. \end{aligned} \tag{A-4}$$

Substituting (A-4) into (A-3) we obtain

$$\begin{aligned} \bar{J}_i(N-1) &= \hat{x}^T(N-1)[P_i + A^T L_i(N)A]\hat{x}(N-1) \\ &\quad + \sum_{j\in\theta_2} u_j^T(N-1)Q_{ij}u_j(N-1) \\ &\quad + \sum_{j\in\theta_2} u_j^T(N-1)B_j^T L_i(N)\sum_{j\in\theta_2} B_j u_j(N-1) \\ &\quad + 2\hat{x}^T(N-1)A^T L_i(N)\sum_{j\in\theta_2} B_j u_j(N-1) \\ &\quad + \operatorname{tr}\,[\Sigma(N/N-1) - \Sigma(N)]L_i(N) \\ &\quad + \operatorname{tr}\,[P_i\Sigma(N-1)] + K_i(N) \end{aligned} \tag{A-5}$$

player i chooses $u_i(N-1)$ to minimize $\bar{J}_i(N-1)$. Since $\bar{J}_i(N-1)$ is convex in $u_i(N-1)$, we have

$$\left.\frac{d\bar{J}_i(N-1)}{du_i(N-1)}\right|_{j\in\theta_2} u_j^*(N-1) = 2\left\{[I + B_i^T L_i(N)B_i]u_i^*(N-1) + B_i^T L_i(N)\left[Ax(N-1) + \sum_{j=1,\, j\neq i}^{M} B_j u_j^*(N-1)\right]\right\} = 0. \tag{A-6}$$

By writing down M equations like (A-6), i.e., i from 1 to M, we have the following system equations:

$$\begin{bmatrix} I + B_1^T L_1(N)B_1 & B_1^T L_1(N)B_2 & \cdots & B_1^T L_1(N)B_M \\ B_2^T L_2(N)B_1 & 1 + B_2^T L_2(N)B_2 & \cdots & B_2^T L_2(N)B_M \\ & \cdots & & \\ B_M^T L_M(N)B_1 & B_M^T L_M(N)B_2 & \cdots & I + B_M^T L_M(N)B_M \end{bmatrix} \cdot \begin{bmatrix} u_1^*(N-1) \\ u_2^*(N-1) \\ \vdots \\ u_M^*(N-1) \end{bmatrix} = -\begin{bmatrix} B_1^T L_1(N) \\ B_2^T L_2(N) \\ \vdots \\ B_M^T L_M(N) \end{bmatrix} A\hat{x}(N-1). \tag{A-7}$$

If a solution exists, $u_i^*(N-1)$ is of the form such that

$$B_i u_i^*(N-1) = G_i(N-1)A\hat{x}(N-1). \tag{A-8}$$

Also from (A-6) we have

$$\begin{aligned} B_i u_i^*(N-1) &= -B_i[I + B_i^T L_i(N)B_i]^{-1}B_i^T L_i(N)\left[A\hat{x}(N-1) + \sum_{j\in\theta_2,\, j\neq 1} B_j u_j^*(N-1)\right] \\ &= -[I + B_iB_i^T L_i(N)]^{-1}B_iB_i^T L_i(N)\left[A\hat{x}(N-1) + \sum_{j\in\theta_2,\, j\neq i} B_j u_j^*(N-1)\right] \end{aligned} \tag{A-9}$$

where we use the fact that

$$X_1(I+X_2X_1)^{-1}X_2=(I+X_1X_2)^{-1}X_1X_2 \qquad \text{(A-10)}$$

provided that the corresponding matrices are invertible. Substituting (A-8) into (A-9)

$$G_i(N-1)=-[I+B_iB_i^TL_i(N)]^{-1}B_iB_i^TL_i(N) \cdot \left[I+\sum_{j\in\vartheta_2} G_j(N-1)-G_i(N-1)\right] \qquad \text{(A-11)}$$

$$G_i(N-1)=-B_iB_i^TL_i(N)\left(I+\sum_{j\in\vartheta_2} G_j(N-1)\right) \qquad \text{(A-12)}$$

$$\sum_{j\in\vartheta_2} G_j(N-1)=-\left[\sum_{j\in\vartheta_2} B_jB_j^TL_j(N)\right]\left(I+\sum_{j\in\vartheta_2} G_j(N-1)\right) \qquad \text{(A-13)}$$

$$I+\sum_{j\in\vartheta_2} G_j(N-1)=\left[I+\sum_{j\in\vartheta_2} B_jB_j^TL_j(N)\right]^{-1}. \qquad \text{(A-14)}$$

Substituting (A-8) into (A-7) and then (A-14) into (A-7), we obtain

$$u_i^*(N-1)=-B_i^TL_i(N)\left[I+\sum_{j\in\vartheta_2} B_jB_j^TL_j(N)\right]^{-1}A\hat{x}(N-1). \qquad \text{(A-15)}$$

This solution is unique if $I+\sum_{j\in\vartheta_2} B_jB_j^TL_j(N)$ is nonsingular. Substituting (A-15) into (A-2) we obtain

$$J_i(N-1)=E[\hat{x}^T(N-1)L_i(N-1)\hat{x}(N-1)]+K_i(N-1) \qquad \text{(A-16)}$$

where $L_i(N-1)$ is given by (10) and $K_i(N-1)$ by (14). As we can see, (A-16) and (A-1) are of the same form. So in going back to stage $N-2$, we will just repeat what we did at stage $N-1$. The general form of the solution and the costs are given by (10)–(14).

Appendix B

a) $X>Y$, premultiplying and postmultiplying both sides by X^{-1} yields the result.

b) $Y^{-1}-X^{-1}=Y^{-1}(X-Y)X^{-1}$

also

$$\begin{aligned} Y^{-1}-X^{-1}&=X^{-1}(X-Y)Y^{-1}\\ &=X^{-1}(X-Y)X^{-1}+X^{-1}(X-Y)(Y^{-1}-X^{-1})\\ &=X^{-1}(X-Y)X^{-1}+X^{-1}(X-Y)Y^{-1}(X-Y)X^{-1}\\ &\geqq 0. \end{aligned}$$

Appendix C

All the diagonal elements y_{ii} of Y are nonnegative and at least one of them is nonzero. Without loss of generality, let $X=\text{diag}(x_1,\cdots,x_n)$, $x_i>0$, $1\leqq i\leqq n$.

$$\text{tr}\,\{XY\}=\sum_{i=1}^{n} x_iy_{ii}>0.$$

Appendix D

$$\begin{aligned} \lambda\min(X^TYX)&=\min_{\|v\|=1} v^TX^TYXv=\min_{\|v\|=1}(Xv)^TY(Xv)\\ &\geqq \lambda\min(Y)\min_{\|v\|=1}\|Xv\|^2\\ &=\lambda\min(Y)\lambda\min(X^TX). \end{aligned}$$

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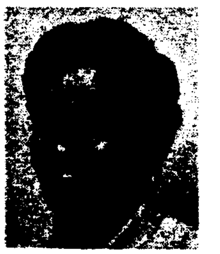

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