JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **103**, 74–82 (1984)

On a Linear Differential Equation of the Advanced Type*

GEORGE P. PAPAVASSILOPOULOS† AND GEERT JAN OLSDER

Department of Applied Mathematics, Twente University of Technology, P.O. Box 217, 7500 AE Enschede, The Netherlands

Submitted by George Leitmann

A linear differential equation of the advanced type is considered. Existence of solutions for any finite interval is shown. Also, a method for generating all the solutions is described and justified.

INTRODUCTION

The purpose of this short paper is to study a simple differential equation of the advantage type, see Eq. (1). Such equations appear in several branches of applied mathematics, for example, see [1] for an application in probability.

In this paper and for the simple type of equation considered, we show existence of a solution on any interval $[0, T]$, where T is any positive finite constant, and provide and justify the procedure for generating all solutions on $[0, T]$. There are several papers in the literature dealing with differential equations of the advanced type. A local existence theorem for equations more general that the one considered here is given in Theorem 2 of [2], by using the Schauder fixed point theorem. This result of [2] generalizes a result of [3]. The existence theorem of [3] is a local existence theorem proved under a hypothesis which does not hold in our case (see Remark 1). In [4] an advanced type of differential equation is studied and several asymptotic results are obtained. The analysis of [4] does not apply to the case considered here (see Remark 1). In [5], analytic solutions are considered and it is shown that for the advanced case, they almost never exist. Finally, [6] and [7] consider analytic solutions on the half axis.

* This work was supported in part by the U.S. Air Force Office of Scientific Research under Grants AFOSR–80–0171 and 82–0174.

† On leave of absence from the Department of Electrical Engineering Systems, University of Southern California, Los Angeles, California 90089.

0022-247X/84 $3.00

Problem Statement

Consider the differential equation

$$\dot{y}(t) = ay(bt), \qquad y(0) = c, \qquad t \geqslant 0 \tag{1}$$

where a, b, c are real constants, $a \neq 0$ and $b > 1$. We say that a function y is a solution of (1) in the interval $[0, T]$ (T is a positive finite constant), if y is a real-valued, continuous function defined on $[0, T]$, satisfies $\dot{y}(t) = ay(bt)$ for $t \in [0, T/b)$ and $y(0) = c$. Notice that since $b > 1$, (1) is of the advanced type.

An alternative formulation of (1) is the following.[1] Let

$$\bar{y}(t) = y(e^{\lambda t}), \tag{2}$$

where λ is a nonzero constant. Then $\bar{y}$ satisfies

$$\dot{\bar{y}}(t) = a\lambda e^{\lambda t}\bar{y}\left(t + \frac{\ln b}{\lambda}\right), \qquad \begin{cases} \lim\limits_{t \to +\infty} \bar{y}(t) = c, \ t \in \left[\dfrac{\ln T}{\lambda}, +\infty\right), & \text{if } \lambda < 0 \\ \lim\limits_{t \to -\infty} \bar{y}(t) = c, \ t \in \left(-\infty, \dfrac{\ln T}{\lambda}\right], & \text{if } \lambda > 0. \end{cases} \tag{3}$$

If $\lambda < 0$, then the differential equation for $\bar{y}(t)$ is a retarded differential equation (recall: $b > 1$ and thus $(\ln b)/\lambda < 0$). Obviously, the study of (1) on $[0, T]$ with $y(0) = c$ is equivalent to studying the limit of $\bar{y}(t)$ as $t \to +\infty$, if $\lambda < 0$.

Existence of Solutions

A way of studying (1) comes from the following considerations. Given an arbitrary continuous function y_0 defined on the interval $[T/b, T]$, we can uniquely define a function y_1 on $[T/b^2, T/b)$ by

$$y_1\left(\frac{T}{b}\right) = y_0\left(\frac{T}{b}\right), \qquad \dot{y}_1(t) = ay_0(bt), \qquad \frac{T}{b^2} \leqslant t \leqslant \frac{T}{b}. \tag{4}$$

We can similarly define y_2 on $[T/b^3, T/b^2]$ and continue backwards, so that at the nth step we define y_n on $[T/b^{n+1}, T/b^n]$ by

$$\begin{aligned} y_n\left(\frac{T}{b^n}\right) &= y_{n-1}\left(\frac{T}{b^n}\right) \\ \dot{y}_n(t) = ay_{n-1}(bt), &\qquad \frac{T}{b^{n+1}} \leqslant t \leqslant \frac{T}{b^n}. \end{aligned} \tag{5}$$

[1] This reformulation of the problem was suggested to us by Professor M. L. J. Hautus.

Piecing together $y_0, y_1, y_2, \ldots$ we have a function y defined on $(0, T]$ which is continuous on $(0, T]$ and satisfies $\dot{y}(t) = ay(bt)$ on $(0, T]$, with the possible exception of the point T/b; had we chosen y_0 as to satisfy $\dot{y}_0^+(T/b) = ay_0(T)$, (here $\dot{y}_0^+(T/b)$ stands for the right derivative of y_0 at T/b), then y would be continuously differentiable at T/b as well. Actually, as $t \to 0$, y becomes more and more smooth, as is also the case with functional retarded differential equations when $t \to +\infty$ (recall (2)–(3)). The only thing that is not obvious is what the behaviour of $y(t)$ will be as $t \to 0$. We will show that this limit exists for any choice of (y_0, T) and that there are infinitely many (y_0, T)'s which result to the same limit of $y(t)$ as $t \to 0$.

Our first objective is to show existence of a solution of (1) for sufficiently small T. Let ϕ be an element of the space of continuous functions on $[0, T]$, $C[0, T]$. $C[0, T]$ is equipped with the usual sup norm. We define the mapping $Q : C \to C$ as follows:

$$[Q\phi](t) = \begin{cases} c + \dfrac{a}{b}\displaystyle\int_0^{bt} \phi(s)\, ds, & 0 \leqslant t \leqslant \dfrac{T}{b} \\ c + \dfrac{a}{b}\displaystyle\int_0^{T} \phi(s)\, ds - \phi\left(\dfrac{T}{b}\right) + \phi(t), & \dfrac{T}{b} \leqslant t \leqslant T. \end{cases} \tag{6}$$

Obviously $Q\phi \in C[0, T]$. The idea is to create a sequence $\{\phi_n = Q^n\phi\}$, which is Cauchy; then its limit ϕ^* exists in $C[0, T]$ and satisfies $Q\phi^* = \phi^*$ or equivalently $\dot{\phi}^*(t) = a\phi^*(bt)$ for $t \in [0, T/b)$, $\phi^*(0) = c$, i.e., ϕ^* solves (1). Thus, we have to guarantee the Cauchy character of this sequence.

LEMMA 1. *It holds:*

$$[Q\phi](t) - [Q\phi]\left(\frac{T}{b}\right) = \phi(t) - \phi\left(\frac{T}{b}\right), \qquad \frac{T}{b} \leqslant t \leqslant T. \tag{7}$$

Proof. If $T/b \leqslant t \leqslant T$, using (6) we obtain

$$\begin{aligned} &[Q\phi](t) - [Q\phi]\left(\frac{T}{b}\right) \\ &= c + \frac{a}{b}\int_0^T \phi(s)\, ds - \phi\left(\frac{T}{b}\right) + \phi(t) - \left[c + \frac{a}{b}\int_0^T \phi(s)\, ds\right] \\ &= \phi(t) - \phi\left(\frac{T}{b}\right). \quad \blacksquare \end{aligned}$$

Let us now study the sequence $\{\phi_n = Q^n\phi_0\}$, where ϕ_0 is an arbitrary element of $C[0, T]$. For $0 \leqslant t \leqslant T/b$, we have

$$|\phi_{n+1}(t)-\phi_n(t)| = \left| \frac{a}{b}\int_0^{bt} \phi_n(s)\,ds - \frac{a}{b}\int_0^{bt} \phi_{n-1}(s)\,ds \right|$$
$$= \left| \frac{a}{b}\int_0^{bt} [\phi_n(s)-\phi_{n-1}(s)]\,ds \right| \leqslant \frac{|a|}{b}\|\phi_n-\phi_{n-1}\|\cdot T. \tag{8}$$

For $T/b \leqslant t \leqslant T$ we have

$$|\phi_{n+1}(t)-\phi_n(t)|$$
$$= \left| c+\frac{a}{b}\int_0^T \phi_n(s)\,ds + \phi_n(t) - \phi_n\left(\frac{T}{b}\right)\right.$$
$$\left. - \left(c+\frac{a}{c}\int_0^T \phi_{n-1}(s)\,ds + \phi_{n-1}(t) - \phi_{n-1}\left(\frac{T}{b}\right)\right)\right|$$
$$= \left| \frac{a}{b}\int_0^T [\phi_n(s)-\phi_{n-1}(s)]\,ds\right.$$
$$\left. + \left[\phi_n(t)-\phi_n\left(\frac{T}{b}\right) - \left(\phi_{n-1}(t)-\phi_{n-1}\left(\frac{T}{b}\right)\right)\right]\right|$$
$$= \left| \frac{a}{b}\int_0^T [\phi_n(s)-\phi_{n-1}(s)]\,ds + \left\{ [Q\phi_{n-1}](t) - [Q\phi_{n-1}]\left(\frac{T}{b}\right)\right.\right.$$
$$\left.\left. - \left(\phi_{n-1}(t)-\phi_{n-1}\left(\frac{T}{b}\right)\right)\right\}\right|. \tag{9}$$

The second term in (9) is zero by Lemma 1 and thus for $T/b \leqslant t \leqslant T$

$$|\phi_{n+1}(t)-\phi_n(t)| \leqslant \frac{|a|}{b}\left|\int_0^T [\phi_n(s)-\phi_{n-1}(s)]\,ds\right| \leqslant \frac{|a|}{b}\|\phi_n-\phi_{n-1}\|\cdot T. \tag{10}$$

Combining (8) and (10) we have

$$|\phi_{n+1}(t)-\phi_n(t)| \leqslant \frac{|a|}{b} T\|\phi_n-\phi_{n-1}\|, \qquad 0\leqslant t\leqslant T$$

and thus

$$\|\phi_{n+1}-\phi_n\| \leqslant \frac{|a|}{b} T\|\phi_n-\phi_{n-1}\|.$$

We can show now that $\{\phi_n\}$ is a Cauchy sequence if $(|a|/b)\,T<1$. Let n, m be any positive integers, and assume that

$$d = \frac{|a|\,T}{b} < 1. \tag{11}$$

$$\begin{aligned}
&\|\phi_{n+m}-\phi_m\| \\
&\quad \leqslant \|\phi_{n+m}-\phi_{n+m-1}\| + \|\phi_{n+m-1}-\phi_{n+m-2}\| + \cdots + \|\phi_{m+1}-\phi_m\| \\
&\quad \leqslant (d)^{n+m-1}\|\phi_1-\phi_0\| + (d)^{n+m-2}\|\phi_1-\phi_0\| + \cdots + (d)^m\|\phi_1-\phi_0\| \\
&\quad \leqslant d^m\|\phi_1-\phi_0\|(1+d+d^2+\cdots) \\
&\quad = \frac{d^m}{1-d}\|\phi_1-\phi_0\|.
\end{aligned}$$

So, $\|\phi_{n+m}-\phi_m\| \to 0$ as $n, m \to +\infty$. We have thus proved:

PROPOSITION 1. *If $T < b/|a|$, then the sequence $\{Q^n\phi_0\}$ converges in $C[0,T]$ to a unique limit ϕ which satisfies*

$$\dot{\phi}(t) = a\phi(bt), \qquad 0 \leqslant t \leqslant T/b, \qquad \phi(0) = c.$$

On $[T/b, T]$, it holds

$$\phi(t) - \phi\left(\frac{T}{b}\right) = \phi_0(t) - \phi_0\left(\frac{T}{b}\right). \tag{12}$$

Remark 1. Proposition 1 is actually a local existence theorem for (1). In [3] a local existence theorem for $y'(t) = F(t, y(g(t)))$, $y(0) = c$ is given, but the assumptions under which the proof of [3] holds includes the following: for some constants L, h with $h \geqq L \geqq 0$ it holds $|g(t)| \leqslant \max\{|t|, L\}$ for $|t| \leqslant h$. In our case $g(t) = bt$, $b > 1$ and this assumption does not hold. Thus the existence theorem of [3] does not apply here.

Remark 2. In [4] the equation

$$\dot{y}(t) = F(t, y(t), y(t+h_1), \ldots, y(t+h_p))$$

$$0 \leqslant t < +\infty, \qquad 0 < h_1 < h_2 < \cdots < h_p < +\infty,$$

$$y(t_0) = c, \qquad 0 \leqslant t_0 < +\infty$$

is considered. It was shown that studying (1) is equivalent to studying

$$\dot{\bar{y}}(t) = a\lambda e^{\lambda t}\bar{y}\left(t + \frac{\ln b}{\lambda}\right),$$

$$\lim_{t \to -\infty} \bar{y}(t) = c, \qquad t \in \left(-\infty, \frac{\ln T}{\lambda}\right]$$

where $\lambda > 0$. Thus the existence results of [4] do not apply to the case considered here.

Remark 3. Let $T < b/|a|$. Had we started with a different ϕ_0, say, $\bar{\phi}_0$, we would have ended up with a different solution $\bar{\phi}$, which would have the same value c at $t=0$ as ϕ does; i.e., the initial condition $y(0)=c$ in (1) does not determine uniquely the solution as was to be expected since we are dealing with a functional differential equation.

Remark 4. Let $T<b/|a|$. If $c \neq 0$ and we start with $\phi_0(t)=c_1$ (a constant), Proposition 1 guarantees the existence of a solution of (1) which will satisfy $y(0)=c$, and $y(t)=c_2$ for $t \in [T/b, T]$ where c_2 is some constant. c_1 cannot be zero, since if it were zero, y would have to be zero on $[T/b^2, T/b]$ (since $\dot{y}(t)=ay(bt)$), $[T/b^3, T/b^2]$,... and thus $y(0)=c$ would have to be zero, and it is not. Using the linearity of (1), it is easy to see now that $y(0)$ is a linear function of the constant function $y_0(t)$ and that $y(0) \neq 0$ if and only if $y_0 \equiv 0$ on $[T/b, T]$.

Remark 5. $\phi(t)$ differs from $\phi_0(t)$ only by some constant.

Combining Remarks 4 and 5 we can prove the following theorem, which is the central result of this section.

THEOREM 1. *If we employ the procedure described in* (4)–(5) *with an arbitrary fixed finite* T *and an arbitrary continuous function* y_0 *defined on* $[T/b, T]$, *generate the* y_n*'s on* $[T/b^{n+1}, T/b^n]$ *backwards and piece them together, then the resulting* y *has a well-defined limit as* $t \to 0$ *and satisfies* (1).

Proof. Let us first prove this theorem under the assumption $T<b/|a|$. Let ϕ_0 be any element of $C[0, T]$ which coincides with y_0 on $[T/b, T]$. The sequence $Q^n\phi_0$ converges to some $y_1 \in C[0, T]$ which satisfies

$$\dot{y}_1(t)=ay_1(bt), \qquad y_1(0)=c, \qquad t \in \left[0, \frac{T}{b}\right]$$

$$y_1(t)=y_0(t)+c_2, \qquad t \in \left[\frac{T}{b}, T\right]$$

where c_2 is some constant. By Remark 2, there is a solution $y_2(t)$ of (1) on $[0, T]$ which satisfies

$$y_2(t)=c_2, \qquad t \in \left[\frac{T}{b}, T\right].$$

Let $y=y_1-y_2$. Obviously, y satisfies

$$\dot{y}(t)=ay(bt), \qquad t \in \left[0, \frac{T}{b}\right]$$

$$y(t)=y_0(t), \qquad t \in \left[\frac{T}{b}, T\right]$$

$$y(t)=c-c_2.$$

Had we employed (4) and (5) starting with $y_0(t)$ on $[T/b, T]$ we would have obviously generated the y and the limit of $y(t)$ as $t \to 0$ would be $c - c_2$.

We thus proved that Theorem 1 holds if $T < b/|a|$.

If $T \geqslant b/|a|$, then if we start with an arbitrary y_0 on $[T/b, T]$, after a finite number of backward extensions we will be in an interval $[T/b^{k+1}, T/b^k]$ with a y_k defined there on, where $T/b^k < b/|a|$ if $k > 1/\ln b \cdot \ln(T|a|/b)$. Thus, considering that we start on $[T/b^{k+1}, T/b^k]$ with y_k we are back in the first case considered in this proof. ∎

A remaining issue to be settled is how we can generate solution of (1) which satisfies $y(0) = c$, since the procedure of (4) and (5) does not immediately guarantee that. If $y(t)$ satisfies $\dot{y}(t) = ay(bt)$ and $y(0) \neq 0$, then obviously the function $(c/y(0))\, y(t)$ satisfies the differential equation and the initial condition. Another way is the following: we use the procedure (4) and (5) starting with $y_0(t) \equiv 1$ on $[T/b, T]$ and generate the solution $y^1(t)$ which will have $y^1(0) \neq 0$ (recall Remark 4). If $y(t)$ is any other solution of (1) with $y(0) \neq c$, then the function $y(t) - ((y(0) - c)/y^1(0))\, y^1(t)$ is again a solution of (1) satisfying the initial condition. Of course, if $T < b/|a|$ we can also generate solutions of (1) with $y(0) = c$ by generating Cauchy sequences $Q^n \phi_0$ according to (6) with arbitrary ϕ_0's.

Discussion and Conclusions

Given that $y(t)$ solves (1) on $[0, T]$, we could ask whether it can be extended to the right of T. This can be done by defining the function y_{-1} on $[T, bT]$ by

$$y_{-1}(t) = \frac{1}{a} \dot{y}_0 \left(\frac{t}{b}\right), \qquad T \leqslant t \leqslant bT$$

as long as y_0 is differentiable and $y_{-1}(T) = y_0(T)$, $\dot{y}^T_{-1}(T) = \dot{y}_0^-(T)$ which can be easily guaranteed by an appropriate choice of y_0 (+ and − denote right and left derivatives). We can similarly extend the solution to $[bT, b^2T]$, $[b^2T, b^3T]$ and so on, as long as y_0 is sufficiently differentiable and such as to have $y_{-n}(T) = y_{-(n+1)}(T)$, $\dot{y}_{-n}(T) = \dot{y}_{-(n+1)}(T)$ which, as can be easily seen, amounts to

$$y_0^{(n+1)}\left(\frac{T}{b}\right) = ab^n y_0^{(n)}(T) \tag{13}$$

(the superscript denotes the order of the derivative). There are many functions y_0 which satisfy (13) for $n = 0, 1, 2, ..., N$; for example, y_0 can be chosen as a polynomial of sufficiently high degree and appropriate coefficients so as to satisfy (13). If (13) holds for $n = 0, 1, \ldots, N$ and y_0 is at least N times continuously differentiable, we can extend the solution up to $b^N T$. An immediate question is whether there are choices of y_0 on $[T/b, T]$, so that the solution can be extended to $+\infty$. A y_0 which has this property is

$$\exp\left(1\Big/\left[1-\left(\frac{2b}{T(b-1)}t+\frac{1+b}{1-b}\right)^2\right]\right)$$

which is infinitely continuously differentiable and satisfies (13) as $0 = 0$.

As was shown in [5] in a more general setup, Eq. (1) cannot have an analytic solution around zero. This can be directly verified by plugging in (1) $y(t) = \sum_{n=0}^{\infty} d_n t^n$, equating the coefficients of the powers of t on both sides and showing that the radius of convergence of this series is zero.

On could consider linear equations more general than (1), for example,

$$\dot{y}(t) = \sum_{i=1}^{n} a_i y(b_i(t)), \qquad y(0) = c$$

where $b_1 > b_2 > \cdots > b_n > 0$. It is easy to see that considering the differential equations

$$\dot{y}_i(t) = a_i y_i(b_i t), \qquad y_i(0) = c_i, \qquad i = 1, \ldots, n$$

where the c_i's have sum $c_1 + c_2 + \cdots + c_n = c$, but otherwise are arbitrary, enables one to easily prove existence of solutions. There are several issues to be settled concerning Eq. (1) or the above-mentioned generalization. One of the most important ones has to do with the solutions of (1) which exist over the whole half axis $[0, +\infty)$ and remain bounded for any t or go to zero as $t \to +\infty$. For example, it can be shown that if a solution $y(t)$ of (1) exists on $[0, +\infty)$, it cannot go to zero faster than any exponential. (If this were the case, then the Laplace transform $Y(s)$ of $y(t)$ is analytic around zero. Transforming $\dot{y}(t) = ay(bt)$, $y(0) = c$ in the Laplace domain yields $sY(s) - y(0) = (a/b)\, Y(bs)$; substituting $Y(s)$ in a series form in it and calculating the coefficients yields zero radius of convergence.) The study of possible periodic solutions as $t \to \infty$ is another important issue.

References

1. T. S. Ferguson, Lose a dollar or double your fortune, *in* "Proceedings, Sixth Berkeley Symposium," Vol. III, pp. 657–666, Univ. of California Press, Berkeley, 1972.
2. R. J. Oberg, On the local existence of solutions of certain functional differential equations, *Proc. Amer. Math. Soc.* **20** (1969), 285–302.

3. D. R. ANDERSON, An existence theorem for a solution of $f'(x) = F(x, f(g(x)))$", *SIAM Rev.* **8** (3) (1966), 359–362.
4. C. A. ANDERSON, Asymptotic oscillation results for solutions to first-order nonlinear differential difference equations of advanced type, *J. Math. Anal. Appl.* **24** (1968), 430–439.
5. R. J. OBERG, Local theory of complex functional differential equations, *Trans. Amer. Math. Soc.* **161** (1971), 269–281.
6. G. R. MORRIS, A. FELDSTEIN, AND E. W. BOWEN, The Phragmén–Lindelöf principle and a class of functional differential equations, *in* "Ordinary Differential Equations, 1971 NRL–MRC Conference" (L. Weiss, Ed.), pp. 513–540, Academic Press, New York/London, 1972.
7. P. O. FREDERICKSON, Dirichlet series solutions for certain functional differential equations, *in* "Japan–United States Seminar on Ordinary Differential and Functional Equations," pp. 249–254, Springer-Verlag, New York/Berlin, 1971.