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On Linear-Quadratic Gaussian Continuous-Time Nash Games[1]

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Abstract. Two classes of linear-quadratic Gaussian continuous-time Nash games are considered. Their main characteristic is that the σ-fields with respect to which the control actions of the players have to be measurable at each instance of time are not affected by the past controls of the players. We show that, if a solution exists, then there exists a solution linear in the information, and also show how to construct all the solutions. Several conditions guaranteeing the existence of a unique solution are also given.

Key Words. Stochastic Nash games, linear-quadratic games.

1. Introduction

The purpose of this paper is to study two classes of linear quadratic, Gaussian, continuous-time, two-player Nash games. The first class considers that the information of each player at time t is $\{x_\tau^i, 0 \leq \tau \leq t\}$, $i = 1, 2$, and that the stochastic process x_t appears in their costs. x_t^1, x_t^2, x_t are Gaussian processes, not affected by the decisions u_t^1, u_t^2 of the two players. The time horizon $[0, T]$ is fixed. The formulation of this problem is given in (2)–(5). The main result is that, if a solution exists, then there will exist a solution linear in the information. A theoretical way of constructing all the solutions is presented as well as conditions concerning the existence and uniqueness of solutions. Results concerning the canonical correlation coefficients of two Gaussian stochastic processes are developed and used; in particular, it is shown how the problem of finding these coefficients can be set up in a reproducing kernel Hilbert space framework (Refs. 1 and 2). Another

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interesting result provided for this class is a sufficient condition which guarantees existence and uniqueness of the solution and which is very easy to check. The second class involves a stochastic Gaussian Nash game, which is described by a linear stochastic differential equation, quadratic costs, finite-time horizon, and where the players have noise-corrupted measurements of the initial state as their information. We show that, if a solution exists, then there exists a solution linear in the information. We also show that, in order to find all the solutions (linear and nonlinear), one has to solve a finite number of open-loop deterministic Nash games. Our analysis leads to the important conclusion that one can, without much loss of generality, restrict the admissible solutions to those which are linear in the information, since, as we show, nonlinear solutions are highly unlikely to occur and in addition, if they occur, they are not robust with respect to small variations of the parameters of the problem. Conditions under which a unique solution exists are also provided.

The main common characteristic property of these two classes is that the σ-fields with respect to which the control actions of the player have to be measurable at each instant of time are not affected by the past controls of the players. Whether the common conclusion about these problems (namely, the one that, if a solution exists, then there will exist a solution linear in the information) is still valid for more general linear-quadratic Gaussian Nash games which share the above-mentioned property (concerning the σ-fields) or some other more general property is an open question.

The analysis presented here concerns the continuous-time case. The discrete-time analogues were studied in Refs. 3–5. In this paper, we show that many results which hold for the discrete-time case hold also for the continuous-time one. The fact that a finite sequence of open-loop deterministic linear-quadratic Nash games has to be solved in order to find all the solutions (linear and nonlinear) for the second class considered is presented for the first time here.

For the sake of simplicity, we treat the two-player case and also treat the scalar case only, for the first class. Generalizing our results to the N-player case and to vector-valued processes for the first class is rather straightforward.

2. First Class

Problem Statement. Let $(\Omega, \mathscr{F}, P)$ be a complete probability space and x_t^1, x_t^2, x_t, $0 \leq t \leq T$, three scalar-valued, zero-mean, second-order, Gaussian, q.m. continuous stochastic processes defined on $(\Omega, \mathscr{F}, P)$. For fixed $t \in [0, T]$, let $\mathscr{F}_{it}$ be the minimal sub σ-field of $\mathscr{F}$ generated by

$\{x_\tau^i, 0 \le \tau \le t\}, i=1,2$. Let $u_t^i, 0 \le t \le T, i=1,2$, denote a second-order, scalar-valued, stochastic process defined on $(\Omega, \mathscr{F}, P)$, such that:

(i) u_t^i is $\mathscr{F}_{it}$ measurable;

(ii) u^i as a function of (ω, t), where $\omega \in \Omega$, is jointly measurable in ω and t; (on $[0, T]$, we consider Lebesgue-measurable sets);

(iii)

$$E \int_0^T u_t^i u_t^i \, dt < +\infty, \tag{1}$$

where E denotes total expectation.

Let us denote by U^i the set of all these u^i's (U^i is a Hilbert space, see Ref. 6, page 163).

Let

$$J_1(u^1, u^2) = E \int_0^T (\tfrac{1}{2} u_t^1 u_t^1 + u_t^1 r_1(t) u_t^2 + u_t^1 s_1(t) x_t) \, dt, \tag{2}$$

$$J_2(u^1, u^2) = E \int_0^T (\tfrac{1}{2} u_t^2 u_t^2 + u_t^2 r_2(t) u_t^1 + u_t^2 s_2(t) x_t) \, dt, \tag{3}$$

where r_i, s_i are scalar-valued, bounded, Lebesgue-measurable functions of t.

The problem that we intend to solve is the following: Find a pair $(u^{1*}, u^{2*}) \in U^1 \times U^2$, so that

$$J_1(u^{1*}, u^{2*}) \le J_1(u^1, u^{2*}), \qquad \forall u^1 \in U^1, \tag{4}$$

$$J_2(u^{1*}, u^{2*}) \le J_2(u^{1*}, u^2), \qquad \forall u^2 \in U^2. \tag{5}$$

It is easy to see that, for fixed $u^2 \in U^2$, we can add to (2)

$$E \int_0^1 (r_1(t) u_t^2)^2 \, dt + E \int_0^T (s_1(t) x_t)^2 \, dt,$$

which is finite, make the integrand in (2) nonnegative, and then apply Fubini's theorem. Thus, we can interchange expectation and integration in (2); consequently, (4) can be replaced equivalently by

$$\begin{aligned} &E[\tfrac{1}{2} u_t^{1*} u_t^{1*} + u_t^{1*} r_1(t) u_t^{2*} + u_t^{1*} s_1(t) x_t] \\ &\quad \le E[\tfrac{1}{2} u_t^1 u_t^1 + u_t^1 r_1(t) u_t^{2*} + u_t^1 s_1(t) x_t], \qquad \forall t \in [0, T], \forall u^1 \in U^1. \end{aligned} \tag{6}$$

For fixed t, a necessary and sufficient condition for u^{1*} to satisfy (6) is that it satisfies (see Ref. 7).

$$u_t^1 + r_1(t) E[u_t^2 | \mathscr{F}_{1t}] + s_1(t) E|x_t| \mathscr{F}_{1t}] = 0. \tag{7}$$

Similarly, we obtain for (5):

$$u_t^2 + r_2(t)E(u_t^1|\mathscr{F}_{2t}] + s_2(t)E|x_t|\mathscr{F}_{2t}] = 0. \tag{8}$$

Here, $E[\cdot|\mathscr{F}_{it}]$ denotes conditional expectation given $\mathscr{F}_{it}$. Later, we will use the symbol

$$P_{it} = E[\cdot 1 \mathscr{F}_{it}].$$

Substituting u_t^2 from (8) into (7), we obtain

$$u_t^1 - r_1(t)r_2(t)E[E[u_t^1|\mathscr{F}_{2t}]|\mathscr{F}_{1t}]$$
$$= -r_1(t)s_2(t)E[E[x_t|\mathscr{F}_{2t}]|\mathscr{F}_{1t}] + s_1(t)E[x_t|\mathscr{F}_{1t}], \qquad \forall t \in [0, T]. \tag{9}$$

Therefore, the solution of our problem is equivalent to solving (9) with $u^1 \in U^1$.

Solution of Equation (9). Let

$$L_{2t}(x_\tau^i, 0 \le \tau \le t) = L_2^i$$

be the separable Hilbert space of all finite linear combinations $\sum_J c_j x_{\tau_j}^i$, $0 \le \tau_j \le t$, and their q.m. limits. L_{2t}^1, L_{2t}^2 are both closed subspaces of the separable Hilbert space $L_{2t}(x_\tau^1, x_\tau^2, 0 \le \tau \le t)$. The inner product in L_{2t} is defined by

$$\langle \zeta, n \rangle = E[\zeta \cdot \eta]$$

(see Ref. 2). The following lemma ascertains that we can find a complete orthonormal set ζ_{tk}^i, $k = 1, 2, \ldots$ in L_{2t}^i, $i = 1, 2$, where $\{\zeta_{tk}^i\}$ and $\{\zeta_{tk}^2\}$ are canonically related (see Ref. 8 for similar results in a more abstract setup), and it also shows how to find the canonical correlation coefficients. The proof of lemma 1 is relegated to the Appendix.

Lemma 2.1. (i) There are Gaussian random variables $\zeta_{t1}^i, \zeta_{t2}^i, \ldots,$ such that $\{\zeta_{tk}^i\}$ is a complete orthonormal set in L_{2t}^i, $i = 1, 2$, and

$$E[\zeta_{tk}^i \zeta_{tl}^i] = \delta_{kl}, \tag{10a}$$

$$E[\zeta_{tk}^1 \zeta_{tl}^2] = \rho_k \delta_{kl}, \tag{10b}$$

$$\delta_{kl} = 1 \text{ if } k = l, \qquad \delta_{kl} = 0 \text{ if } k \neq l, \tag{10c}$$

$$0 \le \rho_k \le 1. \tag{10d}$$

(ii) The ρ_k's can be found by: (a) solving the system (58), where

$$\sum_n (x_n^i)^2 < +\infty, \qquad i = 1, 2,$$

for ρ and x_n^i; or (b) solving for ρ, g_1, g_2 the formal integral equations (60), where g_1, g_2 are allowed to be generalized functions; or (c) solving (66) or (67), see also (64), for h_1, h_2, ρ, where $h_1 \in H(K_{ii})$, the reproducing kernel Hilbert space corresponding to $\{x_\tau^i, 0 \le \tau \le t\}$, where

$$K_{ij}(t, s) = E[x_t^i x_s^j].$$

It should be pointed out that ρ_k, η_{tk}^i (see Appendix), ζ_{tk}^i, ϕ_k^i, λ_k^i depend on t, but for convenience we drop the index t. A little reflection will convince the reader that, if some ρ is a canonical correlation coefficient of x_t^1, x_t^2, when considered on $[0, t_1]$, it does not necessarily remain so if we consider x_t^1, x_t^2 on $[0, t_2]$, $t_1 \neq t_2$. Obviously, finding the ρ's is quite a difficult task, even if we are interested only in a specific t_1. At least, we know that all of them are in $[0, 1]$, and they are countably many. In the sequel, we will denote the dependence of ζ_k^i, ρ_k on t by a subscript t.

Let us now return to (9) and solve it for each fixed $t \in [0, T]$. It is known (see Ref. 9) that any second-order random variable which is $\mathscr{F}_{1t}$-measurable can be expressed as a sum of products of Hermite polynomials in $\eta_{t1}^1, \eta_{t2}^1, \eta_{t3}^1, \ldots$ or equivalently in $\zeta_{t1}^1, \zeta_{t2}^1, \ldots, 0 \le t1, t2, \ldots \le t$. If h_n denotes the nth-order normalized Hermite polynomial, then (see Refs. 9 and 10)

$$u_t^1 = \sum c^{(t)}_{(n_1 m_1), \ldots, (n_k, m_k)} h_{n_1}(\zeta_{tm_1}^1) h_{n_2}(\zeta_{tm_2}^1) \cdots h_{n_k}(\zeta_{tm_k}^1), \tag{11a}$$

$$\sum [c^{(t)}_{(n_1, m_1), \ldots, (n_k, m_k)}]^2 < +\infty, \tag{11b}$$

$$h_n(z) = [(-1)^n/\sqrt{n!}] \exp(\tfrac{1}{2}z^2)(d^n/dz^n) \exp(-\tfrac{1}{2}z^2), \qquad z \in R. \tag{11c}$$

The right-hand side of (9) can also be expressed in the same form; and, since it is an element of L_{2t}^1, it will have the form

$$\sum_k b_k(t) \zeta_{tk}^1, \tag{12}$$

where

$$\sum_k (b_k(t))^2 < +\infty,$$

and the $b_k(t)$ can be calculated.

We will need the following lemma, which is an easy extension of Lemma 2 of Ref. 3.

Lemma 2.2. Let

$$a^{(t)}_{(n_1, m_1) \cdots (n_k m_k); (\bar{n}_1, \bar{m}_1) \cdots (\bar{n}_{\bar{k}} \bar{m}_{\bar{k}})} = E[h_{n_1}(\zeta_{m_1}^1) \cdots h_{n_k}(\zeta_{m_k}^1) E[E[h_{\bar{n}_1}(\zeta_{\bar{m}_1}^1) \cdots (h_{\bar{n}_{\bar{k}}}(\zeta_{\bar{m}_{\bar{k}}}^1)] | \mathscr{F}_{2t}] | \mathscr{F}_{1t}]. \tag{13}$$

Then,

$$a=0, \qquad \text{if } (n_1, m_1), \ldots, (n_k, m_k) \neq (\bar{n}_1 \bar{m}_1), \ldots, (\bar{n}_{\bar{k}}, \bar{m}_{\bar{k}}),$$
$$a=\rho_{tm_1}^{n_1} \cdots \rho_{tm_k}^{n_k}, \qquad \text{if } (n_1, m_1), \ldots, (n_k, m_k) = (\bar{n}_1, \bar{m}_1) \cdots (\bar{n}_k, \bar{m}_k).$$

Proof. The only adaptation needed to the proof of Lemma 2 of Ref. 3 is taken care of by using the fact that

$$E[h_{n_1}(\zeta^1_{m_1}) \cdots h_{n_k}(\zeta^1_{m_k}) | \zeta^2_1, \zeta^2_2, \ldots]$$
$$= E[h_{n_1}(\zeta^1_{m_1}) \cdots h_{n_k}(\zeta^1_{m_k}) | \zeta^2_{m_1}, \ldots, \zeta^2_{m_k}]. \qquad \square$$

Substituting u^1_t from (11) and the right-hand side of (9) from (12) into (9), multiplying both sides by $h_{n_1}(\zeta^1_{m_1}) \cdots h_{n_k}(\zeta^1_{m_k})$, and taking expectation yields

$$c^{(t)}_{(n_1,m_1),\ldots,(n_k,m_k)} \cdot [1 - r_1(t) r_2(t) \rho_{tn_1}^{m_1} \cdots \rho_{tn_k}^{m_k}]$$
$$= \begin{cases} 0, & \text{if } h_{n_1}(\zeta^1_{m_1}) \cdots h_{n_k}(\zeta^1_{m_k}) \neq \zeta^1_{mj}, j=1,2,\ldots,k, \\ b_j(t), & \text{otherwise.} \end{cases} \tag{14}$$

Thus, it is clear that solving (9), for each fixed t, is equivalent to solving (14) for c's. Analyzing the solvability of (14) is easy and leads to the following proposition concerning the solution of (9).

Proposition 2.1. For t fixed, the following results hold:

(i) (9) has a solution u^1_t if and only if the system (14) has a solution (for the c's), and then u^1_t is given by (11).

(ii) If there exists a solution, then there exists a solution affine in $\{\zeta^1_{tn}\}$, i.e., affine in $\{x^1_\tau, 0 \leq \tau \leq t\}$.

(iii) There exist nonlinear solutions if

$$1 = r_1(t) r_2(t) \rho_{tn_1}^{m_1} \cdots \rho_{tn_k}^{m_k},$$

for some $\rho_{tn_1}, \ldots, \rho_{tn_k}$ and $m_1, \ldots, m_k$ with

$$m_1 + \cdots + m_k \geq 2.$$

(iv) If there exists a nonlinear solution, then there exists infinitely many nonlinear solutions.

It should be pointed out that Proposition 2.1 considers the solution of (9) for each fixed t and guarantees that u^1_t is $\mathscr{F}_{1t}$ measurable, but does not guarantee that $u^1 \in U_1$. Also, in solving (14), one has to check whether $1 - r_1(t) r_2(t) \rho_{tn_1}^{m_1} \cdots \rho_{n_k}^{m_k}$ is nonzero, and thus one needs to calculate the ρ's, which is quite a cumbersome task. Both of these difficulties are partially remedied in the next section.

Sufficient Condition for Unique Solvability of (9) with $u^1 \in U_1$. It is an immediate consequence of (14) that, since all the ρ's are in $[0, 1]$, if

$$r_1(t)r_2(t) < 1, \qquad \text{for every } t \in [0, T],$$

then there exists only one solution of (9) and this solution will be affine in x^1 for every t. This is a very important observation, since the calculation of the ρ's so as to verify whether the coefficient of c in (14) is zero or not as well as the calculation of the b's for every t is quite a task. Also, if we assume the slightly stronger conditions:

$$\inf[r_1(t)r_2(t), t \in [0, T]] > -\infty,$$

and

$$\sup[r_1(t)r_2(t), t \in [0, T]] < 1,$$

then not only (9) has a unique solution for each t, but the solution is also in U_1 (see Corollary 2.1 below). It should also be pointed out that the Gaussian assumption on x, x^1, x^2 can be dispensed with as far as it concerns the results of this section. These two results can be proved by using the following Proposition of Ref. 3.

Proposition 2.2. Let H be a Hilbert space over the reals and P an orthogonal projection in H. Let $Q: H \to H$ be a continuous linear operator (P and Q do not necessarily commute) and v an element of H. Then, a sufficient condition that the equation

$$PQu + Pv = 0, \qquad Pu = u, \tag{15}$$

has a unique solution $u \in H$ is that there exists a continuous linear operator $E: H \to H$ which has a continuous inverse E^{-1}, commutes with P, and that it holds

$$QE^* + EQ^* \geq I, \qquad \text{on } PH. \tag{16}$$

If this sufficient condition holds, then the solution of (9) is given by

$$u = P \sum_{n=0}^{\infty} [(I - \bar{E}^{-1}Q)P]^n \bar{E}^{-1} v, \tag{17}$$

where

$$\bar{E} = \sigma E, \qquad \text{for any } \sigma > \|QQ^*\|.$$

By applying Proposition 2 to (9), we obtain the following corollary.

Corollary 2.1. If

$$\inf[r_1(t)r_2(t), t \in [0, T]] > -\infty,$$

and

$$\sup[r_1(t)r_2(t), t\in[0, T]]<1,$$

then (9) has a unique solution in U_1 which is given by

$$u^1=P_{1t}\sum_{n=0}[1-\epsilon^{-1}(1-r_1(t)r_2(t)P_{2t})]^n\epsilon^{-1}P_{1t}[-r_1(t)s_2(t)P_{2t}x(t)$$
$$+s_1(t)x(t)], \tag{18}$$

where ϵ is any positive number such that

$$2\epsilon>1-\inf[r_1(t)r_2(t), t\in[0, T]]. \tag{19}$$

Proof. Applying Proposition 2.2 to (9) by letting P_{1t} play the role of P, $1-r_1(t)r_2(t)P_{2t}$ play the role of Q, and $\epsilon=E$, we obtain (18) and (19). We still have to show that u^1 as given in (18) is in U_1. First, notice that each term of the summation in (18) lies in U by Theorem 1.1 of Ref. 11. Also, using the assumptions

$$\sup r_1r_2<1, \qquad \inf(r_1r_2)>-\infty,$$

it is easy to show that

$$|1-\epsilon^{-1}(1-r_1(t)r_2(t)P_{2t})|<1,$$

for ϵ chosen as in (19). Thus, $u^1\in U_1$. □

It should be clear from the proof that both Proposition 2.2 and Corollary 2.1 do not depend on the Gaussian assumption about x, x^1, x^2. This corollary guarantees the existence of a unique Nash solution to the game considered in Section 1 under the assumptions stated. It is clear that, in a practical situation, the assumption

$$\inf[r_1(t)r_2(t)]>-\infty$$

will be almost always satisfied, but the assumption

$$\sup[r_1(t)r_2(t)]<1$$

might be violated. Nonetheless,

$$\sup[r_1(t)r_2(t)]<1$$

will also hold in many practical situations for the following reason. Usually, one starts with a single objective J, which is split into two parts, J_1 and J_2, and then a Nash solution is sought. If

$$J=J_1+J_2$$

and J is a convex function of u_1, u_2, then

$$(r_1(t)+r_2(t))^2<1,$$

which implies

$$|r_1(t)r_2(t)|<\tfrac{1}{4}.$$

Thus, even if J_1 and J_2 have a sum approximately equal to some function J convex in (u_1, u_2), the assumption

$$\sup[r_1(t)r_2(t)]<1$$

stands an excellent chance to hold. In addition, if $r_1(t)r_2(t)$ has values in $[1, +\infty)$, then it stands a good chance to be equal or close to some product of powers of correlation coefficients, which would imply nonunique solutions (or an unreliable unique solution, as long as $r_1(t)r_2(t)$ is approximately known in a practical setup and any computer uses finite precision).

Corollary 2.1, as well as the whole analysis presented, can be extended to the case where u^1, u^2, x are vector valued. In this case, $r_1(t), r_2(t)$ will become matrices $R_1(t), R_2(t)$, and the role of $r_1(t), r_2(t)$ in (14), Proposition 2.1, and Corollary 2.1 will be played by the eigenvalues $R_1(t), R_2(t)$ (see also Ref. 3). Corollary 2.1 and its extension to the vector case can be used to provide easy proofs for results concerning existence and uniqueness of a solution in linear-quadratic team problems with delayed information and information exchange among the players under appropriate nestedness conditions on the information of the players.

3. Second Class

Problem Statement. Let $(\Omega, \mathscr{F}, P)$ be a complete probability space over which all the random quantities involved are defined. $w(t)$ is an n-dimensional standard Brownian motion; x_0, v_1, v_2 are Gaussian, zero-mean, random vectors with nonsingular variances, independent of each other and w, and with dimensions n, r_1, r_2, respectively. We assume, without loss of generality, that $r_1 \leq r_2$. Let

$$y_i = C_i x_0 + v_i, \qquad i=1,2, \tag{20}$$

where C_1, C_2 are real constant matrices of appropriate dimensions with full rank equal to r_1, r_2, respectively. Let $\mathscr{F}_i$ be the sub σ-field of $\mathscr{F}$ generated by y_i. Let U_i, $i=1,2$, be the space of all functions

$$u:[t_0, t_f]\times\Omega \to R^{m_i} \tag{21}$$

(m_1, m_2 are some integers), which are jointly measurable with respect to the product σ-field $\mathscr{B} \times \mathscr{F}_i$, where $\mathscr{B}$ is the usual σ-field on $[t_0, t_f]$ and which satisfy

$$E\left[\int_{t_0}^{t_f} u'(t)u(t)\,dt\right] < +\infty.$$

U_i is obviously a Hilbert space with inner product defined by

$$\langle u, \hat{u}\rangle = E\left[\int_{t_0}^{t_f} u'(t)\hat{u}(t)\,dt\right].$$

We can similarly define Hilbert spaces where a different sub σ-field of $\mathscr{F}$ is used and the elements take values in finite-dimensional Euclidean spaces. Between any two such Hilbert spaces, we can define operators of the form

$$u \mapsto \int_{t_0}^{t} K_1(t, s)u(s)\,ds, \qquad u \mapsto K_2(t)u(t),$$

where $K_1(t, s), K_2(t)$ are real matrices which are piecewise continuous functions of their arguments. The adjoints of such operators will be denoted by an asterisk superscript, whereas the transposes of vectors or matrices will be denoted by a prime superscript.

For any $(u_1, u_2) \in U_1 \times U_2$, consider a dynamic system, whose state $x(t)$ takes values in R^n and evolves according to

$$dx(t) = [A(t)x(t) + B_1(t)u_1(t) + B_2u_2(t)]\,dt + dw(t), \tag{22a}$$

$$x(t_0) = x_0, \qquad t \in [t_0, t_f], \tag{22b}$$

and two cost J_1, J_2 defined by

$$\begin{aligned} J_i(u_1, u_2) = E\bigg[x'(t_f)Q_{if}x(t_f) + \int_{t_0}^{t_f} [x'(t)Q_i(t)x(t) \\ + u_i'(t)u_i(t) + 2x'(t)S_{ii}(t)u_i(t) + 2x'(t)S_{ij}(t)u_j(t) \\ + 2u_j'(t)R_i(t)u_i(t)]\,dt\bigg], \qquad i \neq j, \qquad i, j = 1, 2, \end{aligned} \tag{23}$$

where the matrices A, B_i, Q_i, Q_{if}, S_{ij}, R_i are real, piecewise continuous in t, Q_{if} is constant, and Q_i, Q_{if} are symmetric and positive semidefinite. We also assume that

$$\begin{bmatrix} Q_i(t) & S_{ii}(t) \\ S_{ii}'(t) & I \end{bmatrix} \geq 0. \tag{24}$$

With $(u_1, u_2) \in U_1 \times U_2$, the solution of (22) exists over $[t_0, t_f]$, and J_1, J_2 are finite and strictly convex in u_1, u_2, respectively.

Our problem is to characterize and find a pair $(u_1, u_2) \in U_1 \times U_2$ for which the following Nash equilibrium conditions hold:

$$J_1(u_1, u_2) \leq J_1(\bar{u}_1, u_2), \qquad \forall \bar{u}_1 \in U_1, \tag{25}$$

$$J_2(u_1, u_2) \leq J_2(u_1, \bar{u}_2), \qquad \forall \bar{u}_2 \in U_2. \tag{26}$$

Solution. By defining appropriate Hilbert space operators $L_0, L_1, L_2, L, \bar{Q}_i, \bar{S}_{ij}, \bar{R}_i$, we can equivalently transform (22) and (23) into

$$x = L_0 x_0 + L_1 u_1 + L_2 u_2 + Lw, \tag{27}$$

$$J_i(u_1, u_2) = \langle x, \bar{Q}_i x\rangle + \langle u_i, u_i\rangle + 2\langle x, \bar{S}_{ii} u_i\rangle + 2\langle x, \bar{S}_{ij} u_j\rangle + 2\langle u_j, \bar{R}_i u_i\rangle, \qquad i \neq j,\ i, \quad j = 1, 2. \tag{28}$$

Substituting x from (27) into (28) yields

$$\begin{aligned} J_i(u_1, u_2) = {} & \langle u_1, (L_i^* \bar{Q}_i L_i + L_i^* \bar{S}_{ii} + \bar{S}_{ii}^* L_i + I) u_i\rangle \\ & + 2\langle u_i, (L_i^* \ \bar{Q}_i L_0 + \bar{S}_{ii}^* L_0) x_0\rangle \\ & + 2\langle u_i, (L_i^* \bar{Q}_i L_j + \bar{S}_{ii}^* L_j + L_i^* \bar{S}_{ij} + \bar{R}_i^*) u_j\rangle \\ & + \text{quadratic terms in } (x_0, u_j, w), \qquad i \neq j, \quad i, j = 1, 2. \end{aligned} \tag{29}$$

In (29), no cross products between u_i and w appear, since w has zero mean and y_i (and thus u_i) is independent of w. J_i is a strictly convex function of u_i [recall (23)–(24)]; thus, the necessary and sufficient conditions that u_1, u_2 satisfy (25)–(26) is that the pair (u_1, u_2) solves the following system of equations[3]:

$$R_{11} u_1 + R_{12} P_1 u_2 = -S_1 P_1 x_0, \tag{30}$$

$$R_{21} P_2 u_1 + R_{22} u_2 = -S_2 P_2 x_0, \tag{31}$$

where

$$\begin{aligned} & P_i = E[\cdot \mid \mathscr{F}_i], && i = 1, 2, \\ & R_{ii} = I + L_i^* \bar{Q}_i L_i + L_i^* \bar{S}_{ii} + \bar{S}_{ii}^* L_i, && i = 1, 2, \\ & R_{ij} = \bar{R}_i^* + L_i^* \bar{Q}_i L_j + \bar{S}_{ii} L_j + L_i^* \bar{S}_{ij}, && i \neq j, \quad i, j = 1, 2, \\ & S_i = L_i^* \bar{Q}_i L_0 + \bar{S}_{ii}^* L_0, && i = 1, 2. \end{aligned}$$

We will now study these two equations by using an expansion for u_1, u_2 in orthonormal series. Let us first notice that we can premultiply y_1, y_2 by

[3] It is a standard result in linear-quadratic control theory that, under assumption (24), the operator R_{ii} is strictly positive definite, it has a bounded inverse, and is equal to the sum of the unit operator plus a compact operator.

nonsingular matrices, so that the transformed y_1, y_2 have unit variances and diagonal covariance (see Refs. 3 and 12); i.e., we can assume without loss of generality that

$$E[y_1 y_1'] = I(r_1 \times r_1),$$

$$E[y_2 y_2'] = I(r_2 \times r_2),$$

$$E[y_1 y_2'] = \left[\begin{array}{ccc|c} \mu_1 & & 0 & \\ & \mu_2 & & 0 \\ 0 & & \ddots \; \mu_{r_1} & \end{array}\right] (r_1 \times r_2),$$

where

$$0 < \mu_1, \ldots, \mu_{r_1} < 1.$$

The μ_i's are the canonical correlation coefficients of y_1, y_2. Had we allowed dependences among x_0, v_1, v_2, C_i not to have full rank r_i, it would have resulted in

$$0 \leq \mu_i \leq 1;$$

this case can also be studied by the methodologies employed here at the expense of more notational complication.

Let us also denote by y_{il} the components of y_i, i.e.,

$$y_i = \begin{bmatrix} y_{i1} \\ y_{il} \\ \vdots \\ y_{ir_i} \end{bmatrix}.$$

Consider the Hermite polynomial h_n defined in (12). Using the separability of U_1, it is easy to see that u_1 can be expressed as

$$u_i = \sum u^1_{k_1 \cdots k_{r_1}}(t) h_{k_1}(y_{11}) h_{k_2}(y_{12}) \cdots h_{k_{r_1}}(y_{1r_1}), \tag{32}$$

where the summation is taken over all $k_1, k_2, \ldots, k_{r_1}$, ranging in $\{0, 1, 2, 3, \ldots\}$, each $u^1_{k_1 \cdots k_2}$ is in $L_2([t_0, t_f], R^{m_1})$, and

$$\sum \|u^1_{k_1 \cdots k_{r_1}}\|^2 < +\infty.$$

For u_2, we have similarly

$$u_2 = \sum u^2_{k_1 \cdots k_{r_2}}(t) h_{k_1}(y_{21}) h_{k_2}(y_{22}) \cdots h_{k_{r_2}}(y_{2r_2}). \tag{33}$$

Solving (30)–(31) for u_1, u_2, is equivalent to finding the $u^i_{k_1 \cdots k_{r_i}}$'s. Substituting u_1, u_2 by their series expansions (32), (33) in (30), (31), multiplying both sides of (30) by $h_{k_1}(y_{11}) \cdots h_{kr_1}(y_1, r_1)$, and of (31) by

$h_{k_1}(y_{21}) \cdots h_{k_{r_2}}(y_{2r_2})$, taking the total expectation of both sides, and using Lemma 2 of Ref. 3 yields the following characterizations of the $u^i_{k_1 \cdots k_{r_i}}$'s:

$$R_{11}u^1 + \mu R_{12}u^2 = 0, \tag{34a}$$

$$\mu R_{21}u^1 + R_{22}u^2 = 0, \tag{34b}$$

where

$$u^1 = u^1_{k_1 \cdots k_{r_1}}, \tag{34c}$$

$$u^2 = u^2_{k_1 \cdots k_{r_1}, 0 \cdots 0}, \tag{34d}$$

$$\mu = \mu_1^{k_1} \mu_2^{k_2} \cdots \mu_{r_1}^{k_{r_1}}, \tag{34e}$$

$$(k_1, k_2, \ldots, k_{r_1}) \neq (1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1); \tag{34f}$$

$$R_{22}u^2 = 0, \tag{35a}$$

where

$$u^2 = u^2_{k_1 k_2 \cdots k_{r_1} k_{r_1+1} \cdots k_{r_2}}, \tag{35b}$$

$$(k_{r_1+1}, \cdots, k_{r_2}) \neq (0, 0, \ldots, 0); \tag{35c}$$

$$R_{11}u^1 + \mu R_{12}u^2 = -S_1 E[x_0 y_{il}], \tag{36a}$$

$$\mu R_{21}u^1 + R_{22}u^2 = -S_2 E[x_0 y_{2l}], \tag{36b}$$

where

$$u^1 = u^1_{k_1 \cdots k_{r_1}}, \tag{36c}$$

$$u^2 = u^2_{k_1 \cdots k_{r_1}, 0, \ldots, 0}, \tag{36d}$$

$$(k_1, \ldots, k_{l-1}, k_l, k_{l+1}, \ldots, k_{r_1}) = (0, \ldots, 0, 1, 0, \ldots, 0), \tag{36e}$$

$$\mu = \mu_l, \tag{36f}$$

$$l = 1, 2, \ldots, r_1; \tag{36g}$$

$$R_{22}u^2 = -S_2 E[x_0 y_{2l}], \tag{37a}$$

where

$$u^2 = u^2_{k_1 \cdots k_{r_1} \cdots k_{r_2}}, \tag{37b}$$

$$\begin{aligned}(k_1, \ldots, k_{r_1}, k_{r_1+1}, \ldots, k_{l-1}, k_l, k_{l+1}, \ldots, k_{r_2}) \\ = (0, \ldots, 0, 0, \ldots, 0, 1, 0, \ldots, 0),\end{aligned} \tag{37c}$$

$$l = r_1 + 1, r_1 + 2, \ldots, r_2. \tag{37d}$$

These conditions characterize completely the coefficients of the expansions (32)–(33). Notice that the operators R_{ij} are operators from L_2 into L_2; thus, (34)–(37) are integral equations with nothing of stochastic nature being involved in them. Let us analyze them more closely. (34) characterizes the coefficients of those nonlinear terms of the expansions (32)–(33) where

u^2 does not use any information decoupled from the information that is available to u^1; i.e., u^2 does not use $y_{2,r_1+1}, \ldots, y_{2,r_2}$. (35) characterizes the coefficients of the nonlinear terms of (33) which use $y_{2,r_1+1}, \ldots, y_{2,r_2}$. Since R_{22} is invertible, (35) yields that these coefficients have to be zero. (36) and (37) characterize the parts of u_1, u_2 which depend linearly on the information y_1, y_2, respectively. Due to the invertibility of R_{22}, (37) yields that the part of u_2 which depends linearly on information not available to u_1, and on nothing else, can be determined uniquely. (36) characterizes the parts of u_1, u_2 which depend linearly on the coupled information, i.e., on $(y_{11}, \ldots, y_{1,r_1})$ and $(y_{21}, \ldots, y_{2r_1})$, respectively. Before continuing analyzing these conditions, we can state our first important result.

Proposition 3.1. The stochastic Nash game under consideration [(22)–(26)] admits a solution if and only if it admits a solution linear in the information.

Proof. Since (34) and (35) admit the identically zero solution, and since (37) admits a unique solution, we conclude that the game has a solution if and only if (36) admits a solution. The solutions of (36) and (37) provide the coefficients in the expansion (33) which multiply linear functions of y_i. □

It is interesting that (34)–(37) admit the following-game theoretic (in flavor) interpretation, which at the same time provides an alternative way of finding the solution of the stochastic Nash game (22)–(26). Consider the state equation

$$(d/dt)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} I & \mu I \\ \mu I & I \end{bmatrix}\begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}\begin{bmatrix} u^1 \\ u^2 \end{bmatrix} \tag{38}$$

and the costs

$$J_i(u^1, u^2) = x_i'(t_f)Q_{if}x_i(t_f) + \int_{t_0}^{t_f} [x_1'Q_ix_i + (u^i)'(u^i) + 2x_i'S_{ii}u^i + 2\mu x_i'S_{ij}u^j + 2\mu(u^j)'R_iu^i]\,dt, \qquad i, j = 1, 2, \quad i \neq j. \tag{39}$$

A little reflection will persuade the reader that these statements hold:

(i) A pair (u^1, u^2) solves (34) if and only if it is an *open-loop Nash equilibrium of the deterministic Nash game* described by (38)–(39) with initial condition

$$\begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{40}$$

and μ equal to the corresponding μ considered in (34e).

(ii) A pair (u^1, u^2) solves (36) if and only if it is an *open-loop Nash equilibrium of the deterministic Nash game* described by (38)–(39) with the initial condition

$$\begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix} = E\begin{bmatrix} x_0 y_{1l} \\ x_0 y_{2l} \end{bmatrix}, \qquad l = 1, 2, \ldots, r_1, \tag{41}$$

and μ equal to the corresponding μ considered in (36f).

(iii) u^2 solves (35) if and only if it solves the *open-loop control problem*

$$\dot{x}_2 = Ax_2 + B_2 u^2, \tag{42a}$$

$$J_2 = x_2'(t_f)Q_{2f}x_2(t_f) + \int_{t_0}^{t_f} [x_2' Q_2 x_2 + (u^2)'u^2 + 2x_2' S_{22} u^2]\, dt, \tag{42b}$$

$$x_2(t_0) = 0. \tag{42c}$$

(iv) u^2 solves (37) if and only if it solves the *open-loop control problem* described in (iii) with initial condition

$$x_2(t_0) = E[x_0 y_{2l}], \qquad l = r_1 + 1, \ldots, r_2. \tag{43}$$

It should be noticed that (i)–(iv) carry through even if the x_0, v_1, v_2 are dependent and C_1, C_2 do not have full rank. We thus conclude that, in order to solve the stochastic game (22)–(26), we have to solve the control problems (iii), (iv), r_1 in multitude open-loop deterministic Nash games (ii), and a possibly infinite-in-multitude series of open-loop deterministic Nash games (i). The solution of (42) is obviously

$$u^2 = 0$$

[recall (35)], and each problem of type (iv) admits a unique solution [recall (37) and Footnote 3]. Let us now show that only a finite number of problems of type (i) need to be solved, as the rest of them have as only solution

$$u^1 = u^2 = 0.$$

If we had to solve an infinite number of such problems, this would be due to having to consider an infinite number of distinct μ's in (34) and thus in (38)–(39). This can happen if at least one of the $\mu_1, \ldots, \mu_{r_1}$ is strictly between zero and one. The operator on the left-hand side of (34) is

$$\begin{bmatrix} R_{11} & 0 \\ 0 & R_{22} \end{bmatrix} + \mu \begin{bmatrix} 0 & R_{12} \\ R_{21} & 0 \end{bmatrix} \tag{44}$$

and R_{11}, R_{22} have bounded inverses, whereas R_{12}, R_{21} are bounded operators. Thus, as $\mu \to 0$, the operator (44) is invertible and the corresponding u^1, u^2 are identically zero. Thus, only a finite number of equations (34), or equivalently problems of type (i), need to be solved. It should be emphasized that the above argument does not imply that the expansions (32)–(33) have a finite number of terms, but rather that only a finite number of coefficients might be different than each other. Nonetheless, if all the $\mu_1, \ldots, \mu_{r_1}$ are strictly less than one, then the series (32)–(33) will be finite, as a slight extension of the argument just used will persuade the reader. Finally, notice that an open-loop deterministic Nash game of the type (i) always has the identically zero solution; and, if it has a nonzero solution, then it has infinitely many [by inspection of (34)]. Let us formalize this discussion in the following proposition.

Proposition 3.2. Solving the stochastic Nash game (22)–(26) is equivalent to solving a finite number of open-loop deterministic control problems and a finite number of open-loop deterministic Nash games, described in (i)–(iv). The solutions of these problems provide the coefficients of the expansions (32)–(33). The expansions (32)–(33) are finite if

$$0 \le \mu_1, \ldots, \mu_{r_1} < 1.$$

Further study of the stochastic Nash game (22)–(26) is thus reduced to the open-loop deterministic Nash case. Interestingly, the open-loop Nash games that are of interest here have a special structure; see (38)–(39); consequently, some interesting results can be obtained without studying the LQ open-loop deterministic Nash game in its generality. Let $R_1, R_2, S_{11}, S_{12}, S_{21}, S_{22}$ be identically zero, and let

$$B_1 = B_2 = B;$$

then, (u^1, u^2) solves the game (38)–(39) if and only if the following hold:

$$(d/dt)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} I & \mu I \\ \mu I & I \end{bmatrix}\begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}\begin{bmatrix} u^1 \\ u^2 \end{bmatrix}, \tag{45}$$

$$-(d/dt)\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} A' & 0 \\ 0 & A' \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{46}$$

$$\begin{bmatrix} u^1 \\ u^2 \end{bmatrix} + \begin{bmatrix} B' & 0 \\ 0 & B' \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 0, \tag{47}$$

$$\begin{bmatrix} y_1(t_f) \\ y_2(t_f) \end{bmatrix} = \begin{bmatrix} Q_{1f} & 0 \\ 0 & Q_{2f} \end{bmatrix}\begin{bmatrix} x_1(t_f) \\ x_2(t_f) \end{bmatrix}, \tag{48}$$

and

$$\begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{49a}$$

or

$$\begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix} = E\begin{bmatrix} x_0 y_{1l} \\ x_0 y_{2l} \end{bmatrix}. \tag{49b}$$

If $0 \leq \mu < 1$, (47) can be written equivalently

$$\begin{bmatrix} I & \mu I \\ \mu I & I \end{bmatrix}\begin{bmatrix} u^1 \\ u^2 \end{bmatrix} + \left(\begin{bmatrix} I & \mu I \\ \mu I & I \end{bmatrix}\begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}\right)\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 0. \tag{50}$$

We thus conclude that, if

$$0 < \mu_1, \ldots, \mu_{r_1} < 1,$$

(u^1, u^2) solves the open-loop deterministic Nash problem if and only if it solves the control problem

$$(d/dt)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} I & \mu I \\ \mu I & I \end{bmatrix}\begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}\begin{bmatrix} u^1 \\ u^2 \end{bmatrix}, \tag{51a}$$

$$\begin{aligned} \min_{u^1, u^2} & \begin{bmatrix} x_1(t_f) \\ x_2(t_f) \end{bmatrix}\begin{bmatrix} Q_{1f} & 0 \\ 0 & Q_{2f} \end{bmatrix}\begin{bmatrix} x_1(t_f) \\ x_2(t_f) \end{bmatrix} + \int_{t_0}^{t_f} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ & + \begin{bmatrix} u^1 \\ u^2 \end{bmatrix}\begin{bmatrix} I & \mu I \\ \mu I & I \end{bmatrix}\begin{bmatrix} u^1 \\ u^2 \end{bmatrix} dt, \end{aligned} \tag{51b}$$

with initial condition

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{52a}$$

or

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} E[x_0 y_1 l] \\ E[x_0 y_2 l] \end{bmatrix}. \tag{52b}$$

With the initial condition (52a), u^1, u^2 are identically zero; i.e., there is no nonlinear part in the solution of the stochastic Nash game! With initial condition (52b), u^1, u^2 are uniquely determined for each $\mu = \mu_1, \ldots, \mu_{r_1}$ and the solution exists for any finite but arbitrarily large $[t_0, t_f]$; i.e., the part of the solution of the stochastic Nash game that is linear in the information exists and is unique for any finite interval $[t_0, t_f]$. Taking into account the fact that (iii) has as the only solution the identically zero one, and thus (iv) admits a unique solution for each $l = r_1 + 1, \ldots, r_2$, we can summarize our discussion in the following proposition.

Proposition 3.3. If $R_1, R_2, S_{11}, S_{12}, S_{22}, S_{21}$ are identically zero, $B_1 = B_2$, and $0 \leq \mu_i < 1$, then the stochastic Nash game (22)–(26) admits a unique solution which has to be linear in the information, for any arbitrarily large but finite $[t_0, t_f]$.

Finding the solution guaranteed by this proposition can be achieved by solving the control problem (iv) and the control problems (45)–(48), (49b) and can be reformulated in terms of solving two coupled Riccati-type differential equations whose solution is guaranteed by Proposition 3.3 to exist on any finite interval $[t_0, t_f]$. The derivation of these two coupled Riccati-type equations is easy and can also be achieved directly from the initial formulation (22)–(26), by setting

$$u_1 = K_1(t) y_1$$

and solving for

$$u_2 = K_2(t) y_2,$$

and vice versa. Finally, notice that our assumptions on x_0, v_1, v_2, C_1, C_2 [see discussion after (31)] guarantee $0 < \mu_i < 1$, which implies $0 \leq \mu_i < 1$ that Proposition 3.3 requires.

We can also prove existence and uniqueness of a solution of the game by basically assuming that $[t_0, t_f]$ is sufficiently small, without having to assume $R_i, S_{ij} = 0$, $B_1 = B_2$ as in Proposition 3.3. In order to do that, we only need the following assumption.

Assumption 3.1. The matrix

$$\begin{bmatrix} I & \mu R_1(t) \\ \mu R_2(t) & I \end{bmatrix} \tag{53}$$

has an inverse for any $t \in [t_0, t_f]$; this inverse is uniformly bounded in norm by some nonnegative constant C, which might depend on $\mu_1, \ldots, \mu_{r_1}$ but does not depend on the particular $\mu = (\mu_1^{k_1}, \ldots, \mu_{r_1}^{k_{r_1}})$ or on the interval $[t_0, t_f]$.

Let us first consider some cases where this assumption holds. If R_1, R_2 are constant matrices, then the inverse of (53) exists if

$$\rho\mu^2 \neq 1,$$

where ρ is any eigenvalue of $R_1 R_2$ and equals

$$\begin{bmatrix} (I - \mu^2 R_1 R_2)^{-1} & 0 \\ 0 & (I - \mu^2 R_2 R_1)^{-1} \end{bmatrix} \begin{bmatrix} I & -\mu R_1 \\ -\mu R_2 & I \end{bmatrix}. \tag{54}$$

It can easily be seen that this inverse is bounded in norm by a constant C, which depends on $R_1, R_2, \mu_1, \ldots, \mu_{r_1}$, and C does not vary with the particular $\mu = (\mu_1^{k_1}, \ldots, \mu_{r_1}^{k_{r_1}})$ used in (54) as long as no eigenvalue of $R_1 R_2$ equals any $(\mu_1^{k_1}, \ldots, \mu_{r}^{k_{r_1}})^{-2}$; this is the case, for example, if no eigenvalue of R_1R_2 lies in $[1+\infty)$. If R_1, R_2 are functions of time, whose norms $\|R_1(t)\|, \|R_2(t)\|$ are smaller than some $\delta, 0<\delta<1$, for any t, then again Assumption 3.1 holds. Let us now state and prove the following proposition.

Proposition 3.4. If Assumption 3.1 holds, then the stochastic Nash game (22)–(26) admits a unique solution which is linear in the information, if $[t_0, t_f]$ is sufficiently small.

Proof. The operator (44) [see also (34)–(37)] is of the form

$$\begin{bmatrix} I & \mu R_1(t) \\ \mu R_2(t) & I \end{bmatrix} + \int_{t_0}^{t_f} \begin{bmatrix} K_{11}(t, s) & \mu K_{12}(t, s) \\ \mu K_{21}(t, s) & K_{22}(t, s) \end{bmatrix} (\cdot)\, ds, \tag{55}$$

where the $K_{ij}(t, s)$ are piecewise continuous functions of (t, s) and can be calculated in terms of $A, B_i, Q_i, S_{ij}, Q_{if}$. Invertibility of this operator is equivalent to the invertibility of

$$I + \begin{bmatrix} I & \mu R_1(t) \\ \mu R_2(t) & I \end{bmatrix}^{-1} \int_{t_0}^{t_f} \begin{bmatrix} K_{11}(t, s) & \mu K_{12}(t, s) \\ \mu K_{21}(t, s) & K_{11}(t, s) \end{bmatrix} (\cdot)\, ds,$$

which under Assumption 3.1 is invertible if $[t_0, t_f]$ is sufficiently small. Invertibility of (55) guarantees the existence of a unique solution of (34)–(37), and thus of the game, which is linear in the information by Proposition 3.1. □

We will conclude our discussion by showing that, although nonlinear solutions might exist, their existence is highly unlikely and that they are subject to disappearance by perturbing slightly the matrices involved in the description of the game. Let us assume for simplicity that $R_1 = 0$, $R_2 = 0$ and that all the matrices B_i, Q_i, S_{ij} are constant. A pair (u^1, u^2) solves the open-loop deterministic Nash game (38)–(39) with initial condition (40) if and only if x_1, x_2, together with two adjoint variables y_1, y_2, solve the two-point boundary-value problem

$$(d/dt)\begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} = \hat{A}(\mu) \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix}, \tag{56a}$$

$$x_1(t_0) = x_2(t_0) = 0, \qquad y_1(t_f) = Q_{1f} x_1(t_f), \qquad y_2(t_f) = Q_{2f} x_2(t_f), \tag{56b}$$

where $\hat{A}(\mu)$ is a linear function of μ and thus is analytic. Let $t_0 = 0$, and let Φ be the transition matrix of $\hat{A}(\mu)$. Let

$$\Phi(t, \mu) = \begin{bmatrix} \Phi_{11}(t, \mu) & \Phi_{12}(t, \mu) \\ \Phi_{21}(t, \mu) & \Phi_{22}(t, \mu) \end{bmatrix}, \qquad Q_f = \begin{bmatrix} Q_{1f} & 0 \\ 0 & Q_{2f} \end{bmatrix}.$$

(56) admits a nonzero solution if and only if

$$\delta(\mu)=\det(\Phi_{22}(t_f, \mu)-Q_f\Phi_{12}(t_f, \mu))$$

is zero. Since $\hat{A}$ is an analytic function of μ, so are $\Phi(t, \mu)$ and $\delta(\mu)$. $\delta(0)$ is different than zero, since with $\mu=0$ the open-loop deterministic Nash game (38)–(39) decomposes into two open-loop convex control problems with initial conditions zero, and thus it has a unique identically zero solution. Since $\delta(0)\neq 0$ and δ is analytic in μ, it has a finite number of zeros for $\mu\in[0, 1]$. Thus, in order for the Nash game to have nonlinear solutions, it must hold that the choice of x_0, v_1, v_2, C_1, C_2 is such that some $\mu_1^{k_1}, \ldots, \mu_{r_1}^{k_{r_1}}$ equals some of these finite zeros of $\delta(\mu)$, for some $k_1, \ldots, k_r=0, 1, 2, \ldots$. It is obvious now that this is highly unlikely to happen. Also, the zeros of δ change if we perturb A, B_i, S_{ij}, Q_i, Q_{if}; thus, even if a nonlinear solution exists, it is subject to disappearance by slightly perturbing the matrices A, B_i, S_{ij}, Q_i, Q_{if}. Notice that this line of reasoning can be used to prove that only a finite number of open-loop deterministic Nash games need to be solved, as Proposition 3.2 states, since as $\mu\to 0$ we do not need to assume $R_1=0$, $R_2=0$, and thus we can solve the necessary conditions that u^1, u^2 solve (38)–(39), and end up studying a two-point boundary-value problem like (56), where $\hat{A}$ is not linear in μ but is still analytic in some interval $[0, \bar{\mu}]$, $0<\bar{\mu}<1$.

4. Conclusions

One of the central features of the present paper is that it demonstrated the importance of linear solutions for two classes of linear-quadratic stochastic Nash games. As has been shown, if a solution exists, then there will exist a solution linear in the information; but, if there exists a nonlinear solution, then there will exist infinitely many nonlinear solutions. The possible existence of infinitely many nonlinear solutions questions the credibility of the linear ones; but, as has been shown for the second class considered, nonlinear solutions are not only highly unlikely to exist, but also disappear under small perturbations of the parameters describing the problem.

There are several problems worthy of attention, emanating from the analysis presented. For example, it would be interesting to find conditions more general than those of Propositions 3.3 and 3.4, so that the game admits a unique solution linear in the information for any measurements of the type (20); i.e., conditions that guarantee the invertibility of the operator (49) for any $\mu\in[0, 1]$. As has been shown, this is equivalent to studying the deterministic open-loop Nash games (38)–(39).

The results presented can also be used as a starting point in studying Nash games with more interesting information structures. For example, one could consider a Nash game like (22)–(23) where the information is described by the so-called one-step delay observation sharing pattern; see Ref. 1 for a team problem with this information structure. This is an analogue of a problem considered in Refs. 4 and 5 in the discrete-time setup.

5. Appendix: Proof of Lemma 2.1

Using the Karhunen–Loéve expansion for $x^i_\tau, 0 \le \tau \le t$, we obtain

$$x^i_\tau = \sum_n \phi^i_n(\tau)\eta^i_n \overline{\lambda^i_n} \qquad \text{(q.m. limit)}, \tag{57a}$$

$$\int_0^t \phi^i_n(\sigma)\phi^i_m(\sigma)\, d\sigma = \delta_{nm}, \tag{57b}$$

$$\int_0^t K_{ii}(\tau, \sigma)\phi^i_n(\sigma)\, d\sigma = \lambda^i_n \phi^i_n(\tau), \qquad 0 < \cdots \le \lambda^i_{n+1} \le \lambda^i_n \le \cdots \le \lambda^i_1, \tag{57c}$$

$$K_{ii}(\tau, \sigma) = \sum_n \lambda^i_n \phi^i_n(\tau)\phi^i_n(\sigma), \qquad i = 1, 2, \tag{57d}$$

$$\eta^i_n = (\sqrt{\lambda^i_\eta})^{-1} \int_0^t \phi^i_n(\tau)x^i_\tau\, d\tau, \tag{57e}$$

$$E[\eta^i_n \eta^i_m] = \delta_{nm}. \tag{57f}$$

For convenience, we dropped the subscript t. Clearly,

$$L^i_{2t} = L_2(\eta^i_n, n = 1, 2, 3, \ldots).$$

By considering complete orthonormal sets for $L^1_{2t} \cap L^2_{2t}, L^1_{2t} \cap (L^2_{2t})^\perp, (L^1_{2t})^\perp \cap L^2_{2t}$, we end up with the complete orthonormal sets $\{\zeta^i_k\}$; see (10); see Ref. 11 for a similar analysis in a more abstract setup; i.e., we can find unitary transformations V_1 from L^i_{2t} onto L^i_{2t} so that

$$\zeta^i_n = V_i(\eta^i_n)$$

satisfy the condition (i). If l^i_2 denotes the Hilbert space of square summable sequences,

$$x^i = (x^i_1, x^i_2, \ldots), \qquad \sum_n (x^i_n)^2 < +\infty,$$

with dimension equal to the one of L^i_{2t}, then V_i can be considered as been defined on l^i_2 if to ζ^i_n we correspond $(0, \ldots, 0, 1, 0, \ldots)$ (nth position has

1). Obviously, the ρ_k's can be obtained by solving the system

$$\rho_n x_n^1 = \sum_m \langle \eta_n^1, \eta_m^2 \rangle x_m^2, \tag{58a}$$

$$\rho_n x^2 = \sum_m \langle \eta_n^2, \eta_m^1 \rangle x_m^1, \tag{58b}$$

$$x^i = (x_1^i, x_2^i, \ldots) \in l_2^i, \qquad i = 1, 2. \tag{58c}$$

Using (57), we can write equivalently

$$\rho x_n^i = \sum_m \left[\int_0^t \int_0^t K_{ij}(\tau, s)[\phi_n^i(\tau)/\sqrt{\lambda_n^i}][\phi_m^j(s)/\sqrt{\lambda_m^j}]\, d\tau\, ds \right] x_m^j,$$
$$i \neq j, \qquad i, j = 1, 2. \tag{59}$$

If the summation on m is infinite, then

$$g^j(s) = \sum_m [x_m^j \phi_m^j(s)/\sqrt{\lambda_m^j}]$$

might not exist (the limit) in the H_2 sense, where H_2 is the Hilbert space of square-integrable functions on $[0, t]$ with the usual inner product. Nonetheless, if we accept g^j formally as a generalized function and use (10), we obtain the formal equivalent to (58):

$$-\rho \int_0^t K_{11}(t, s) g_1(s)\, ds + \int_0^t K_{12}(t, s) g_2(s)\, ds \in [H_2(\phi_n^1, n = 1, 2, \ldots)]^\perp, \tag{60a}$$

$$\int_0^t K_{21}(t, s) g_1(s)\, ds - \rho \int_0^t K_{22}(t, s) g_2(s)\, ds \in [H_2(\phi_n^2, n = 1, 2, \ldots)]^\perp, \tag{60b}$$

where $H_2(\phi_n^1, n = 1, 2, \ldots)$ is the closed subspace of H_2 spanned by $\phi_1^1(t)\phi_2^1(t), \ldots$ and $\perp$ denotes its orthogonal complement in H_2. (60) is postulated in Ref. 8 as the integral equation one has to solve in order to find the canonical correlation of two Gaussian processes [it is tacitly assumed in Ref. 8 that the $\{\phi_n^i(t)\}$ span H_2 so that the quantities in (60) are set equal to zero].

To translate (58) into a reproducing kernel Hilbert space (RKHS) setup, we proceed as follows. Let

$$x_n^i = \bar{x}_n^i / \sqrt{\lambda_n^i}.$$

We can write (59) equivalently as

$$\rho(\bar{x}_n^i / \lambda_n^i) = \sum_m \left[\int_0^t \int_0^t K_{ij}(\tau, s)(\phi_n^i(\tau)/\lambda_n^i)(\phi_m^j(s)/\lambda_m^j)\, d\tau\, ds \right] \bar{x}_m^j,$$
$$i \neq j, \qquad i, j = 1, 2. \tag{61}$$

Let $H(K_{ii})$ be the RKHS corresponding to $x^i_\tau, 0 \leq \tau \leq t$ (see Ref. 2). Let

$$h_i(t) = \sum_{n=1}^{\infty} \bar{x}^i_n \phi^i_n \cdot h_i \in H(K_{ii})$$

and

$$\langle \phi^i_n, h_i \rangle_i = \bar{x}^i_n / \lambda^i_n; \tag{62}$$

here, $\langle \ , \ \rangle_i$ denotes the inner product in $H(K_{ii})$. The right-hand side of (61) can be written as

$$\langle \theta^n_j, h_j \rangle_j, \tag{63}$$

where

$$\theta^n_j = \sum_m a^j_{nm} \phi^j_m(\tau), \qquad j = 1, 2, \tag{64a}$$

$$a_{nm} = \int_0^t \int_0^t K_{ij}(\tau, s)[\phi^i(\tau)/\lambda^i_n] \phi^j_m(s) \, d\tau \, ds, \ i \neq j. \tag{64b}$$

For (63) to hold, we have to prove that

$$\theta^n_j \in H(K_{jj}),$$

which is equivalent to

$$\sum_m [(a^j_{nm})^2 / \lambda^j_m] < +\infty, \tag{65}$$

or

$$\sum_m \left[\int_0^t \int_0^t K_{ij}(\tau, s) \phi^i_n(\tau) \phi^j_m(s) \, d\tau \, ds / \lambda^i_n \right]^2 (1/\lambda^j_m) < +\infty,$$

or

$$\sum_m \left[\int_0^t \left[\int_0^t K_{ij}(\tau, s) \phi^i_n(\tau) \, d\tau \cdot (1/\lambda^i_n) \right] \phi^j_m(s) \, ds \right]^2 (1/\lambda^j_m) < +\infty,$$

or equivalently that

$$(1/\lambda^i_n) \int_0^t K_{ij}(\tau, s) \phi^i_n(\tau) \, d\tau \in H(K_{jj})$$

(see Ref. 10). Since

$$(1/\lambda^i_n) \int_0^t K_{ij}(\tau, s) \phi^i_n(\tau) \, d\tau = E\left[\int_0^t x^i_\tau \phi^i_n(\tau) \, d\tau x^j_s \right] (1/\lambda^i_n)$$

$$= E[x^j_s \cdot \eta^i_n] \cdot (1/\sqrt{\lambda^i_n}) = (1/\sqrt{\lambda^i_n}) E\left[\sum_m \sqrt{\lambda^j_m} \phi^j_m(s) \eta^j_m \cdot \eta^i_n \right]$$

$$= (1/\sqrt{\lambda^i_\eta}) \sum_m \sqrt{\lambda^j_m} \langle \eta^j_m, \eta^i_n \rangle \phi^j_m(s),$$

we have to show equivalently that

$$\sum_m (1/\lambda_m^j) \cdot (\sqrt{\lambda_m^j} \langle \eta_m^j, \eta_n^i \rangle)^2 < +\infty,$$

or

$$\sum_m \langle \eta_m^j, \eta_n^i \rangle^2 = (E[\eta_n^i \mid \eta_1^j, \eta_2^j, \ldots])^2 < +\infty,$$

which holds. We conclude that the problem of finding the ρ's is equivalent to solving for ρ, h_1, h_2 $[h_i \in H(K_{ii})]$ the equations

$$\rho\langle \phi_n^i, h_i \rangle_i = \langle \theta_j^n, h_j \rangle_j, \qquad i \neq j, \quad i, j = 1, 2, \qquad n = 1, 2, \ldots. \tag{66}$$

Since $\{\phi_n^i\}$ spans $H(K_{ii})$, we can also write

$$\rho h_1 = \sum_n \langle \theta_2^n, h_2 \rangle 1 \phi_n^1, \tag{67a}$$

$$\rho h_2 = \sum_m \langle \theta_1^n, h_1 \rangle 1 \phi_m^2. \tag{67b}$$

□

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