$$\det\left(\frac{\partial(\Phi_1,\Phi_2,\cdots,\Phi_p)}{\partial(\theta_1,\theta_2,\cdots,\theta_p)}\right).$$

so that we can apply the implicit function theorem to (2.3). Let $\alpha \triangleq (\theta_1,\theta_2,\cdots,\theta_p)$ and $\beta \triangleq (\beta_1,\beta_2,\cdots,\beta_{\eta-p})=(\theta_{p+1},\theta_{p+2},\cdots,\theta_\eta)$, so that $\theta=(\alpha,\beta)$ and $\mathring{\theta}=(\mathring{\alpha},\beta)$. Hence there are open neighborhoods Ω_α and Ω_β of $\mathring{\alpha}$ and β, respectively, and a unique continuously differentiable function $f:\Omega_\beta\to\Omega_\alpha$ such that $f(\beta)=\mathring{\alpha}$ and

$$\Phi_i(f(\beta),\beta)=\mathring{\Sigma}_i, \qquad i=1,2,\cdots,p, \tag{2.4}$$

for all $\beta\in\Omega_\beta$. Let us define a new function $\Psi:\Omega_\beta\to\Omega_\Sigma$

$$\Psi_i(\beta)=\Phi_i(f(\beta),\beta); \qquad i=1,2,\cdots,s.$$

Equation (2.4) implies that $\Psi_1,\Psi_2,\cdots,\Psi_p$ are constants. We will show that $\Psi_{p+1},\Psi_{p+2},\cdots,\Psi_s$ are also constant. We have

$$0=\frac{\partial\Psi_i}{\partial\beta_j}=\Phi_{i,j+p}(f(\beta),\beta)+\sum_{k=1}^{p}\Phi_{i,k}(f(\beta),\beta)\frac{\partial f_k}{\partial\beta_j},$$

$$1\le i\le p,\ 1\le j\le\eta-p$$

$$\frac{\partial\Psi_i}{\partial\beta_j}=\Phi_{i,j+p}(f(\beta),\beta)+\sum_{k=1}^{p}\Phi_{i,k}(f(\beta),\beta)\frac{\partial f_k}{\partial\beta_j},$$

$$p+1\le i\le s,\ 1\le j\le\eta-p \tag{2.5}$$

where $\Phi_{i,j}$ denotes the derivative of Φ_i with respect to its jth argument, in other words it is the (i,j)th element of the Jacobian $J(\cdot)$. Let us define the neighborhood $\Omega_1=\Omega\cap(\Omega_\alpha\times\mathbb{R}^{\eta-p})\cap(\mathbb{R}^p\times\Omega_\beta)$ of θ. Then, by the regularity condition in Ω_1 rank $\Phi=p$ throught Ω_1. Hence, each row $\{\Phi_{i,j};\ 1\le j\le\eta\}$, $p+1\le i\le s$, is linearly dependent on the rows $\{\Phi_{i,j};\ 1\le j\le\eta\}$, $1\le i\le p$ throughout Ω_1. Then, by (2.5), each row $\{\partial\Psi_i/\partial\beta;\ 1\le j\le\eta-p\}$, $p+1\le i\le s$ is linearly dependent on the rows $\{\partial\Psi_i/\partial\beta;\ 1\le j\le\eta-p\}$, $1\le i\le p$, which are identically zero by construction. It follows that $\partial\Psi_i/\partial\beta_j=0$, $l\le j\le\eta-p$, $1\le i\le s$, so that $\Psi_i(\beta)$ is constant for $1\le i\le s$. As a result Φ is constant on the manifold $\{\alpha,\beta|\alpha=f(\beta)\}\cap\Omega_1$. Therefore, Φ cannot be one-to-one for any neighborhood of the point θ, and θ is not locally identifiable. ∎

Lemma 1 and Theorems 1 and 2 combine to form the following theorem.

Theorem 3: Let $\mathring{\theta}\in\mathfrak{R}$ and let $\Phi(\cdot):\Omega\to\Omega_\Sigma$ be continuously differentiable. Then $H^n(\mathring{\theta})\to M$ a.s. as $n\to\infty$ and $\mathring{\theta}$ is locally identifiable if and only if, $M>0$. ∎

We remark again that if Φ has rank η at $\mathring{\theta}$ then $\mathring{\theta}\in\mathfrak{R}$.

We have the following specialization to the case where Φ is analytic.

Theorem 4: Assume that Φ is analytic throughout an open connected set $\Omega\subset\mathbb{R}^\eta$ and that rank Φ is not identically less than η for $\theta\in\Omega$, i.e., $\Omega\cap\mathfrak{D}\neq\varnothing$, where $\mathfrak{D}=\{\theta|\text{rank}\,\Phi=\eta\}$. Then in Ω we have $\mathfrak{D}=\mathfrak{R}$ and $\mathfrak{R}^C$ has Lebesgue measure zero.

Proof: Let $\mathring{\theta}\in\mathfrak{R}\cap\mathfrak{D}^C$. Thus, the analytic function formed as the determinant of any $\eta\times\eta$ submatrix of $J(\theta)$ is zero in a neighborhood of $\mathring{\theta}$, and, by analyticity, must be zero throughout Ω. This implies that $\Omega\subset\mathfrak{D}^C$, contrary to assumption; therefore, $\mathfrak{R}\cap\mathfrak{D}^C$ must be empty and $\mathfrak{R}\subseteq\mathfrak{D}$. Now assume $\mathring{\theta}\in\mathfrak{D}$ so that there is an $\eta\times\eta$ submatrix $M(\theta)$ of $J(\theta)$ with rank η. By continuity $M(\theta)$ has rank η for all θ in a neighborhood $N_{\mathring{\theta}}$; hence $\mathring{\theta}\in\mathfrak{R}$ and so $\mathfrak{D}\subset\mathfrak{R}$. It follows that $\mathfrak{R}=\mathfrak{D}$. Finally, the set $\mathfrak{D}^C$ has Lebesgue measure zero since it consists of the zeroes of a set of analytic functions that are not identically zero. ∎

Consider a zero-mean p-component second-order stationary stochastic process y with rational spectral density matrix $\Phi_\theta(z)$ parameterized by $\theta\in\Omega\subset\mathbb{R}^\eta$. Let $\Sigma_y(\theta)$ and $\Sigma_\nu(\theta)$ denote, respectively, the $(\infty\times\infty)$ covariance matrix of the process y and the $(p\times p)$ covariance matrix of the orthogonal innovations process ν. Then, subject to reasonable technical conditions, it can be shown that the maps $\theta\to\Phi_\theta(z)$, $\theta\to\{$a linearly spanning set of entries of $\Sigma_y(\theta)\}$, $\theta\to\Sigma_\nu(\theta)$ are all continuously differentiable, and the first two are one-to-one if the third is. This is true since the third map factors through each of the first two. Hence, local second-order identifiability of $\Sigma_y(\mathring{\theta})$ at $\mathring{\theta}$ holds if it holds for $\Sigma_\nu(\mathring{\theta})$. By Theorem 3 $\mathring{\theta}$ is locally identifiable if and only if the a.s. limit of $\{H^n(\Sigma_\nu(\theta)),\ n>1\}$ is positive definite. In the Gaussian case $H^n(\Sigma_\nu(\theta))$ is both the (approximate) scaled log-likelihood function on $\{\nu_i;\ i>1\}$ and on $\{y_i;\ i>1\}$ (see [11]) and is frequently used as a convenient indirect method to compute the likelihood on the process y.

The ideas in the paragraph above will be spelled out in detail in a future paper.

We conclude by remarking that since the results in this paper are of a local nature in $\mathbb{R}^\eta$ we may rephrase them all as results on an η-dimensional once continuously differentiable (denoted C^1) manifold. This is significant because the space of canonical forms of all finite-dimensional linear systems of fixed internal state dimension is an analytical (and hence C^1) manifold [12].

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On the Existence of Solutions to Coupled Matrix Riccati Differential Equations in Linear Quadratic Nash Games

G. P. PAPAVASSILOPOULOS AND J. B. CRUZ, JR.

***Abstract*—Sufficient conditions for the existence of closed-loop linear Nash strategies for a linear quadratic game are derived through use of differential inequalities.**

I. Introduction

Differential games have attracted considerable attention as a method for studying large scale, decomposed, or hierarchical systems and, of course, systems where many controllers with nonidentical aims operate. The concept of Nash equilibrium is well known; see [1]. There are many results concerning Nash games, but many questions related to existence and uniqueness of solutions, within certain strategy spaces, are still to be answered. In [2], sufficient conditions for the existence of closed-loop, linear in the state, equilibrium Nash strategies are given, for a linear plant with quadratic cost functionals over $[0,+\infty)$. Conditions for

Manuscript received June 12, 1978. This work was supported in part by the National Science Foundation under Grant ENG-74-20091, by the Department of Energy, Electric Energy Systems Division under Contract U. S. ERDA EX-76-01-2088, and in part by the Joint Services Electronics Program under Contract DAAG-29-78-C-0016.

The authors are with the Decision and Control Laboratory, Coordinated Science Laboratory, Department of Electrical Engineering, University of Illinois, Urbana, IL 61801.

existence of solutions to a pair of coupled algebraic Riccati equations were derived via Brower's fixed-point theorem. The stability of the resulting closed-loop matrix was also studied.

In the present paper we consider a linear quadratic Nash game over a finite period of time $[0,T]$. The matrices involved are piecewise-continuous functions of time. The existence of linear closed-loop Nash strategies depends on the existence of continuous solutions to an associated system of two coupled Riccati differential equations over $[0,T]$. Sufficient conditions for existence are derived in the next section by using a simple result from the theory of differential inequalities.

II. Statement of the Problem

Let us consider the dynamic system

$$\dot{x}=Ax+B_1u_1+B_2u_2,\quad x(0)=x_0,\qquad t\in[0,T] \tag{1}$$

the two cost functionals

$$J_i(u_1,u_2)=\frac{1}{2}\left\{x(T)'K_ix(T)+\int_0^T(x'Q_ix+u_i'R_{ii}u_i+u_j'R_{ij}u_j)\,dt\right\},\qquad i,j=1,2,\ i\neq j \tag{2}$$

and the associated Nash game, see [1]. The state x and the strategies u_1, u_2 take values in R^n, R^{m_1}, and R^{m_2}, respectively. The matrices A, B_i, Q_i, R_{ij}, are real valued piecewise-continuous functions of time and of appropriate dimensions. We also assume $K_i=K_i'>0$ constant real matrices, $Q_i(t)=Q_i'(t)>0$, $R_{ij}(t)=R_{ij}'(t)>0$, $R_{ii}(t)>0$, $\forall t\in[0,T]$, where the time interval $[0,T]$ is assumed fixed.

We restrict the admissible strategies to those which are linear in x, i.e.,

$$u_i(t)=L_i(t)x(t),\qquad i=1,2.$$

It can be shown [1] that if such an equilibrium Nash pair of strategies exist, it will be given by

$$u_i=-R_{ii}^{-1}B_i'P_ix,\qquad i=1,2, \tag{3}$$

where P_1, P_2 satisfy a system of two coupled differential Riccati equations. This system can be written as

$$-\dot{P}=F'P+PF+Q-PSP-PJSPJ-JPSJP+JPJS_oJPJ \tag{4}$$
$$P(T)=K_o,\qquad t\in[0,T]$$

where

$$F=\begin{bmatrix}A & 0\\ 0 & A\end{bmatrix}$$
$$S=\begin{bmatrix}B_1R_{11}^{-1}B_1' & 0\\ 0 & B_2R_{22}^{-1}B_2'\end{bmatrix}$$
$$S_0=\begin{bmatrix}B_2R_{22}^{-1}R_{12}R_{22}^{-1}B_2' & 0\\ 0 & B_1R_{11}^{-1}R_{21}R_{11}^{-1}B_1'\end{bmatrix} \tag{5}$$
$$Q=\begin{bmatrix}Q_1 & 0\\ 0 & Q_2\end{bmatrix}$$
$$J=\begin{bmatrix}0 & I\\ I & 0\end{bmatrix},\qquad I=n\times n\ \text{unit matrix}$$
$$P=\begin{bmatrix}P_1 & 0\\ 0 & P_2\end{bmatrix}.$$

The purpose of the present paper is to give sufficient conditions under which (4) has a solution over $[0,T]$.

III. Derivation of the Sufficiency Conditions

By setting

$$\bar{P}(t)=P(T-t) \tag{6}$$

we can consider equivalently to (4):

$$\dot{\bar{P}}=\bar{F}'\bar{P}+\bar{P}\bar{F}+\bar{Q}-\bar{P}\bar{S}\bar{P}-\bar{P}J\bar{S}\bar{P}J-J\bar{P}\bar{S}J\bar{P}+J\bar{P}J\bar{S}_0J\bar{P}S \tag{7}$$
$$\bar{P}(0)=K_0,\qquad t\in[0,T]$$

where

$$\bar{X}(t)=X(T-t),\qquad t\in[0,T].$$

If $\bar{P}(t)$ is a solution of (7) on $[0,t')$ where $t'<T$, then

$$\|\dot{\bar{P}}(t)\|<\beta\|\bar{P}(t)\|^2+\alpha\|\bar{P}(t)\|+q \tag{8}$$

where $\|\cdot\|$ denotes the usual sup norm of a square matrix calculated for fixed t, and

$$\alpha=\max\{2\|A(t)\|;\quad 0<t<T\}$$
$$\beta=\max\{3\|S(t)\|+\|S_0(t)\|;\quad 0<t<T\} \tag{9}$$
$$q=\max\{\|Q(t)\|;\quad 0<t<T\}.$$

The α, β, q are finite due to the piecewise continuity of the matrices and the finiteness of $[0,T]$. Clearly $\alpha,\beta,q>0$. We assume $\beta\neq 0$, since if $\beta=0$ then (4) is a linear differential equation and the solution exists for T arbitrarily large.

Consider the scalar differential equation

$$\dot{y}=\beta y^2+\alpha y+q,\qquad y(0)=y_0,\quad t>0. \tag{10}$$

Using [3, p. 32, Corollary 6.3][1], we obtain that

> if $y(0)>\|\bar{P}(0)\|$ and $y(t)$ is a solution of (10) on $[0,T]$ then the solution of (7) exists on $[0,T]$ and $\|P(t)\|<y(t)$, $t\in[0,T]$.

We thus conclude that a sufficient condition for the existence of a continuous solution $P(t)$ of (4) over $[0,T]$, is that

$$y(0)>\|K_0\|\quad\text{and}\quad T<t_f \tag{11}$$

where $[0,t_f)$ is the maximal interval of existence of the continuous solution of (10).

A straightforward investigation of the behavior of the solution of (10) yields the conditions under which (11) is satisfied. We state the results of this investigation in the form of a proposition.

Proposition: Let $\beta\neq 0$ and set

$$\Delta=\alpha^2-4\beta q,\quad \rho_1=\frac{-\alpha+\sqrt{\Delta}}{2\beta},\quad \rho_2=\frac{-\alpha-\sqrt{\Delta}}{2\beta}.$$

1) If $\Delta=0$, and

$$T<\frac{2}{\alpha+2\beta\|K_0\|} \tag{12-1}$$

then the solution of (4) exists and

$$\|P(t)\|<\rho_1+\frac{1}{C-\beta(T-t)},\quad C=\frac{2\beta}{\alpha+2\beta\|K_0\|},\qquad t\in[0,T]. \tag{12-2}$$

2) If $\Delta>0$, $\rho_2<\rho_1<0$, and

$$T<\frac{1}{\sqrt{\Delta}}\ln\left(\frac{\|K_0\|-\rho_2}{\|K_0\|-\rho_1}\right) \tag{13-1}$$

then the solution of (4) exists and

$$\|P(t)\|<\frac{\rho_1-\rho_2Ce^{\beta(\rho_1-\rho_2)(T-t)}}{1-Ce^{\beta(\rho_1-\rho_2)(T-t)}}$$
$$C=\frac{\|K_0\|-\rho_1}{\|K_0\|-\rho_2},\qquad t\in[0,T]. \tag{13-2}$$

[1] Although Corollary 6.3, page 32 of [3], is stated for the vector case, its extension to the matrix case is trivial.

3) If $\Delta<0$, $\rho_1=k+i\lambda$, $\rho_2=k-i\lambda$, $k=-\alpha/2\beta$, $\lambda=\sqrt{|\Delta|}/2\beta$, $i=\sqrt{-1}$ and

$$T<\frac{1}{\sqrt{|\Delta|}}\left[\pi-2\tan^{-1}\left(\frac{\|K_0\|-k}{\lambda}\right)\right] \tag{14-1}$$

then the solution of (4) exists and

$$\|P(t)\|\leq k+\lambda\tan(\lambda\beta(T-t)+C)$$

$$C=\tan^{-1}\left(\frac{\|K_0\|-k}{\lambda}\right), \qquad t\in[0,T] \tag{14-2}$$

where $\tan^{-1}$ is the inverse tan on $(-\pi/2,\pi/2)$.

If $A\equiv0$, $Q_i\equiv0$, $K_i=0$, and B_i, R_{ij} are constant, then case 1) holds and T can be taken arbitrarily large.

III. Conclusions

In the present paper we derived sufficient conditions for the existence of continuous solutions of the coupled Riccati differential equations arising in closed-loop Nash games. The conditions are given in terms of upper bounds on the length of the time interval of interest and do not depend on controllability or observability assumptions. Note also that the positive (semi-) definiteness assumptions on Q_i, R_{ij} were not used in proving the existence of solutions of (4). The basic tool in deriving these conditions was a simple differential inequality-type result. Although the conditions give only a partial answer to the question of existence of solutions, they can nonetheless provide a positive answer for a certain class of problems. The extension of the present results to the N-players case is straightforward.

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On the Internal Stability of Two-Dimensional Filters

E. FORNASINI AND G. MARCHESINI

***Abstract*—The internal stability concept for state-space representations of two-dimensional filters is introduced and an algebraic stability criterion is presented. The connections among internal stability, input–output stability, and coprimeness of the realization are also clarified.**

I. Introduction

The stability problem for two-dimensional filters in input–output form has been investigated by several authors [1]–[6]. The aim of this correspondence is to provide a first insight into the "internal" stability problem which arises when we consider state-space realizations of the filters.

State-space models of two-dimensional discrete-time filters constitute a recent field of investigation [7]–[14]. It has been shown [14] that all state-space models so far considered can be viewed as special cases of the following "doubly indexed dynamical system" $\Sigma=(A_1,A_2,B_1,B_2,C)$:

Manuscript received January 3, 1978. This work was supported by the Consiglio Nazionale delle Ricerche: Gruppo Nazionale Automatica e Sistemistica.

The authors are with the Department of Electrical Engineering, University of Padova, Padova, Italy.

$$x(h+1,k+1)=A_1x(h+1,k)+A_2x(h,k+1)$$
$$+B_1u(h+1,k)+B_2u(h,k+1) \tag{1}$$
$$y(h,k)=Cx(h,k)$$

where $u(h,k)$, the input value at (h,k), and $y(h,k)$, the output value at (h,k), are in $\mathbb{R}$, and h, $k\in Z$, $A_i\in\mathbb{R}^{n\times n}$, $B_i\in\mathbb{R}^{n\times 1}$, $C\in\mathbb{R}^{1\times n}$, $i=1,2$, and $x\in X=\mathbb{R}^n$ (local state space).

A first attempt to investigate the internal stability was made by Attasi [8] in the special case of separable filters. The class of filters we deal with is the whole class of filters having rational transfer function, so we shall consider the stability of dynamical systems represented by (1).

II. Stability Criterion

Introduce the following notation:

$$\mathfrak{X}_r=\{x(h,k):x(h,k)\in X,\ h+k=r\}.$$

Let $\|x\|$ denote the Euclidean norm of x in X and let

$$\|\mathfrak{X}_r\|=\sup_{n\in Z}\|x(r-n,n)\|.$$

We therefore have the following definition.

Definition: Let Σ be described by (1). The system Σ is asymptotically stable if, assuming $u=0$ and $\|\mathfrak{X}_0\|$ finite, $\|\mathfrak{X}_i\|\to0$ as $i\to+\infty$.

As is well known, the asymptotic stability analysis of discrete-time linear systems is reduced to investigate the zero's position of the characteristic polynomial of the matrix A.

The asymptotic stability of Σ is related to the algebraic curve defined in $\mathbf{C}\times\mathbf{C}$ by the equation

$$\det(I-z_1A_1-z_2A_2)=0$$

as stated in the following proposition.

Proposition 1: Let Σ be as in (1). Then Σ is asymptotically stable if and only if the polynomial $\det(I-A_1z_1-A_2z_2)$ is devoid of zeros in the closed polydisk:

$$\mathbf{P}_1=\{(z_1,z_2)\in\mathbf{C}\times\mathbf{C}:|z_1|\leq1,\ |z_2|\leq1\}$$

sufficiency. Let $\det(I-z_1A_1-z_2A_2)\neq0$ in $\mathcal{P}_1$ and call V the algebraic curve defined by $\det(I-A_1z_1-A_2z_2)=0$. Since V and $\mathcal{P}_1$ are closed, $V\cap\mathcal{P}_1=\varnothing$ implies that there exists $\epsilon>0$ such that the polydisk

$$\mathcal{P}_{1+\epsilon}=\{(z_1,z_2)\in\mathbf{C}\times\mathbf{C}:|z_1|\leq1+\epsilon,\ |z_2|\leq1+\epsilon\}$$

does not intersect V.

Then the rational matrix $(I-A_1z_1-A_2z_2)$ can be inverted in $\mathcal{P}_{1+\epsilon}$ and its McLaurin series expansion, given by

$$(I-A_1z_1-A_2z_2)^{-1}=\Sigma_{ij}M_{ij}z_1^iz_2^j$$

converges normally in the interior of $\mathcal{P}_{1+\epsilon}$ [15].

It follows that the series $\Sigma_{ij}\|M_{ij}\|$ converges. Consequently, $\Sigma_{i+j=r}\|M_{ij}\|\to0$ as $r\to\infty$, [16]. This implies the asymptotic stability of Σ. For, assume $\|\mathfrak{X}_0\|$ finite and pick in $\mathfrak{X}_r$, $r>0$ any local state $x(m,r-m)$, then

$$\|x(m,r-m)\|=\left\|\sum_{i+j=r}M_{ij}x(m-i,r-m-j)\right\|$$
$$\leq\sum_{i+j=r}\|M_{ij}\|\,\|x(m-i,r-m-j)\|\leq\|\mathfrak{X}_0\|\sum_{i+j=r}\|M_{ij}\|$$

necessity. Assume Σ be asymptotically stable. Then for any $x\in X$, $M_{ij}x\to0$ as $i+j\to\infty$. This fact and

$$\|M_{ij}\|\leq\sum_1^n{}_k\|M_{ij}e_k\|,$$

(with $\{e_k\}_1^n$ the standard basis in $X=\mathbb{R}^n$) imply