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Yu-Chi Ho (S'54–M'55–SM'62) was born in China on March 1, 1934. He received the B.S. and M.S. degrees in electrical engineering from the Massachusetts Institute of Technology, Cambridge, in 1953 and 1955, respectively, and the Ph.D. degree in applied mathematics from Harvard University, Cambridge, Mass., in 1961.

From 1955 to 1958 he worked on numerical control systems at the Research Laboratory Division, Bendix Corporation, Mich. Since 1961 he has been on the faculty of Harvard University, where he is presently Gordon McKay Professor of Engineering and Applied Mathematics. He is a consultant to various industrial and research organizations and coinventor of four U.S. patents on various aspects of numerical and digital control systems. He is coauthor of the book *Applied Optimal Control: Optimization, Estimation and Control* (Blaisdell, 1969).

Dr. Ho is a member of the Army Scientific Advisory Panel, a member of the Editorial Board of the IEEE Press, and Associate Editor of the *Journal of Optimization Theory and Applications*. In 1969 he was the Chairman of the First International Conference on the Theory and Application of Differential Games. In 1970 he was a Guggenheim Fellow at Imperial College, London, England, and Cambridge University, Cambridge, England.

K'ai-Ching Chu (S'69–M'71) was born in Szechuan, China, on November 19, 1944. He received the B.S. degree in electrical engineering from the National Taiwan University, Taipei, Taiwan, China, in 1966, and the M.S. and Ph.D. degrees in applied mathematics from Harvard University, Cambridge, Mass., in 1968 and 1971, respectively.

From 1968 to 1971 he was a Research Assistant and Teaching Fellow at Harvard University. He has been a consultant to Bolt, Beranek, and Newman, Inc., Cambridge, Mass. Since September 1971, he has been a Mathematician with Systems Control, Inc., Palo Alto, Calif. His current research interests involve the theory and application of optimal control, statistical decision analysis, and game theory.

Team Decision Theory and Information Structures in Optimal Control Problems—Part II

K'AI-CHING CHU, MEMBER, IEEE

Abstract—General dynamic team decision problems with linear information structures and quadratic payoff functions are studied. The primitive random variables are jointly Gaussian. No constraints on the information structures are imposed except causality.

Equivalence relations in information and in control functions among different systems are developed. These equivalence relations aid in the solving of many general problems by relating their solutions to those of the systems with "perfect memory." The latter can be obtained by the method derived in Part I. A condition is found which enables each decision maker to infer the information available to his precedents, while at the same time the controls which will affect the information assessed can be proven optimal. When this condition fails, upper and lower bounds of the payoff function can still be obtained systematically, and suboptimal controls can be obtained.

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The author was with the Division of Engineering and Applied Physics, Harvard University, Cambridge, Mass. He is now with Systems Control, Inc., Palo Alto, Calif. 94306.

I. Introduction

In Part I of this paper, Ho and Chu [1] have discussed the information structures in a general organization and their relation to team decision problems. It is found that in a general causal system a partially ordered precedence relation $\{$ can be defined among all the members. This precedence relation then specifies the nature of the solution.

A linear-quadratic-Gaussian (LQG) team problem (Q, S, c, H_i, $D_{ij} | i, j = 1, \cdots, N$) is an optimal decision problem with payoff function

$$J = E[g] = E[\tfrac{1}{2}u^TQu + u^TS\xi + u^Tc] \qquad (1)$$

where $u^T = (u_1^T, \cdots, u_N^T)$ and u_i is the action variable of team member i; matrices Q, S and vector c are fixed and of appropriate dimensions, Q is symmetric positive definite; the random variable of the external world ξ is *a priori* Gaussian with distribution $N(0, X)$. The information z_i

received by the ith member is a linear function η_i of ξ and u,

$$z_i = \eta_i(\xi, u)$$
$$= H_i\xi + \sum_{j \wr i} D_{ij}u_j. \tag{2}$$

Finally, the actions of the team members are to be determined as the control laws γ_i,

$$u_i = \gamma_i(z_i) \tag{3}$$

where $\gamma_i \in \Gamma_i$, the set of all k_i-dimensional Borel-measurable functions of z_i.

In Part I it has been shown that for both static and dynamic LQG teams with partially nested information structures, the optimal decision $\gamma_i{}^*$ is linear function of z_i. Part II of this paper will extend the study to general dynamic LQG teams in which the information structures are not necessarily partially nested. It has been shown [2] that in general the optimal action of each member may not be linear in the information. Generally, each z_i, and hence u_i, will depend on all the control laws of his precedents explicitly; and this dependence is itself affected by the nature of the past control laws which are part of the solution to be decided.

In the following sections dealing with nonnested information structures, we shall bypass this dependence difficulty by defining an auxiliary problem which will be more easily solvable. By considering some equivalent controls, we shall try to relate the solution of the original problem to that of the auxiliary problem. For any given team problem, a systematic approach is adopted for either solving it or giving natural upper and lower bounds to its optimal payoff function.

II. Equivalent Controls

We again index the members in such a way that if member i is a precedent of member j, then $i < j$.

In a given LQG team problem Ω, for the ith member the information function η_i is a mapping from the Cartesian production of Ξ and U_{i-1} to set Z_i such that

$$z_i = \eta_i(\xi, u_{i-1}) \tag{4}$$

where

$$z_i \in Z_i,\ \xi \in \Xi \text{ and } u_j \in U_j \text{ for } j = 1, 2, \cdots, i-1$$
$$u_{i-1} = (u_1, u_2, \cdots, u_{i-1})$$
$$U_{i-1} = U_1 \times U_2 \times \cdots \times U_{i-1}.$$

The control function γ_i is a mapping from set Z_i to set U_i such that

$$u_i = \gamma_i(z_i) \tag{5}$$

where $z_i \in Z_i$ and $u_i \in U_i$. Then the composite function $\gamma_i\eta_i$ gives the recursive relation for u_i, following the precedence diagram as defined in Part I, such that

$$u_i = \gamma_i[\eta_i(\xi, u_{i-1})] = \gamma_i\eta_i(\xi, u_{i-1}), \qquad i = 1, \cdots, N. \tag{6}$$

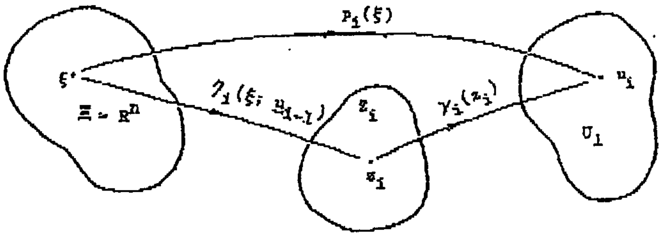

Fig. 1.

In our problem Ξ is the Euclidean space R^n, U_i is the Euclidean space R^{k_i}, Z_i is the Euclidean space R^{q_i}. The control γ_i is assumed to be chosen from Γ_i, which is the set of all Borel-measurable functions from Z_i to U_i. Since on Ξ a Gaussian probability distribution has been defined, it is measurable. The information function η_i is known to be linear; therefore, a measurable space U_{i-1} will imply a measurable z_i, and hence a measurable U_i.

For any given set of control laws $\gamma_1, \cdots, \gamma_N$ and causal information functions $\eta_1, \cdots, \eta_N$, we define a set of *composite control functions* p_i from Ξ to U_i such that

$$p_1(\xi) = \gamma_1[\eta_1(\xi)] \tag{7}$$

$$p_i(\xi) = \gamma_i[\eta_i(\xi, p_1(\xi), \cdots, p_{i-1}(\xi))], \quad i = 2, \cdots, N. \tag{8}$$

Naturally,

$$u_i = \gamma_i[\eta_i(z_i)] = p_i(\xi), \qquad i = 1, \cdots, N. \tag{9}$$

Schematically, p_i is a composite function which directly maps points of Ξ space into U_i space. (See Fig. 1).

Definition: In a team with two sets of information-control design, (1) $(\eta_1, \cdots, \eta_N; \gamma_1, \cdots, \gamma_N)$ and (2) $(\tilde{\eta}_1, \cdots, \tilde{\eta}_N; \tilde{\gamma}_1, \cdots, \tilde{\gamma}_N)$, the control laws $(\gamma_1, \cdots, \gamma_N)$ and $(\tilde{\gamma}_1, \cdots, \tilde{\gamma}_N)$ are said to be *equivalent* if $p_i = \tilde{p}_i$ for all i; where p_i and $\tilde{p}_i$ are the composite control functions of (1) and (2), respectively.

With equivalent control laws, the same value of ξ will imply the same value of u_i for all i, and hence we have Theorem 1.

Theorem 1: The Problems Ω and $\tilde{\Omega}$ with equivalent controls have the same payoff.

Proof:

$$u_i = p_i(\xi) = \tilde{p}_i(\xi) = \tilde{u}_i, \qquad i = 1, \cdots, N$$
$$J = E[\mathcal{J}(\xi, u_N)] = E[\mathcal{J}(\xi, p_N(\xi)] = E[\mathcal{J}(\xi, \tilde{p}_N(\xi)]$$
$$= E[\mathcal{J}(\xi, \tilde{u}_N)] = \tilde{J}. \qquad \text{Q.E.D.}$$

The relation of control equivalence is decided by information structure and control functions *independently* of the payoff function.

III. Construction of the Auxiliary Problem

To facilitate the solution of given team Problem Ω with information structure

$$z_i = \eta_i(\xi, u_1, \cdots, u_{i-1}) = H_i\xi + \sum_{j \wr i} D_{ij}u_j, \qquad \forall i, \tag{10}$$

we shall construct an auxiliary team Problem $\tilde{\Omega}$ with infor-

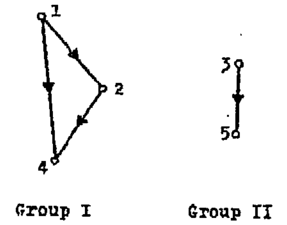

Fig. 2.

mation structure $\tilde\eta_1, \tilde\eta_2, \cdots, \tilde\eta_N$ such that the following conditions hold.

Condition 1: $\tilde\eta_i$ is *more informative* than η_i for all i in the sense that knowing $\tilde z_i$ implies knowing z_i for all i.

Condition 2: $\tilde\eta_1, \tilde\eta_2, \cdots, \tilde\eta_N$ is partially nested.

The information functions $\tilde\eta_i$ such that

$$\tilde z_i = \tilde\eta_i(\xi, u_1, \cdots, u_{i-1}) = (\{z_j | j \langle i \text{ or } j = i\}) \qquad (11)$$

satisfy both Conditions 1 and 2.

Example: Given is the Problem Ω with

$$z_1 = H_1\xi \qquad z_2 = H_2\xi + D_{21}u_1 \qquad z_3 = H_3\xi$$

$$z_4 = H_4\xi + D_{42}u_2 + D_{41}u_1 \qquad z_5 = H_5\xi + D_{53}u_3.$$

The precedence diagram of this problem is as displayed in Fig. 2. Member one is precedent to members two and four; member two is precedent to member four; member three is the precedent of member five. The auxiliary Problem $\tilde\Omega$, which is partially nested and more informative for each of the members, has

$$\tilde z_1 = z_1 = H_1\xi \qquad \tilde z_2 = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}\xi + \begin{bmatrix} 0 \\ D_{21} \end{bmatrix}u_1$$

$$\tilde z_4 = \begin{bmatrix} z_1 \\ z_2 \\ z_4 \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \\ H_4 \end{bmatrix}\xi + \begin{bmatrix} 0 \\ D_{21} \\ D_{41} \end{bmatrix}u_1 + \begin{bmatrix} 0 \\ 0 \\ D_{42} \end{bmatrix}u_2$$

$$\tilde z_3 = z_3 = H_3\xi \qquad \tilde z_5 = \begin{bmatrix} z_3 \\ z_5 \end{bmatrix} = \begin{bmatrix} H_3 \\ H_5 \end{bmatrix}\xi + \begin{bmatrix} 0 \\ D_{53} \end{bmatrix}u_3.$$

In the auxiliary problem, the precedent's, and only the precedent's, information is included as extra components of the old information for each of the members. Thus, this process of adding information results in a unique and more informative Problem $\tilde\Omega$.

Condition 1 guarantees that the optimal payoff value of Problem $\tilde\Omega$ is no worse than that of Problem Ω, or $\tilde J^* \leq J^*$, because any control law in Problem Ω is also available in Problem $\tilde\Omega$.[1] Condition 2 tells us that $\tilde\gamma_i^*$, the optimal control of Problem $\tilde\Omega$, are linear and are solvable by Theorem 2 of Part I of this paper. Thus, we can always get a lower bound for the payoff for any given problem by this procedure.

Suppose we can find controls $(\gamma_1, \gamma_2, \cdots, \gamma_N)$ for Problem Ω such that they are equivalent to the optimal linear

[1] In the more general many-person game problems, when the payoff function of each of the members may not be the same, a more informative information structure does not necessarily imply a better payoff for each of the members.

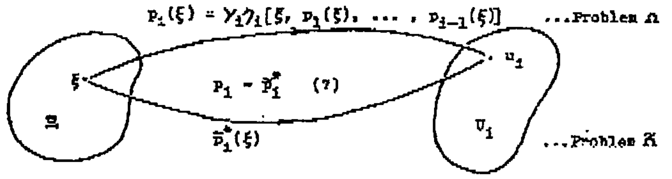

Fig. 3.

controls of Problem $\tilde\Omega$; in other words, for all i, the p_i function constructed is equal to the $\tilde p_i^*$ for Problem $\tilde\Omega$; the payoff function for both problems will be the same as asserted in Theorem 1. (See Fig. 3.) However, the payoff of Ω is bounded below by that of $\tilde\Omega$, hence $(\gamma_1, \gamma_2, \cdots, \gamma_N)$ chosen for Ω must be optimal for it.

Now, the main problem we are concerned with is: Under what conditions can we find $\gamma_1, \gamma_2, \cdots, \gamma_N$ for Ω such that the resulting p_i will equal the known $\tilde p_i^*$ of $\tilde\Omega$ for all i.

Theorem 2: Define $\tilde p_i^*$ as the composite control corresponding to the optimal control $\tilde\gamma_i^*$ of the auxiliary Problem $\tilde\Omega$ for each i; and define functions

$$g_i(\xi) = \eta_i(\xi, \tilde p_1^*(\xi), \tilde p_2^*(\xi), \cdots, \tilde p_{i-1}^*(\xi)), \qquad \forall\, i. \quad (12)$$

If there exists some functions r_i from R^{q_i} to R^{k_i} such that

$$\tilde p_i^* = r_i g_i, \qquad \forall\, i, \qquad (13)$$

then $(r_1, \cdots, r_N)$ is optimal for Problem Ω and they are equivalent to $(\tilde\gamma_1^*, \cdots, \tilde\gamma_N^*)$ of the auxiliary Problem $\tilde\Omega$.

Proof: Let $\gamma_i = r_i$ for all i.

$$p_1(\xi) = \gamma_1\eta_1(\xi) = r_1\eta_1(\xi) = \tilde p_1^*$$

$$\vdots$$

$$p_N(\xi) = \gamma_N\eta_N(\xi; p_1, \cdots, p_{N-1}) = r_N\eta_N(\xi; \tilde p_1^*, \cdots, \tilde p_{N-1}^*)$$
$$= r_N g_N(\xi) = \tilde p_N^*(\xi).$$

So, $(r_1, \cdots, r_N)$ is equivalent to $(\gamma_1^*, \cdots, \gamma_N^*)$; and J by $(r_1, \cdots, r_N)$ is equal to $\tilde J^*$ of the auxiliary Problem $\tilde\Omega$. But $\tilde\eta_i$ is more informative than η_i for all i, and $\tilde J^* \leq J^*$, $(r_1, \cdots, r_N)$ must be optimal for Problem Ω.

Corollary: If g_i is invertible for all i, the optimal control laws for Problem Ω can be found, and they are equivalent to those of Problem $\tilde\Omega$.

Proof: Suffice to let $\gamma_i = \tilde p_i^* g_i^{-1}$ for all i.

$$p_1(\xi) = \gamma_1\eta_1(\xi) = [\tilde p_1^* g_1^{-1}]g_1(\xi) = \tilde p_1^*(\xi)$$
$$p_2(\xi) = \gamma_2\eta_2(\xi, p_1(\xi)) = [\tilde p_2^* g_2^{-1}]g_2(\xi) = \tilde p_2^*(\xi)$$

$$\vdots$$

$$p_N(\xi) = \gamma_N\eta_N(\xi, p_1(\xi), p_2(\xi), \cdots, p_{N-1}(\xi))$$
$$= \gamma_N\eta_N(\xi, \tilde p_1^*(\xi), \tilde p_2^*(\xi), \cdots, \tilde p_{N-1}^*(\xi))$$
$$= [\tilde p_N^* g_N^{-1}]g_N(\xi) = \tilde p_N^*(\xi).$$

Composite control function $\tilde p_i^*$ partitions the space Ξ into equivalent classes such that for any ξ_1 and ξ_2 in Ξ, $\tilde p_i^*(\xi_1) = \tilde p_i^*(\xi_2)$ if and only if ξ_1 and ξ_2 belong to the same class.

Theorem 2 provides a condition under the assumed in-

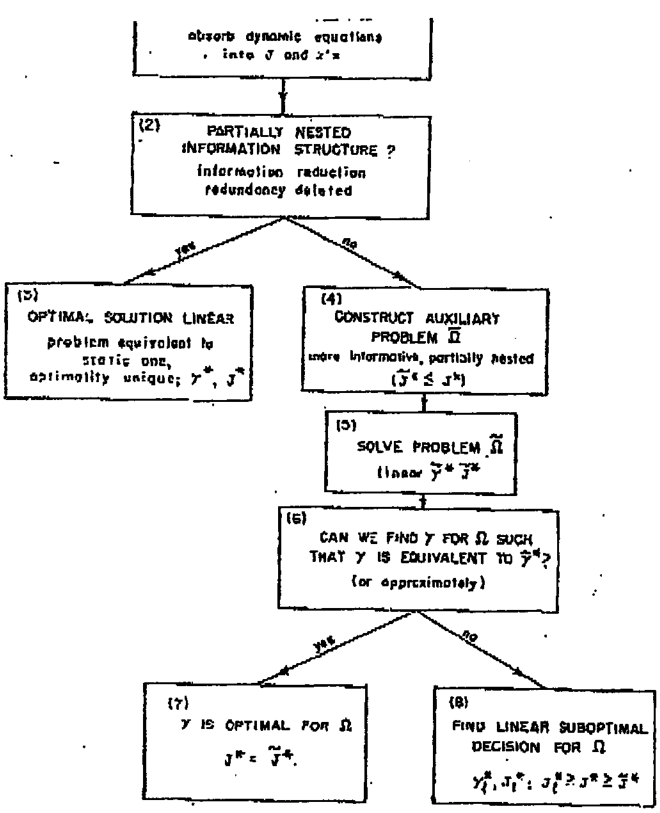

Fig. 4. General analysis flow chart.

formation structure when the members have enough knowledge to distinguish differing ξ's up to the classes they belong to. Thus, to each ξ they are able to assign specific $u_i = \bar{p}_i^*(\xi)$ necessary for a payoff $\tilde{J}^*$.

The corollary is more special than Theorem 2 in the sense that each member is able to distinguish each individual ξ-element in the space Ξ through function g_i^{-1}.

IV. Examples

With the help of Theorem 2 and its corollary, we can solve many team problems for their optimal controls and payoff functions. The general analysis flow chart is as displayed in Fig. 4.

Block 1: Normalize Problem Ω into the standard form (1) and (2). Any proper discrete linear dynamic process with payoff function quadratic in state and controls and with Gaussian *a prior* random variables fits our model.

Block 2: Draw the information precedence diagram of Problem Ω according to (2). After reducing (2) to that of row-reduced echelon matrices, check if it is partially nested.

Block 3: If the answer to Block 2 is *yes*, Problem Ω is static or is equivalent to a static one. The optimal control functions are linear and unique. (See Theorem 2 of Part I of this paper.)

Block 4: If the answer to Block 2 is *no*, construct auxiliary Problem $\tilde{\Omega}$ with partially nested and more informative information structure.

Block 5: Solve Problem $\tilde{\Omega}$, the optimal control $\tilde{\gamma}^*$ of which is linear.

... whether we can find γ_i for Problem Ω such that $(\gamma_1, \cdots, \gamma_N)$ is equivalent to $(\tilde{\gamma}_1^*, \cdots, \tilde{\gamma}_N^*)$. (See Theorem 2 of Part II.)

Block 7: If the answer to Block 6 is *yes*, $(\gamma_1, \cdots, \gamma_N)$ is optimal for Problem Ω and $J^* = \tilde{J}^*$.

Block 8: If the answer to Block 6 is *no*, $\tilde{J}^*$ of Problem $\tilde{\Omega}$ is a lower bound of J^* of Problem Ω. An upper bound of J^* can be obtained as J_l^*, the payoff of the linear suboptimal controls of Problem Ω. Generally in this case, γ^* is nonlinear and not solvable by the existing methods.

We give some examples of this systematic design approach.

Problem A:

$$N = 3 \qquad Q = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \qquad S = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad c = 0$$

$$\Pr(\xi) = N(0, X)$$

$$z_1 = \xi \qquad z_2 = u_1 \qquad z_3 = \xi + u_2. \tag{14}$$

From (14) we shall keep in mind the fact that member one is the precedent of member two; while both member one and member two are the precedents of member three. Member one knows the value of ξ; member two knows only what member one has done, not what member one has known; member three knows only $\xi + u_2$, but does not know exactly what the second member has done. This information structure is clearly not a partially nested one according to our definition. Following the approach outlines in Fig. 4, we have the following steps.

Step 1: Construct the auxiliary Problem $\tilde{A}$ for which the information structure is

$$\tilde{z}_1 = z_1 = \xi$$

$$\tilde{z}_2 = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \xi \\ u_1 \end{bmatrix}$$

$$\tilde{z}_3 = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} \xi \\ u_1 \\ \xi + u_2 \end{bmatrix} \tag{15}$$

or equivalently

$$\hat{z}_1 = \xi \qquad \hat{z}_2 = \xi \qquad \hat{z}_3 = \xi$$

after deleting the redundancy. (See Theorem 1 of Part I.) The optimal solution of Problem $\tilde{A}$ is

$$\begin{bmatrix} \tilde{u}_1^* \\ \tilde{u}_2^* \\ \tilde{u}_3^* \end{bmatrix} = -Q^{-1}S\xi = -\tfrac{1}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\xi = -\tfrac{1}{4}\begin{bmatrix} \hat{z}_1 \\ \hat{z}_2 \\ \hat{z}_3 \end{bmatrix}$$

or

$$\tilde{u}_1^* = -\tfrac{1}{4}\tilde{z}_1 \qquad \tilde{u}_2^* = -\tfrac{1}{4}[1 \quad 0]\tilde{z}_2$$
$$\tilde{u}_3^* = -\tfrac{1}{4}[1 \quad 0 \quad 0]\tilde{z}_3. \tag{16}$$

From (16) we have

$$\tilde{p}_1^*(\xi) = \tilde{p}_2^*(\xi) = \tilde{p}_3^*(\xi) = -\tfrac{1}{4}\xi \tag{17}$$

and

$$\bar{J}^* = E\left[\tfrac{1}{2}(\bar{u}_1^*\bar{u}_2^*\bar{u}_3^*)\begin{bmatrix}2&1&1\\1&2&1\\1&1&2\end{bmatrix}\begin{bmatrix}\bar{u}_1^*\\\bar{u}_2^*\\\bar{u}_3^*\end{bmatrix} + (\bar{u}_1^*\bar{u}_2^*\bar{u}_3^*)\begin{bmatrix}1\\1\\1\end{bmatrix}\xi\right]$$

$$= E\left[\tfrac{1}{2}\xi^2(-\tfrac{1}{4}\ -\tfrac{1}{4}\ -\tfrac{1}{4})\begin{bmatrix}2&1&1\\1&2&1\\1&1&2\end{bmatrix}\begin{bmatrix}-\tfrac{1}{4}\\-\tfrac{1}{4}\\-\tfrac{1}{4}\end{bmatrix} + \xi^2(-\tfrac{1}{4}\ -\tfrac{1}{4}\ -\tfrac{1}{4})\begin{bmatrix}1\\1\\1\end{bmatrix}\right]$$

$$= -\tfrac{3}{8}X. \tag{18}$$

Step 2: Substitute (17) into (14),

$$g_1(\xi) = z_1 = \xi$$
$$g_2(\xi) = z_2 = \bar{p}_1^*(\xi) = -\tfrac{1}{4}\xi$$
$$g_3(\xi) = z_3 = \xi + \bar{p}_2^*(\xi) = \tfrac{3}{4}\xi. \tag{19}$$

All the functions g_i in (19) are invertible; therefore, by the corollary the optimal controls for Problem A are

$$u_1^* = \gamma_1^*(z_1) = \bar{p}_1^* g_1^{-1}(z_1) = -\tfrac{1}{4}z_1$$
$$u_2^* = \gamma_2^*(z_2) = \bar{p}_2^* g_2^{-1}(z_2) = -\tfrac{1}{4}(-4)z_2 = z_2$$
$$u_3^* = \gamma_3^*(z_3) = \bar{p}_3^* g_3^{-1}(z_3) = -\tfrac{1}{4}(\tfrac{4}{3})z_3 = -\tfrac{1}{3}z_3 \tag{20}$$

and the optimal payoff function J^* corresponds to (20) is the same as $\bar{J}^*$, which is $-(3/8)X$. Problem A is a team problem with result located at Block 7 of Fig. 4.

In this example, since g_2 and g_3 are invertible, members two and three are able to infer from their own information the information available to their precedents; at the same time, the precedents' controls, which will affect the information transmitted to the followers, are shown to be optimal. However, optimality and the passing of information to the followers are not always compatible tasks. Theorem 2 and its corollary give the conditions under which this can be done.

Problem B:

$$N = 3 \qquad Q = \begin{bmatrix}2&1&1\\1&2&1\\1&1&2\end{bmatrix} \qquad S = \begin{bmatrix}\tfrac{2}{3}\\1\\1\end{bmatrix} \qquad c = 0$$

$$\Pr(\xi) = N(0, X)$$

$$z_1 = \xi \qquad z_2 = u_1 \qquad z_3 = \xi + u_2. \tag{21}$$

Note that except for matrix S, Problem B is exactly the same as Problem A.

Step 1: Construct the auxiliary Problem $\bar{B}$ for which the information structure is

$$\bar{z}_1 = z_1 = \xi$$

$$\bar{z}_2 = \begin{bmatrix}z_1\\z_2\end{bmatrix} = \begin{bmatrix}\xi\\u_1\end{bmatrix}$$

$$\bar{z}_3 = \begin{bmatrix}z_1\\z_2\\z_3\end{bmatrix} = \begin{bmatrix}\xi\\u_1\\\xi + u_2\end{bmatrix} \tag{22}$$

or, equivalently,

$$\hat{z}_1 = \xi \qquad \hat{z}_2 = \xi \qquad \hat{z}_3 = \xi$$

after deleting the redundancy. (Theorem 1 of Part I.) The optimal solution of Problem B is

$$\begin{bmatrix}\bar{u}_1^*\\\bar{u}_2^*\\\bar{u}_3^*\end{bmatrix} = -Q^{-1}S\xi = -\tfrac{1}{3}\begin{bmatrix}0\\1\\1\end{bmatrix}\xi = \begin{bmatrix}0\cdot\hat{z}_1\\-\tfrac{1}{3}\hat{z}_2\\-\tfrac{1}{3}\hat{z}_3\end{bmatrix}$$

or

$$\bar{u}_1^* = 0\cdot\bar{z}_1 \qquad \bar{u}_2^* = [-\tfrac{1}{3}\quad 0]\bar{z}_2$$
$$\bar{u}_3^* = [-\tfrac{1}{3}\quad 0\quad 0]\bar{z}_3. \tag{23}$$

From (23) we have

$$\bar{p}_1^*(\xi) = 0 \qquad \bar{p}_2^*(\xi) = -\tfrac{1}{3}\xi \qquad \bar{p}_3^*(\xi) = -\tfrac{1}{3}\xi$$

and

$$\bar{J}^* = E\left[\tfrac{1}{2}(\bar{u}_1^*\bar{u}_2^*\bar{u}_3^*)\begin{bmatrix}2&1&1\\1&2&1\\1&1&2\end{bmatrix}\begin{bmatrix}\bar{u}_1^*\\\bar{u}_2^*\\\bar{u}_3^*\end{bmatrix} + (\bar{u}_1^*\bar{u}_2^*\bar{u}_3^*)\begin{bmatrix}\tfrac{2}{3}\\1\\1\end{bmatrix}\xi\right]$$

$$= E\left[\tfrac{1}{2}\xi^2(-\tfrac{1}{3}\ -\tfrac{1}{3})\begin{bmatrix}2&1\\1&2\end{bmatrix}\begin{bmatrix}-\tfrac{1}{3}\\-\tfrac{1}{3}\end{bmatrix} + \xi^2(-\tfrac{1}{3}\ -\tfrac{1}{3})\begin{bmatrix}1\\1\end{bmatrix}\right]$$

$$= -\tfrac{1}{3}X. \tag{24}$$

Step 2: Substitute (23) into (21),

$$g_1(\xi) = z_1 = \xi$$
$$g_2(\xi) = z_2 = \bar{u}_1^* = 0$$
$$g_3(\xi) = z_3 = \xi + \bar{u}_2^* = \tfrac{2}{3}\xi. \tag{25}$$

Since function g_2 in (25) is not an invertible one, the second member cannot obtain the necessary information about ξ by just knowing the value of z_2 which is zero. Therefore, we cannot have a solution for the original problem such that the control functions are equivalent to those from (23). However, we can have solutions which are as close an approximation to (23) as possible. The process is to change the control laws (23) slightly, such that the resulting functions g_i in (25) are all invertible.

Let

$$\bar{u}_1^\circ = \bar{u}_1^* + \epsilon\bar{z}_1 = \epsilon\bar{z}_1$$
$$\bar{u}_2^\circ = \bar{u}_2^* = [-\tfrac{1}{3}\quad 0]\bar{z}_2$$
$$\bar{u}_3^\circ = \bar{u}_3^* = [-\tfrac{1}{3}\quad 0\quad 0]\bar{z}_3 \tag{23'}$$

where $0 < \epsilon \ll 1$. Then, $\bar{p}_1^\circ(\xi) = \epsilon\xi$, $\bar{p}_2^\circ(\xi) = -\tfrac{1}{3}\xi$, $\bar{p}_3^\circ(\xi) = -\tfrac{1}{3}\xi$, and

$$g_1^\circ(\xi) = z_1 = \xi$$
$$g_2^\circ(\xi) = z_2 = \bar{u}_1^\circ = \epsilon\xi$$
$$g_3^\circ(\xi) = z_3 = \xi + \bar{u}_2^\circ = \tfrac{2}{3}\xi \tag{25'}$$

g_i^o functions in (25′) are invertible now, for Problem B we then have

$$u_1^o = \bar{p}_1^o g_1^{o-1}(z_1) = \epsilon z_1$$

$$u_2^o = \bar{p}_2^o g_2^{o-1}(z_2) = -\frac{1}{3}\left(\frac{1}{\epsilon}\right) z_2 = -\frac{1}{3\epsilon} z_2$$

$$u_3^o = \bar{p}_3^o g_3^{o-1}(z_3) = -\frac{1}{3}\left(\frac{3}{2}\right) z_3 = -\frac{1}{2} z_3. \tag{26}$$

The payoff function of the controls (26) is

$$J^o = E\left[\tfrac{1}{2}(u_1^o u_2^o u_3^o)\begin{bmatrix}2 & 1 & 1\\ 1 & 2 & 1\\ 1 & 1 & 2\end{bmatrix}\begin{bmatrix}u_1^o\\ u_2^o\\ u_3^o\end{bmatrix} + (u_1^o u_2^o u_3^o)\begin{bmatrix}\frac{2}{3}\\ 1\\ 1\end{bmatrix}\xi\right]$$

$$= E\left[\tfrac{1}{2}\xi^2(\epsilon \ -\tfrac{1}{3} \ -\tfrac{1}{3})\begin{bmatrix}2 & 1 & 1\\ 1 & 2 & 1\\ 1 & 1 & 2\end{bmatrix}\begin{bmatrix}\epsilon\\ -\frac{1}{3}\\ -\frac{1}{3}\end{bmatrix} + \xi^2(\epsilon \ -\tfrac{1}{3} \ -\tfrac{1}{3})\begin{bmatrix}\frac{2}{3}\\ 1\\ 1\end{bmatrix}\right]$$

$$= [-\tfrac{1}{3} + \epsilon^2]X \to -\tfrac{1}{3}X \text{ as } \epsilon \to 0. \tag{27}$$

For Problem B, since $-(1/3)X$ is the lower bound of the payoff function proved in (24), it is the best that can be done. This J value of $-(1/3)X$ can be approached as closely as we wish, but is never attainable, if control law (26) is used with $\epsilon \to 0$ but $\neq 0$.

The invertibility of g_i, of dimension at least n, allows member i to have sufficient information to distinguish individual ξ in Ξ. The approximation technique to solve the problem from the auxiliary problem can also be extended to some multidimensional cases by analogy. If g_i is noninvertible with dimension n, we can make it nonsingular by changing slightly the precedent's control as in (23′).

Let us now consider some LQG team problems with the result not located in either Block 3 or Block 7 of Fig. 4; i.e., the result of this category of problems is located in Block 8 of Fig. 4. The general optimal solution of this case is not now known however, upper and lower bounds for J^* can be obtained in a rather systematic way. Witsenhausen [2] has mentioned a very interesting example which just fits into this category. We shall restate his example in our normalized standard form (1) and (2), which is the following.

Problem C:

$$N = 2 \qquad Q = \begin{bmatrix}1 + \frac{1}{X} & -1\\ -1 & 1\end{bmatrix} \qquad S = \begin{bmatrix}-\frac{1}{X} & 0\\ 0 & 0\end{bmatrix}$$

$$\Pr\begin{bmatrix}x\\ v\end{bmatrix} = N\left[\begin{bmatrix}0\\ 0\end{bmatrix}, \begin{bmatrix}X & 0\\ 0 & 1\end{bmatrix}\right]$$

where $X > 0$ and

$$\begin{bmatrix}x\\ v\end{bmatrix} = \xi, \qquad c = 0$$

and

$$z_1 = x, \qquad z_2 = v + u_1. \tag{28}$$

Member one is member two's precedent; however, what member two knows is only a noisy information about what member one has done, i.e., $v + u_1$. Clearly, this is not a partially nested structure. The auxiliary problem which bounds Problem C in payoff from below is Problem $\bar{C}$ with

$$\bar{z}_1 = x$$

$$\bar{z}_2 = \begin{bmatrix}x\\ v + u_1\end{bmatrix} \tag{29}$$

or, equivalently,

$$\hat{z}_1 = x$$

$$\hat{z}_2 = \begin{bmatrix}x\\ v\end{bmatrix}.$$

(See Theorem 1 of Part I.) The optimal solution of Problem $\bar{C}$ can be obtained as

$$\bar{u}_1^* = \hat{z}_1 = \bar{z}_1 \qquad \bar{u}_2^* = [1 \quad 0]\hat{z}_2 = [1 \quad 0]\bar{z}_2$$

$$\bar{J}^* = -\tfrac{1}{2} \tag{30}$$

and

$$\bar{p}_1^*(\xi) = [1 \quad 0]\xi \qquad \bar{p}_2^*(\xi) = [1 \quad 0]\xi. \tag{31}$$

Substituting (31) into (28), we get

$$g_1(\xi) = z_1 = [1 \quad 0]\xi$$

$$g_2(\xi) = z_2 = v + \bar{p}_1^*(\xi) = [1 \quad 1]\xi. \tag{32}$$

If there are control functions γ_i for Problem C and they are equivalent to those $\bar{\gamma}_i^*$ for Problem $\bar{C}$, then we require

$$p_1(\xi) = \gamma_1[g_1(\xi)] = \gamma_1[(1 \quad 0)\xi] \to$$
$$= \bar{p}_1^*(\xi) = [1 \quad 0]\xi \tag{33a}$$

when, γ_1 is the identity function and $u_1 = z_1$;

$$p_2(\xi) = \gamma_2[g_2(\xi)] = \gamma_2[(1 \quad 1)\xi] \to$$
$$= \bar{p}_2^*(\xi) = [1 \quad 0]\xi. \tag{33b}$$

However, (33b) is not possible since matrix $[1 \quad 1]$ is not nonsingular, nor can it be made nonsingular by approximation if its dimensions are fixed. Therefore, we cannot have controls for Problem C that are equivalent to those of Problem $\bar{C}$.

Linear Suboptimal Controls for Problem C: Let

$$u_1 = az_1 + b = ax + b$$

$$u_2 = dz_2 + f = dv + dax + db + f \tag{34}$$

where a, b, d, and f are constants to be determined. The payoff function of the linear controls (34) is

$$J_l = \frac{1}{2}\left(1 + \frac{1}{X}\right)(a^2X + b^2) - da^2X - db^2 - bf + \frac{1}{2}(d^2b^2 + d^2a^2X + f^2 + 2dbf + d^2) - a.$$

To find the optimal a, b, d, and f, let $\partial J_l/\partial i = 0$ for $i = a$, b, d, and f. Then $b = 0$, $f = 0$, and

$$a = \frac{1}{1 + (1-d)^2X} \tag{35}$$

$$d = \frac{a^2X}{1 + a^2X}. \tag{36}$$

In solving (35) and (36) together, there are three pairs of real roots. They are the following.

Pair 1:

$$a = \frac{1}{2}\left(1 + \sqrt{1 - \frac{4}{X}}\right) \qquad d = \frac{1}{2}\left(1 + \sqrt{1 - \frac{4}{X}}\right)$$

$$\text{as } X \to \infty, \quad \begin{cases} a \to 1 - \dfrac{1}{X} + o\left(\dfrac{1}{X^2}\right) \\ d \to 1 - \dfrac{1}{X} + o\left(\dfrac{1}{X^2}\right) \\ J_l \to -\dfrac{1}{2X} \to 0 \text{ (local minimum).} \end{cases}$$

Pair 2:

$$a = \frac{1}{2}\left(1 - \sqrt{1 - \frac{4}{X}}\right) \qquad d = \frac{1}{2}\left(1 - \sqrt{1 - \frac{4}{X}}\right)$$

$$\text{as } X \to \infty, \quad \begin{cases} a \to 1 + \dfrac{1}{X} + o\left(\dfrac{1}{X^2}\right) \\ d \to 1 + \dfrac{1}{X} + o\left(\dfrac{1}{X^2}\right) \\ J_l \to -\dfrac{1}{2X} \to 0 \text{ (local minimum).} \end{cases}$$

Pair 3:

$$a = \frac{2}{\sqrt{3X}} \sinh\frac{\phi}{3} \qquad d = 1 - \frac{2}{\sqrt{3X}} \sinh\frac{\phi}{3},$$

$$\text{where } \sinh\phi = \frac{3\sqrt{3X}}{2}$$

$$\text{as } X \to \infty, \quad \begin{cases} a \to \left(\dfrac{1}{X}\right)^{\frac{1}{3}} + o\left(\dfrac{1}{X^4}\right) \\ d \to 1 - \left(\dfrac{1}{X}\right)^{\frac{1}{3}} + o\left(\dfrac{1}{X^4}\right) \\ J_l \to \dfrac{1}{2} \text{ (local maximum).} \end{cases}$$

These are the only three stationary points for the payoff function (local minimum, local maximum, and saddle point) under the linear class of controls. Hence, the linear suboptimal (global minimum) payoff is

$$J_l^* = \min\left(0, 0, \tfrac{1}{2}\right) = 0, \qquad \text{as } X \to \infty. \tag{37}$$

Equations (30) and (37) serve as the lower and upper bounds for J^*, respectively, as $X \to \infty$,

$$-\tfrac{1}{2} = \bar{J}^* \le J^* \le J_l^* = 0. \tag{38}$$

Many nonlinear controls with payoff between these two bounds can be easily designed. The best one is not known.

To find the linear suboptimal solution to a team problem is an optimization problem in finite-dimensional Euclidean space only; while the optimal solution of the same problem is a problem in measurable functional space. The former can be handled by many of the well-known numerical optimization techniques.

V. Conclusion

General causal dynamic team decision problems have been considered through the control actions of members, each one of which is responsible for a single action at a single time moment. Communication links are developed in a relation of partial ordering. All problems with a quadratic payoff function, linear dynamics, and Gaussian *a priori* random variables are examined in normalized form. Because of the interaction of information, estimation, and control functions, the quantities to be estimated are no longer Gaussian, and in general, the payoff function is not convex for each decision maker. Those standard results of classical stochastic optimal control do not generalize in an obvious way.

Equivalence relation in control functions among different systems is developed. This equivalence relation aids in the solving of a class of difficult problems by relating their solutions to those of some auxiliary problems with partially nested structure of "perfect memory." The latter can be immediately obtained in an easier way. For each member of the team a condition is found which enables him to infer the information available to other persons in the team. At the same time, the controls, which will affect the information assessed, can be proven optimal. In other problems when this condition fails to hold, upper and lower bounds of the payoff function can still be found and suboptimal controls can be obtained.

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K'ai-Ching Chu (S'69–M'71), for a photograph and biography see this issue, page 22.