QUALITATIVE THEORY OF DIFFERENTIAL EQUATIONS

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CHAPTER VI

Systems with an Integral Invariant

1. Definition of an Integral Invariant

We shall consider the motions of a dynamical system given by the differential equations

$$(1.01) \qquad \frac{dx_i}{dt} = X_i(x_1, x_2, \ldots, x_n) \qquad (i = 1, 2, \ldots, n).$$

The functions X_i are defined in some closed domain R of the "phase space" $(x_1, x_2, \ldots, x_n)$; we shall regard them as being continuously differentiable with respect to all the arguments. Then the initial values $x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}$ for $t = t_0$ determine a motion of the system (1.01):

$$(1.02) \quad x_i = \varphi_i(t - t_0;\ x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}) \qquad (i = 1, 2, \ldots, n),$$

where the φ_i have continuous partial derivatives with respect to the initial values $(x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)})$. We shall denote the motion (1.02) more concisely thus:

$$x = f(x_0, t).$$

An *integral invariant* (of the nth order), according to Poincaré, is an expression of the form

$$(1.03) \qquad \int\int \underset{D}{\cdots} \int M(x_1, x_2, \ldots, x_n)\, dx_1\, dx_2 \ldots dx_n,$$

where the integration is extended over any domain D, if this expression possesses the property

$$(1.04) \qquad \int\int \underset{D}{\cdots} \int M(x_1, x_2, \ldots, x_n) dx_1\, dx_2 \ldots dx_n$$

$$= \int\int \underset{D_t}{\cdots} \int M(x_1, x_2, \ldots, x_n)\, dx_1\, dx_2 \ldots dx_n.$$

Here $D_t = f(D, t)$ is the domain occupied at the instant t by the points which for $t = 0$ occupy the domain D.

Poincaré also gave a simple dynamical interpretation of the condition (1.04) characterizing an integral invariant. We shall consider the system in three-dimensional space

$$\frac{dx}{dt} = X(x, y, z), \quad \frac{dy}{dt} = Y(x, y, z), \quad \frac{dz}{dt} = Z(x, y, z), \tag{1.05}$$

and shall interpret it as a system of equations determining the velocity of the steady state motion of a fluid filling the space R. If $\varrho(x, y, z)$ denotes the density of the fluid at the point (x, y, z), then the integral

$$\iiint_D \varrho(x, y, z)\, dx\, dy\, dz \tag{1.06}$$

represents the mass of the fluid filling the domain D; the expression (1.06) is an integral invariant since this mass of fluid remains unaltered when the particles of the fluid, having undergone a displacement along their trajectories for a time interval t, occupy the domain D_t. Thus, for the system (1.05) defining a steady state motion of the fluid, there exists an integral invariant in which the function M of the formula (1.03) is the density of the fluid. If the fluid is incompressible, then $\varrho(x, y, z) = \text{const.}$ and we have

$$\iiint_D dx\, dy\, dz = \iiint_{D_t} dx\, dy\, dz. \tag{1.07}$$

Thus, *in the case of an incompressible fluid, the volume is an integral invariant.*

We now introduce a partial differential equation satisfied by the function M of the formula (1.03) — the "density of the integral invariant". We first take a local point of view. We choose a closed domain D lying wholly within R and a time interval $(-T, T)$ so small that $D_t \subset R$ for $-T < t < T$. Next we take within this interval a fixed instant t and a small increment h such that $t + h \in (-T, T)$. We set

$$I(t) = \iint_{D_t} \cdots \int M(x_1, x_2, \ldots, x_n)\, dx_1 dx_2 \ldots dx_n \tag{1.08}$$

Then one can write

(1.09) $$I(t+h) = \iint\limits_{D_{t+h}} \cdots \int M(x_1', x_2', \ldots, x_n')dx_1' dx_2' \ldots dx_n',$$

where $(x_1', x_2', \ldots, x_n')$ is a point of the domain D_{t+h}. By virtue of the uniqueness of the solution and the continuous dependence on the initial conditions, the formulas which are obtained from (1.02) if one sets $t - t_0 = h$ and denotes the coordinates for the value t by x_i and for the instant $t + h$ by x_i', i.e.

(1.10) $$x_i' = \varphi_i(h;\ x_1,\ x_2, \ldots, x_n) \qquad (i = 1, 2, \ldots, n),$$

determine a one-one correspondence between the points of the domains D_t and D_{t+h}. Therefore in the expression (1.09) one can pass from the variables x_i' to the variables x_i, at the same time replacing the domain D_{t+h} by the domain D_t. From the conditions imposed on X_i there follows the existence of the continuous partial derivatives

$$\frac{\partial x_i'}{\partial x_j} = \frac{\partial \varphi_i}{\partial x_j} \qquad (i,\ j = 1, 2, \ldots, n);$$

therefore, by the formula for a change of variables, the multiple integral (1.09) takes the form

(1.11) $$I(t+h) = \iint\limits_{D_t} \cdots \int M[\varphi_1(h;\ x_1, x_2, \ldots, x_n), \ldots, \varphi_n(h;\ x_1, x_2, \ldots, x_n)] \cdot \frac{D(x_1', x_2', \ldots, x_n')}{D(x_1, x_2, \ldots, x_n)} dx_1\, dx_2 \ldots dx_n.$$

We shall assume that the function M also admits continuous partial derivatives with respect to all its arguments. Under this assumption we shall compute the integrand in formula (1.11). First of all we note that because of the differentiability with respect to h of the functions φ_i we have

$$M[\varphi_1(h;\ x_1, x_2, \ldots, x_n), \ldots, \varphi_n(h;\ x_1, x_2, \ldots, x_n)]$$
$$= M\left[x_1 + h\left(\frac{\partial \varphi_1}{\partial h}\right)_{h=0} + o(h), \ldots, x_n + h\left(\frac{\partial \varphi_n}{\partial h}\right)_{h=0} + o(h)\right],$$

where $o(h)$ denotes in general functions whose quotients when divided by h tend to zero uniformly with respect to $x_1, x_2, \ldots, x_n$ in the domain D_t as $h \to 0$. Noting that the expressions (1.02) are solutions of the system (1.01) and, consequently,

$$\left(\frac{\partial \varphi_i}{\partial h}\right)_{h=0} = X_i(x_1, x_2, \ldots, x_n) \qquad (i = 1, 2, \ldots, n),$$

we have, because of the existence of the complete differential of M (and this follows from the existence of the continuous partial derivatives),

$$(1.12) \qquad M(x_1', x_2', \ldots, x_n')$$
$$= M(x_1, x_2, \ldots, x_n) + h \sum_{i=1}^{n} X_i(x_1, x_2, \ldots, x_n) \frac{\partial M(x_1, x_2, \ldots, x_n)}{\partial x_i} + o(h).$$

Furthermore, the functions $\partial x_i'/\partial x_j$ $(i, j, = 1, 2, \ldots, n)$, as already mentioned, are continuous functions of $h, x_1, x_2, \ldots, x_n$; moreover, they admit continuous derivatives with respect to h; and for $h = 0$ the functions are equal to δ_{ij} (*i.e.* 0 for $i \neq j$ and 1 for $i = j$). Their derivatives satisfy variational equations [1]

$$\frac{d}{dh}\frac{\partial x_i'}{\partial x_j} = \sum_{k=1}^{n} \frac{\partial X_i(x_1', x_2', \ldots, x_n')}{\partial x_k'} \frac{\partial x_k'}{\partial x_j} \qquad (i, j = 1, 2, \ldots, n);$$

whence

$$\frac{\partial x_i'}{\partial x_j} = \delta_{ij} + h \sum_{k=1}^{n} \frac{\partial X_i(x_1, x_2, \ldots x_n)}{\partial x_k} \left(\frac{\partial x_k'}{\partial x_j}\right)_{h=0} + o(h)$$
$$= \delta_{ij} + h \frac{\partial X_i(x_1, x_2, \ldots, x_n)}{\partial x_j} + o(h).$$

Substituting these values into the Jacobian of the transformation and selecting in the determinant the term not containing h and the terms of the first order with respect to h, we obtain

$$(1.13) \quad \frac{D(x_1', x_2', \ldots, x_n')}{D(x_1, x_2, \ldots, x_n)} = 1 + h \sum_{i=1}^{n} \frac{\partial X_i(x_1, x_2, \ldots, x_n)}{\partial x_i} + o(h).$$

Substituting the values (1.12) and (1.13) into (1.11), we find

$$I(t+h) = \int\int_{D_t}\cdots\int \left\{M + h\left[\sum_{i=1}^{n} X_i \frac{\partial M}{\partial x_i} + M \sum_{i=1}^{n} \frac{\partial X_i}{\partial x_i}\right] + o(h)\right\} dx_1\, dx_2 \ldots dx_n.$$

[1]See V. V. Stepanov, *Course in Differential Equations*, Chap. VII, § 3.

We compute the derivative $I'(t)$:

$$I'(t) = \lim_{h\to 0}\frac{I(t+h) - I(t)}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\iint\underset{D_t}{\cdots}\int\{h\left[\sum_{i=1}^{n} X_i\frac{\partial M}{\partial x_i} + M\sum_{i=1}^{n}\frac{\partial X_i}{\partial x_i}\right] + o(h)\}\,dx_1\ldots dx_n$$

$$= \iint\underset{D_t}{\cdots}\int\left[\sum_{i=1}^{n}\frac{\partial(MX_i)}{\partial x_i}\right]dx_1\ldots dx_n.$$

Now let $I(t)$ be an integral invariant; then the equality

$$I'(t) = 0$$

holds for any (sufficiently small) domain D. From this we obtain a necessary condition for the density M:

$$\sum_{i=1}^{n}\frac{\partial(MX_i)}{\partial x_i} = 0. \tag{1.14}$$

It is easily seen that if, conversely, the condition (1.14) is fulfilled identically, then the expression (1.03) is an integral invariant.

The condition (1.14) is the partial differential equation for M. From the existence theorem for such an equation one can assert that for the system under investigation a *local invariant integral always exists*. However, this fact does not enable us to draw the desired conclusions of a qualitative character concerning the dynamical system. In fact, we shall consider only positive integral invariants (or at least non-negative). For a sufficiently small domain D we can satisfy this restriction in the following manner. For a unique determination of the solution of the partial differential equation (1.14) one must assign the Cauchy data. If the point $(x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)})$ is not critical for the system (1.01), it be assumed that for at least one i, $x_i \neq 0$ in some neighborhood of it. Let $i = 1$. Then we can assign for $x_1 = x_1^{(0)}$ the initial condition $M = \varphi(x_2, x_3, \ldots, x_n)$, where $\varphi > 0$; hence, because of the continuity of M as a solution of (1.14), M will be positive for values of x_1 in a sufficiently small neighborhood of the point $x_1 = x_1^{(0)}$.

But we are considering a system of the form (1.01) in some domain R of the space $(x_1, x_2, \ldots, x_n)$ (or in some n-dimensional manifold) which is an invariant set of the system; *i.e.*, if the initial point lies in R, so does the entire trajectory. We shall call an expression of the form (1.03) an *integral invariant* only in the case when $M > 0$ in the whole domain R; then the equality (1.04) holds for any domain D and for any value of $t(-\infty < t < +\infty)$. Furthermore, we shall introduce the added restriction

$$\int\int_R \cdots \int M \, dx_1 \, dx_2 \ldots dx_n < +\infty.$$

If such an integral invariant exists, then its density M satisfies the equation (1.14). But it is impossible to prove its existence for any system satisfying only the condition of differentiability. The existence of an integral invariant in this sense is an additional restriction imposed on the system (1.01).

1.15. NOTES: The case when the right-hand sides of the equations depend on t is reduced to the form (1.01) by a preliminary replacement of t by x_{n+1} and the introduction of the supplementary equation $dx_{n+1}/dt = 1$. The condition (1.14), after a return to the variable t, takes the form

$$\frac{\partial M}{\partial t} + \sum_{i=1}^{n} \frac{\partial (MX_i)}{\partial x_i} = 0. \tag{1.16}$$

1.17. A function M, satisfying equation (1.14) or (1.16), is called a *multiplier of Jacobi.*

1.18. The condition that the system (1.01) admit volume as an n-dimensional integral invariant, *i.e.* $M = 1$, is

$$\sum_{i=1}^{n} \frac{\partial X_i}{\partial x_i} = 0.$$

The equations in the Hamiltonian form,

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \qquad (i = 1, 2, \ldots, n),$$

where

$$H = H(p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n),$$

obviously belong to this class.

1.19. Any system (1.01) with an integral invariant in which $M > 0$ can be reduced to the case $M = 1$. For this it is necessary to transform the independent variable (time) by means of the formula $d\tau = dt/M$; then the equations take the form

$$\frac{dx_i}{d\tau} = MX_i = X_i'(x_1, x_2, \ldots, x_n);$$

and because of (1.14)

$$\sum_{i=1}^{n} \frac{\partial X_i'}{\partial x_i} = 0.$$

The gist of the transformation of the type indicated lies in the fact that, while not altering the trajectories of the particles, we multiply the velocity at the point $(x_1, x_2, \ldots, x_n)$ by the value of the function M at this point. However, this transformation does not simplify the theory significantly.

1.20. If the right-hand sides of the equations (1.01) are subject only to Lipschitz conditions with respect to $x_1, x_2, \ldots, x_n$, and if an integral invariant exists, then the function M has partial derivatives almost everywhere (and has bounded derived numbers everywhere); the condition (1.14) is fulfilled almost everywhere (V. V. Stepanov, *Compositio Math.* 3).

We consider in particular a dynamical system for $n = 2$:

$$\frac{dx}{dt} = X(x, y), \quad \frac{dy}{dt} = Y(x, y). \tag{1.21}$$

We shall deduce the necessary and sufficient condition under which area will be an integral invariant for the system (1.21). From equation (1.14) the condition $M = 1$ gives

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = 0. \tag{1.22}$$

Obviously, this is the condition that the expression

$$Y dx - X dy \tag{1.23}$$

be an exact differential.

In the general case $M \not\equiv 1$ equation (1.14) gives

$$\frac{\partial (MX)}{\partial x} + \frac{\partial (MY)}{\partial y} = 0.$$

This is the equation for an integrating factor M of the expression

(1.23); thus the system (1.21) in the presence of an integral invariant must possess an integrating factor which is continuous and positive over the entire invariant set under consideration.

As another example, we consider the system of linear equations

$$\frac{dx}{dt} = ax + by, \qquad \frac{dy}{dt} = cx + dy, \tag{1.24}$$

where a, b, c, d are constants, with the critical point (0, 0). The condition (1.22) for invariance of area gives

$$a + d = 0.$$

Reducing the system (1.24) to normal form [2], we obtain for λ the equation

$$\lambda^2 - (a + d)\lambda + ad - bc = 0,$$

or, in our case,

$$\lambda = \pm\sqrt{-ad + bc}.$$

Thus, invariance of area for the system (1.24) holds only in case the critical point is a center (imaginary roots) or a saddle, where for the latter case there must be $\lambda_2 = -\lambda_1$.

From what has been said it follows that the methods of the local theory of differential equations do not enable one, in the general case, to establish the existence of a non-negative integral invariant; and these will have great significance in this chapter for the investigation of dynamical systems. There exists a series of investigations giving conditions under which a dynamical system will have an integral invariant in the sense mentioned above. These conditions appear as restrictions imposed on the system.

We shall proceed along a more abstract route. As in the preceding chapter, we shall consider here dynamical systems in the metric space R. Frequently we shall assume that this space has a countable base. In certain cases we shall regard R as compact or locally compact. The role of the integral invariant in these abstract dynamical systems is played by invariant measure. But before the introduction of invariant measure it is necessary to establish a theory of measure in general. The following section is devoted to an exposition of one such theory.

[2]See V. V. Stepanov, *Course in Differential Equations*, Chap. II, § 2.

2.36. Note. We have carried out the proof of Fubini's theorem for the topological product of a metric space with a countable base and a one-dimensional Euclidean space because this case will be encountered in the sequel. The proof would not be altered if we were to have two metric spaces with countable bases, R_1 and R_2, with Carathéodory measures μ_1 and μ_2, each R being the union of not more than a countable number of sets of finite measure.

1. Recurrence Theorems.

Let a dynamical system $f(p, t)$ be given in a metric space R. A measure μ defined in the space R is called *invariant* (with respect to the system $f(p, t)$) if for any μ-measurable set A there holds the equality

$$\mu f(A, t) = \mu A \qquad (-\infty < t < +\infty). \tag{3.01}$$

From the property (3.01) it follows that images of a measurable set are measurable. This invariant measure is the natural generalization of the integral invariant considered in section 1 for systems of differential equations.

Systems with an invariant measure possess a series of properties distinguishing them from general dynamical systems. In this section we shall consider the theorem of Poincaré-Carathéodory on recurrence.

Let there exist in a space R of a dynamical system an invariant measure μ. Suppose that the measure of the entire space is finite; for simplicity we set $\mu R = 1$. The recurrence theorem separates naturally into two parts.

3.02. Theorem (recurrence of sets). *Let $A \subset R$ be a measurable set, $\mu A = m > 0$. Then there can be found positive and negative values of t ($|t| \geqq 1$) such that $\mu[A \cdot f(A, t)] > 0$.*

For the proof we consider the positions of the set $f(A, t)$ for integral values of $t (t = 0, \pm 1, \pm 2, \ldots)$ and introduce the notation

$$A_n = f(A, n) \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Because of invariance we have

$$\mu A_n = \mu A = m > 0.$$

If it be assumed, for example, that the sets $A_0, A_1, \ldots, A_k$ intersect in pairs only in sets of measure zero, we then obtain

$$\mu(A_0 + A_1 + \ldots + A_k) = km,$$

which, when $k > 1/m$, leads to a contradiction of the assumption that $R = 1$.

Thus there exist two sets A_i, A_j $(i \neq j)$ such that

$$\mu(A_i \cdot A_j) > 0. \tag{3.03}$$

Suppose that $i < j$; then $0 \leqq i < j \leqq k$. On applying to the set $A_i \cdot A_j$ the transformation $f(p, -i)$, we obtain from (3.03) that

$$\mu(A_0 \cdot f(A_0, j-i)) > 0,$$

which proves the assertion, since $j - i \geqq 1$; moreover, we can choose

$$j - i \leqq \left[\frac{1}{m}\right] + 1.$$

If one applies the transformation $f(p, -j)$, to (3.03), one obtains

$$\mu(A_0 \cdot f(A_0, i-j)) > 0, \qquad i - j \leqq 1.$$

The theorem is proved.

3.04. Note. By the same method it is easy to prove that the values of t for which $\mu(A \cdot f(A, t)) > 0$ can be arbitrarily large in absolute value. In fact, let T, $T > 0$ be any preassigned number; we choose an integer $N > T$ and consider the sequence of sets

$$A_0, A_N, A_{2N}, \ldots, A_{kN}, \ldots.$$

The preceding argument leads to the relation

$$\mu(A_0 \cdot f(A_0, N(j-i))) > 0, \qquad N(j-i) \geqq N > T,$$

and analogously for values $t < -T$.

3.05. Theorem (recurrence of points). *If in a space R with a countable base we have $\mu R = 1$ for an invariant measure μ, then almost all points $p \in R$ (in the sense of the measure μ) are stable according to Poisson, i. e.* denoting the set of points unstable according to Poisson by $\mathscr{E}$, we have $\mu\mathscr{E} = 0$.

We first take any measurable set A such that $\mu A = m > 0$. As in the preceding theorem, we set

$$A_n = f(A, n) \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Next we construct the sets