STOCHASTIC STABILITY
(OUTLINE OF PRESENTATION)

STOCHASTIC DYNAMIC DISCRETE TIME EQUATION
RECALL LYAPUNOV METHOD
STOCHASTIC CONVERGENCE CONCEPTS
MARTINGALES AND SUPERMARTINGALES
RICHARD BUCY’S PAPER (OVERVIEW)
REVIEW (FROM B.T.POLYAK’S BOOK) OF STOCHASTIC STABILITY THEOREMS
EXAMPLES APPLICATIONS
STOCHASTIC DYNAMIC DISCRETE EQUATION

- In Stochastic Approximation, in Markovian Learning models, as well as in many Estimation or Identification Algorithms and in many Control problems where a Controller is chosen, we are led to study the convergence of a Stochastic Difference Equation of the form:

\[ x_{n+1} = f_n(x_n, y_n), \quad y_n \text{ is a random variable with distribution } H_n(y / x_n), \]

\[ x_0 \text{ is a given deterministic or random variable.} \]

Clearly the \( x_n \)'s constitute a sequence of random variables, i.e. a stochastic process.

In order to study the convergence-stability of this sequence we need to review some concepts and results.
LYAPUNOV’S METHOD

• Stable, Asymptotically Stable, Unstable Equilibria

For continuous time:
\[
\frac{dx}{dt} = f(x,t), x(t_0) = x_0, 0 = f(x^e),
\]
\[
V(x^e) \geq 0, V(x^e) = 0, x^e = 0, \text{ without loss of generality.}
\]
\[
V_x(x)^T f(x) \leq 0, \text{ (locally), Stable: } |x(t) - x^e| \leq \delta(\varepsilon), \text{if } |x_0 - x^e| \leq \varepsilon
\]
\[
V_x(x)^T f(x) < 0, \text{ (locally), Asymptotically Stable: Stable and } x(t) \to x^e
\]
Thus for \(\varepsilon > 0\), small:
\[
\frac{V(x(t+\varepsilon)) - V(x(t))}{\varepsilon} \approx \frac{dV(x(t))}{dt} = V_x(x(t))^T \frac{dx(t)}{dt} = V_x(x(t))^T f(x(t)) \leq 0
\]
i.e.
\[
V(x(t+\varepsilon)) \leq V(x(t))
\]

For discrete time, similar results (see LaSalle's book):
\[
x_{n+1} = f(x_n), x^e = f(x^e), x_0 = \text{given initial condition}
\]
\[
V(f(x)) \leq V(x) \Rightarrow
\]
\[
V(x_{n+1}) \leq V(x_n) \quad \text{IMPORTANT RELATIONSHIP}
\]
• To study the Stability of the Stochastic Difference Equation we need to use the important relationship of last page, for Random Variables and the Inequality will be done in a stochastic setup using the notion of Supermartingale.

• We also need to review first the basic notions of Stochastic convergence in order to make sense of going to an equilibrium.
CONVERGENCE OF SEQUENCES OF RANDOM VARIABLES

Let \( \{X_n\} \) be a sequence of RV's defined on \((\Omega, F, P)\). Let \( X \) be a RV on \((\Omega, F, P)\).

1. ALMOST SURE CONVERGENCE, [a.s.], (or ALMOST EVERYWHERE CONVERGENCE, [a.e.],
or CONVERGENCE WITH PROBABILITY 1):
\[
X_n \to X \text{[a.s.]} \iff \exists \Omega_0 \in F \text{ such that } P(\Omega_0) = 1 \text{ and } \lim_{n \to \infty} X_n(\omega) = X(\omega) \forall \omega \in \Omega_0
\]

2. CONVERGENCE IN PROBABILITY [prob]
\[
X_n \to X \text{[prob]} \iff \forall \varepsilon > 0 \lim_{n \to \infty} P(|X_n - X| \geq \varepsilon) = 0
\]

3. CONVERGENCE IN MEAN OF ORDER \( r \), [mean], \( r = 1, 2, 3... \)
\[
X_n \to X \text{[mean] } \iff \lim_{n \to \infty} E|X_n - X|^r = 0
\]
\((r = 1, \text{[mean]} \text{ and } r = 2, \text{[mean}^2] \text{ are most important})

4. CONVERGENCE IN DISTRIBUTION[dist.]
Let \( F_n \) be the D.F. for \( X_n \) and \( F \) be the D.F. for \( X \).
\[
X_n \to X \text{[dist.]} \iff \lim_{n \to \infty} F_n(x) = F(x) \text{ at all points } x \in R \text{ where } F \text{ is continuous.}
\]

RELATIONSHIPS
\([\text{mean}^2] \Rightarrow [\text{prob}] \Rightarrow [\text{dist}], \ [\text{a.s.}] \Rightarrow [\text{prob}], \ [\text{mean}^2] \Rightarrow [\text{mean}]\)
SUPERMARTINGALE
(J. Doob)

DEFINITION
A sequence of random variables $y_0, y_1, y_2, \ldots$ is called supermartingale if for every $n$

$$E[y_n / y_0, y_1, \ldots, y_{n-1}] \leq y_{n-1}, \text{ a.e.}$$

THEOREM
If $\{y_n\}$ is a sequence of random variables that is a supermartingale and $y_n \geq 0$ a.e., then there is a random variable $y^*$ such that: $y_n \to y^*$ a.e..
This is the basic tool for extending Lyapunov Theory to Stochastic Systems. It was done by R.Bucy in a paper, that we present briefly for historical reasons. We also do examples 1 and 2 from this paper.

We will present material from Chapter 2 from B.Polyak’s book “Introduction to Optimization” Optimization Software, Inc. 1987, (concise, clear and general presentation) (pp. 43-50)
Let $X_{n+1} = AX_n + Bu(X_n)$ be a linear system with control $u(x) = Cx + b$, where the control $u$ drives $X_n$ to a point $p$, and $\{X_n\}$ is asymptotically stable about that point. Let $\{\xi_n\}$ be mutually independent with covariance $G$ and mean zero. Define $Y_{n+1} = AY_n + Bu(Y_n) + \xi_n$. Show that there is some ellipse with center $p$ which is reached w.p.1. for any $Y_0 = y$. [Hint: The $\{X_n\}$ system has a Lyapunov function of the form $V(x) = (x - p)^T P(x - p)$, where $P = P^T > 0$ and $V(X_1) - V(x) = -(x - p)^T Q(x - p)$, $Q = Q^T > 0$ also. Use the same Lyapunov function for the $\{Y_n\}$ process and show that $E_y V(Y_1) - V(y) < 0$ outside a suitable ellipse.]