STOCHASTIC STABILITY

(OUTLINE OF PRESENTATION)

STOCHASTIC DYNAMIC DISCRETE TIME EQUATION

RECALL LYAPUNOV METHOD

STOCHASTIC CONVERGENCE CONCEPTS

MARTINGALES AND SUPERMARTINGALES

RICHARD BUCY'S PAPER (OVERVIEW)

REVIEW (FROM B.T.POLYAK'S BOOK) OF STOCHASTIC STABILITY THEOREMS

EXAMPLES APPLICATIONS

STOCHASTIC DYNAMIC DISCRETE EQUATION

- In Stochastic Approximation , in Markovian Learning models, as well as in many Estimation or Identification Algorithms and in many Control problems where a Controller is chosen, we are led to study the convergence of a Stochastic Difference Equation of the form:

$x_{n+1} = f_n(x_n, y_n),\ \ y_n$ is a random variable with distribution $H_n(y / x_n)$,

x_0 is a given deterministic or random variable.

Clearly the x_n's constitute a sequence of random variables, i.e. a stochastic proccess.

In order to study the convergence-stability of this sequence we need to review some concepts and results.

LYAPUNOV'S METHOD

- Stable, Asymptotically Stable, Unstable Equilibria

For continous time:

$$\frac{dx}{dt} = f(x,t), x(t_0) = x_0, 0 = f(x^e),$$

$V(x^e) \geq 0, V(x^e) = 0, x^e = 0,$ without loss of generality.

$\mathrm{V_x(x)^T f(x)} \leq 0,$ (locally), Stable: $\left|x(t) - x^e\right| \leq \delta(\varepsilon), if \left|x_0 - x^e\right| \leq \varepsilon$

$V_x(x)^T f(x) < 0,$ (locally),AsymptoticallyStable: Stable and $x(t) \to x^e$

Thus for $\varepsilon > 0$, small:

$$\frac{V(x(t+\varepsilon)) - V(x(t))}{\varepsilon} \approx \frac{dV(x(t))}{dt} = V_x(x(t))^T \frac{dx(t)}{dt} = V_x(x(t))^T f(x(t)) \leq 0$$

i.e.

$\mathrm{V(x(t+\varepsilon)) \leq V(x(t))}$

For discrete time ,similar results(see LaSalle's book):

$x_{n+1} = f(x_n), x^e = f(x^e), x_0 =$ given initial condition

$V(f(x)) \leq V(x) \Rightarrow$

$V(x_{n+1}) \leq V(x_n)$ IMPORTANT RELATIONSHIP

- To study the Stability of the Stochastic Difference Equation we need to use the important relationship of last page, for Random Variables and the Inequality will be done in a stochastic setup using the notion of Supermartingale.

- We also need to review first the basic notions of Stochastic convergence in order to make sense of going to an equilibrium.

CONVERGENCE OF SEQUENCES OF RANDOM VARIABLES

Let $\{X_n\}$ be a sequence of RV's defined on(Ω, F, P) .Let X be a RV on (Ω, F, P).

1. ALMOST SURE CONVERGENCE,[a.s.],(or ALMOST EVERYWHERE CONVERGENCE,[a.e.], or CONVERGENCE WITH PROBABILITY 1):

$X_n \to X$[a.s.] iff $\exists \Omega_0 \in F$ such that $P(\Omega_0) = 1$ and $\lim_{n\to\infty} X_n(\omega) = X(\omega) \forall \omega \in \Omega_0$

2. CONVERGENCE IN PROBABILITY [prob]

$X_n \to X$[prob] iff $\forall \varepsilon > 0 \ \lim_{n\to\infty} P[|X_n - X| \geq \varepsilon) = 0$

3.CONVERGENCE IN MEAN OF ORDER r,[mean^r], $r = 1, 2, 3...$

$X_n \to X$[mean^r] iff $\lim_{n\to\infty} E|X_n - X|^r = 0$

($r = 1$,[mean] and $r = 2$,[mean^2] are most important)

4. CONVERGENCE IN DISTRIBUTION[dist.]

Let F_n be the $D.F.$ for X_n and F be the D.F. for X.

$X_n \to X$[dist.] iff $\lim_{n\to\infty} F_n(x) = F(x)$ at all points $x \in R$ where F is continuous.

RELATIONSHIPS

[mean^2] $\Rightarrow$ [prob] $\Rightarrow$ [dist], [a.s.] $\Rightarrow$ [prob], [mean^2] $\Rightarrow$ [mean]

SUPERMARTINGALE
(J.Doob)

DEFINITION

A sequence of random variables $y_0, y_1, y_2, ...$ is called supermartingale if for every n

$$E[y_n / y_0, y_1, ..., y_{n-1}] \leq y_{n-1}, \quad \text{a.e.}$$

THEOREM

If $\{y_n\}$ is a sequence of random variables thats is a supermartingale and $y_n \geq 0$ a.e., then there is a random variable y^* such that: $y_n \to y^*$ a.e..

This is the basic tool for extending Lyapunov Theory to Stochastic Systems.

It was done by R.Bucy in a paper, that we present briefly for historical reasons.We also do examples 1 and 2 from this paper.

We will present material from Chapter 2 from B.Polyak's book "Introduction to Optimization"Optimization Software,Inc.1987, (concise ,clear and general presentation)

(pp.43-50)

HOMEWORK

(from H.Kushner's book: "Introduction to Stochastic Control")

Let $X_{n+1} = AX_n + Bu(X_n)$ be a linear system with control $u(x) = Cx + b$, where the control u drives X_n to a point p, and $\{X_n\}$ is asymptotically stable about that point. Let $\{\xi_n\}$ be mutually independent with covariance G and mean zero. Define $Y_{n+1} = AY + Bu(Y_n) + \xi_n$. Show that there is some ellipse with center p which is reached w.p.1. for any $Y_0 = y$. [Hint: The $\{X_n\}$ system has a Lyapunov functon of the form $V(x) = (x - p)^T P(x - p)$, where $P = P^T > 0$ and $V(X_1) - V(x) = -(x - p)^T Q(x - p), Q = Q^T > 0$ also. Use the same Lyapunov function for the $\{Y_n\}$ process and show that $E_y V(Y_1) - V(y) < 0$ outside a suitable ellipse.]