DYNAMIC PROGRAMMING

(OUTLINE OF PRESENTATION)

BASIC IDEA-HISTORICAL ELEMENTS
GENERAL DISCRETE TIME STOCHASTIC CASE
LINEAR QUADRATIC WITH NOISY MEASUREMENTS
INFINITE TIME AVERAGE COST

BASIC IDEA-HISTORICAL ELEMENTS

- We hope that the following hold:

$f : R^3 \rightarrow R$, if $\min f(x, y, z)$, has $solution : x^*, y^*, z^*$
$then :$

$$\min[f(x, y, z) : x, y, z] = \min[\min[\min f(x, y, z) : x] : z] : \text{y}]$$

$\min[$ f(x,y,z):x} has solution $\hat{x}(y, z)$

$\min[f(\hat{x}(y, z), y, z) : z]$ has solution $\hat{z}(y)$

$\min[f(\hat{x}(y, \hat{z}(y)), y, \hat{z}(y)) : y]$ has solution y^*

$z(y^*) = z^*, \hat{x}(y^*, z^*) = x^*$

$Assumptions\ \ needed!$

Caratheodory (Hamilton-Jacobii) for Cont.Time, (War) Operations Research, (Programming/Optimization: Linear, Nonlinear, Dynamic), Differential Games (Tenet of Transition) R.Isaacs/RAND,R.Bellman,Computers

1. For infima the interchanges are OK but for minima we need assumptions.
2. Order of minimization dictated by convenience. For example a known time/space ordering of actions or calculations.
3. In Stochastic cases we also have the interchange of conditional expectations and minima as well.
4. Discrete time. Continuous time. Partially ordered parameters.
4. Theoretical difficulties for both discrete and continuous time.
5. We focus on discrete time.

Discrete Time Stochastic Model

$$x_{k+1} = f_k(x_k, u_k, w_k), k = 0, 1, 2, \ldots, N$$

$$z_k = h_k(x_k, u_{k-1}, v_k), k = 1, 2, \ldots, N, z_0 = h_0(x_0, v_0)$$

$$v_k \sim P(v_k / x_k, \ldots, x_0, u_{k-1}, \ldots, u_0, w_{k-1}, \ldots, w_0, v_{k-1}, \ldots, v_0)$$

$$w_k \sim P(w_k / x_k, u_k)$$

$$I_k = \{z_0, z_1, \ldots, z_k, u_0, \ldots u_{k-1}\}$$

$$I_0 = \{z_0\}$$

$$u_k = \mu_k(I_k) \in U_k$$

$$\pi = (\mu_0, \ldots, \mu_N)$$

$$\min J_\pi = E\{g_{N+1}(x_{N+1}) + \sum_{k=0}^{N} g_k(x_k, \mu_k(I_k), w_k)\}$$

$$\text{new state:} I_{k+1} = (I_k, z_{k+1}, u_k), I_0 = z_0$$

Dynamic Programming Algorithm

Assuming that the interchanges of minima and conditional expectations are OK, we have:

$$\min J_\pi = \min_{\mu_0} E\{g_0 + ... + \min_{\mu_{N-1}} E\{g_{N-1} + \min_{\mu_N} E\{g_{N+1} + g_N / I_N\} / I_{N-1}\} ... / I_0\}$$

Cases to be worked out

CASE 1. (pages 28-32,in D.P.Bertsekas)

$x_{k+1} = \max(0, x_k + u_k - w_k), x_k, u_k, w_k \in \{0,1,2,...\}, x_k + u_k \leq 2, U_k(x_k) = [0, 2 - x_k]$

$w_{k:}$: are independent random variables(iid),with

$w_k = 0,1,2,$ with probabilities: 0.1, 0.7, 0.2, respectively

x_0 :given number

$J=E[\sum_{k=0}^{N-1} cu_k + H(x_k + u_k - w_k)$

$c = 1, H(\theta) = \max(0,\theta) + 3\max(0,-\theta), N = 3$

CASE 2.LQG Full State Measurements,(pages 148-154,in D.P.Bertsekas)

$x_{k+1} = Ax_k + Bu_k + w_k, k = 0,1,..., N-1, I_k = \{x_0, x_1, ..., x_k\}$

$J = E\{x_N Q x_N + \sum_{k=0}^{N-1} (x_k Q x_k + u_k R u_k)\}$

x_0, w_k are 0-mean finite variance, for example.ind. Gaussian

$Q \geq 0$,R>0,symmetric

CASE 3. LQG Partial State Measurements,(pages 229-236,in D.P.Bertsekas)

As in Case 2 but with:$I_k=\{z_0,z_1,...z_k\}$,$z_k=Cx_k+v_k$

v,w,x,ind.0-mean finite variance,for example ind.Gaussian

Infinite Time Average Cost (pages 421-435 in D.P.Bertsekas)

We will consider:Average Cost,Infinite Time,Stationary Model and Policies

$x_k \in S = \{1,2,...,n\}, u_k \in C_i = \{1,2,...,M_i\}, I_k = \{x_0, x_1, ..., x_k\}$, but stationary $\rightarrow\ u_k = \mu(x_k)$

State evolution $p_{ij}(u) = P(x_{k+1} = j / x_k = i, u_k = u)$

$x_k = i$, and $u_k = u$ give cost $g(i,u), J(x_0) = \lim_{N\to+\infty}(1/N)E[\sum_{k=o}^{N-1} g(x_k, u_k)]$

Example:$P_\mu(u) = \begin{pmatrix} p_{11}(u_1) & p_{12}(u_1) & p_{13}(u_1) \\ p_{21}(u_2) & p_{22}(u_2) & p_{23}(u_2) \\ p_{31}(u_3) & p_{32}(u_3) & p_{33}(u_3) \end{pmatrix}, n = 3, u_1 \in C_1, u_2 \in C\ \ , u_3 \in C_3, g_\mu = \begin{bmatrix} g(1,u_1) \\ g(2,u_2) \\ g(3,u_3) \end{bmatrix}$

$u_i = \mu(i), J_\mu = \lim_{N\to\infty} \frac{1}{N}[\sum_{k=0}^{N-1}(P_\mu)^k\ \ g_\mu]$

Simplifying Assumption:The whole State Space is a Single Ergodic Class for any Policy

ROLE OF INFORMATION: NESTEDNESS,NONCLASSICAL INFORMATION PATTERNS

MATERIAL CHOSEN FROM THE FOLLOWING:

- Withenhausen’s paper and Ho and Chu’s papers

- InformationTheory model with Gaussian channel (Ho Kastner Wong)

- Game Theory Information Patterns (Basar and Olsder)

REFERENCES

1. D.P.Bertsekas,"Dynamic Programming and Optimal Control",
Athena Scientific,2005.

2.H.Witsenhausen,"Acounterexample in stochastic optimal Control"SIAM JOC,Vol.6,No.1,pp.131-147,1968.

3.Ho&Chu "Team Decision Theory and Information structures in Optimal Control problems-Part I,II",IEEE-AC -17,No.1,pp.15-22 and 22-28,Feb.1972.

4.T.Basar&G.J.Olsder,"Dynamic Noncooperative Game Theory",SIAM Classics.

5.C.Caratheodory,"Calculus of Variations and Partial Differential Equations",Chelsea.

6.R.Bellman,"Dynamic Programming"Princeton Univ.Press.

Homework

1. From D.P.Bertsekas,problems:1.1,1.14,

2. In conjunction with the papers of Ho and Chu, study/discuss a three stage scalar LQG problem and compare the solutions of the following three cases with:

$$I_0 = \{z_0\}, I_1 = \{\varepsilon_0 z_0, \varepsilon_1 z_1\}, I_2 = \{z_0, \varepsilon_2 z_1, \varepsilon_3 z_2\}$$

Case 1. All the ε_i's are nonzero.

Case 2. $\varepsilon_0 = 0, \varepsilon_i \neq 0, i = 1, 2, 3$

Case 3. $\varepsilon_0 \to 0, \varepsilon_i \neq 0, i = 1, 2, 3$

3. Solve using Dynamic Programming

$$x_{i+1} = x_i - u_i, i = 0,1,2,3$$

$$\min \sum_{i=0}^{3} (x_{i+1}^2 + 2u_i^2)$$

$$x_0 = 5, u_i \in \{0,1,2\}$$