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On the Stackelberg Strategy in Nonzero-Sum Games[1]

M. SIMAAN[2] AND J. B. CRUZ, JR.[3]

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Abstract. The properties of the Stackelberg solution in static and dynamic nonzero-sum two-player games are investigated, and necessary and sufficient conditions for its existence are derived. Several game problems, such as games where one of the two players does not know the other's performance criterion or games with different speeds in computing the strategies, are best modeled and solved within this solution concept. In the case of dynamic games, linear-quadratic problems are formulated and solved in a Hilbert space setting. As a special case, nonzero-sum linear-quadratic differential games are treated in detail, and the open-loop Stackelberg solution is obtained in terms of Riccati-like matrix differential equations. The results are applied to a simple nonzero-sum pursuit–evasion problem.

1. Introduction

The solution of a nonzero-sum game is generally defined in terms of the rationale that each player adopts as a means of describing optimality. One of the most commonly known rationales is the Nash strategy (Ref. 1), first introduced in dynamic games in Ref. 2–3. The Nash strategy safeguards each player against attempts by any one player to further improve on his individual performance criterion. This solution

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[2] Research Associate, Coordinated Science Laboratory and Department of Electrical Engineering, University of Illinois, Urbana, Illinois.

[3] Professor, Coordinated Science Laboratory and Department of Electrical Engineering, University of Illinois, Urbana, Illinois.

generally assumes that the players know each other's performance functions and that, when the strategies have been calculated, they are announced at the same instant of time. However, because these assumptions may not always hold, many games cannot be modeled and solved in this manner.

For example, in a two-player game, if the first assumption does not hold and one player does not have information about the other's performance function, then it is no longer possible for this player to calculate his Nash strategy. Instead, allowing for the worst possible behavior on his rival's part, he may choose to play a minimax strategy whose calculation requires only knowledge of his own performance function. Or, instead of risking such a pessimistic strategy, he may select to play the game passively, that is, by waiting until the other player's strategy is announced and then solving an ordinary optimization problem for his corresponding strategy. Similarly, the same situation arises in games where, due to faster means of information processing, one player is capable of announcing his strategy before the other. These cases are only a few examples of a class of games that are formulated in such a way that the strategies are announced sequentially. The main question that this paper is concerned with is the following: for the player that has to announce his strategy first, what will be the best strategy to choose? Assuming that the sole objective of the players is to minimize their respective cost functions, a solution concept most reasonable for games of this nature is known as the Stackelberg strategy. This strategy is well known in static competitive economics (Refs. 4–6) and was recently introduced in dynamic games (Ref. 7). It will be shown that, if the player that has to announce his strategy first follows a Stackelberg strategy, he will do no worse, in terms of obtaining lower cost, than the corresponding Nash solution.

As an example of the type of problems considered, let us examine the following simple matrix game. Assume that a government G wants to select a tax rate from the following set of allowable rates: $\{a_1\%, a_2\%, a_3\%\}$ for taxing a certain firm F, which in turn has to decide on manufacturing one out of three possible varieties $\{v_1, v_2, v_3\}$ of the products that it can manufacture. Let the objectives (for example, representing net income for the firm and a combination of income and price stability for the government) of the government and the firm be measured quantitatively for every pair of tax rate and product variety by the entries in Fig. 1.

Here, the first entries correspond to the firm and the second entries correspond to the government. It is assumed that the only desire of F and G is to maximize their individual objective measures. If, due to the

		G		
		a_1	a_2	a_3
F	v_1	8,10	5,10	8,11
	v_2	7,5	8,6	11,7
	v_3	5,6	9,9	12,6

Fig. 1

nature of the game,[4] the government were to fix the tax rate before the firm decides on its product, then, by choosing a Stackelberg strategy, the government will actually be selecting the most advantageous tax rate as well as imposing some influence in the selection of the variety of product to be manufactured by the firm.

The purpose of this paper is to study some of the important characteristics of the Stackelberg strategy and derive necessary and sufficient conditions under which its existence is guaranteed in static as well as dynamic two-player nonzero-sum games. In the case of dynamic problems, linear-quadratic games are formulated and solved in a Hilbert space setting, thus including continuous-time, discrete-time, distributed-parameter and delay-differential systems. A continuous-time differential game problem is then treated in detail as a special case, and the solution for the open-loop Stackelberg strategies is obtained in terms of Riccati-like differential equations. Finally, a pursuit–evasion differential game is solved as an illustrative example.

2. Definition and Properties of the Stackelberg Strategy

Let U_1 and U_2 be the sets of admissible strategies for players 1 and 2, respectively. Let the cost functions $J_1(u_1, u_2)$ and $J_2(u_1, u_2)$ be two functions mapping $U_1 \times U_2$ into the real line such that Player 1 wishes to minimize J_1 and Player 2 wishes to minimize J_2. Following the terminology in Ref. 5, the player that selects his strategy first is called *the leader* and the player that selects his strategy second is called *the follower*. Unless otherwise stated, for the rest of this paper a Stackelberg strategy will always refer to a Stackelberg strategy with Player 2 as leader.

Definition 2.1. If there exists a mapping $T\colon U_2 \to U_1$ such that, for any fixed $u_2 \in U_2$, $J_1(Tu_2, u_2) \leqslant J_1(u_1, u_2)$ for all $u_1 \in U_1$, and if

[4] For instance, the firm might only know the first entries of the table in Fig. 1, corresponding to its own objectives, and thus chooses to wait until a tax rate is fixed before deciding on its product.

there exists a $u_{2s2} \in U_2$ such that $J_2(Tu_{2s2}, u_{2s2}) \leqslant J_2(Tu_2, u_2)$ for all $u_2 \in U_2$, then the pair $(u_{1s2}, u_{2s2}) \in U_1 \times U_2$, where $u_{1s2} = Tu_{2s2}$, is called a Stackelberg strategy pair with Player 2 as leader and Player 1 as follower.

In other words, the Stackelberg strategy is the optimal strategy for the leader when the follower reacts by playing optimally. A Stackelberg strategy with Player 1 as leader is also defined in a similar way. Let the graph $D_1 = \{(u_1, u_2) \in U_1 \times U_2 : u_1 = Tu_2\}$ of the mapping T be called *the rational reaction set* of Player 1. This set represents the collection of strategy pairs in $U_1 \times U_2$ according to which Player 1 reacts to every strategy $u_2 \in U_2$ that Player 2 may choose. By playing according to the set D_1, Player 1 is referred to as being a rational player. In the Stackelberg strategy, the follower is always assumed to be rational. Similarly, let D_2 denote the rational reaction set of Player 2 when Player 1 is the leader. The sets D_1 and D_2 have significant importance in characterizing both the Stackelberg and the Nash strategies as demonstrated in the following two propositions.

Proposition 2.1. A strategy pair (u_{1s2}, u_{2s2}) is a Stackelberg strategy with Player 2 as leader iff $(u_{1s2}, u_{2s2}) \in D_1$ and

$$J_2(u_{1s2}, u_{2s2}) \leqslant J_2(u_1, u_2), \qquad \forall (u_1, u_2) \in D_1. \tag{1}$$

Proposition 2.2. A strategy pair (u_{1N}, u_{2N}) is a Nash strategy pair iff $(u_{1N}, u_{2N}) \in D_1 \cap D_2$.

The proofs of these propositions are straightforward and follow directly from the definitions of the Nash and Stackelberg strategies and the sets D_1 and D_2.

Several interesting properties relating the Nash and Stackelberg strategies can be derived from these propositions. From (1) and Proposition 2.2, it is seen that

$$J_2(u_{1s2}, u_{2s2}) \leqslant J_2(u_{1N}, u_{2N}), \tag{2}$$

which means that the leader in the Stackelberg solution achieves at least as good (possibly better) a cost function as the corresponding Nash solution. This is so because, by choosing a Stackelberg strategy, the leader is actually imposing a solution which is favorable to himself. If the Stackelberg strategies with either player as leader coincide, then they both coincide with the Nash strategy and, clearly, in this case the leader loses its advantage. It is also evident that, in zero-sum games with saddle point, the Nash strategy, the Stackelberg strategy with either player as leader, and the minimax strategy coincide. Similarly, in identical goal games, the Nash and the Stackelberg strategies are the same. The following examples are presented to illustrate the basic idea and related properties.

Example 2.1. In the matrix game of Fig. 1, both F and G want to maximize J_1 and J_2. The rational reaction set of F when G is the leader is the set of pairs $D = \{(a_1, v_1), (a_2, v_3), (a_3, v_3)\}$, and the Stackelberg strategy with G as leader is the element of D that maximizes J_2. This is achieved by the pair (a_1, v_1). Thus, by selecting a tax rate of a_1%, the government leaves no choice for the firm but to manufacture the product v_1; hence, the resulting J_2 is more than what it would have been, had a_2 or a_3 been chosen instead. The Stackelberg strategy with F as leader (a_3, v_2) and the Nash strategy (a_2, v_3) are also easily computed. Note that, in this example, both leaders in the Stackelberg solution obtain better results than in the Nash solution and that the followers are worse off.

Example 2.2. Consider the following single-state two-stage matrix game. The 2×2 matrix games shown in Fig. 2 are to be played consecutively. The first player controls u_1 and u_2 in game (a) and p_1 and p_2 in game (b), while the second player controls v_1 and v_2 in game (a) and q_1 and q_2 in game (b). The first entries in the tables are the costs borne by Player 1 and the second entries are those borne by Player 2. The game may be played starting with either (a) or (b), and the costs to every player, as shown in Fig. 3, are the sum of the costs borne in (a) and (b).

The Stackelberg strategy with Player 2 as leader as obtained from Fig. 3 is $\{(u_1 p_2), (v_2 q_1)\}$. The Stackelberg strategies with Player 2 as leader of the subgames (a) and (b) are (u_1, v_2) and (p_2, q_1), respectively. Thus, it is seen that the Stackelberg strategies of the individual subgames are components of the Stackelberg strategies of the composite game. Stated in more general terms, we have the following proposition.

Proposition 2.3. If a game is composed of N simultaneous separate subgames, where the cost functions are the sum of the corre-

	v_1	v_2
u_1	3,0	3,1
u_2	2,3	5,4

(a)

	q_1	q_2
p_1	5,0	9,9
p_2	4,4	0,5

(b)

Fig. 2

	v_1q_1	v_1q_2	v_2q_1	v_2q_2
u_1p_1	8,0	12,9	8,1	12,10
u_1p_2	7,4	3,5	7,5	3,6
u_2p_1	7,3	11,12	10,4	14,13
u_2p_2	6,7	2,8	9,8	5,9

Fig. 3

sponding cost functions of the subgames, then a composite strategy is a Stackelberg strategy pair for the composite game iff its components are Stackelberg strategy pairs for the component subgames.

Proof. Let the ith subgame be defined by $u_j^{(i)} \in U_j^{(i)}$ and $J_j^{(i)}(u_1^{(i)}, u_2^{(i)})$, $j = 1, 2$. Then,

$$J_j(u_1, u_2) = \sum_{i=1}^{N} J_j^{(i)}(u_1^{(i)}, u_2^{(i)}), \qquad j = 1, 2, \tag{3}$$

where

$$u_j = \{u_j^{(i)}\} = \{u_j^{(1)}, \ldots, u_j^{(i)}, \ldots, u_j^{(N)}\}, \qquad U_j = \prod_{i=1}^{N} U_j^{(i)}, \qquad j = 1, 2.$$

(a) Let $(u_{1s2}^{(i)}, u_{2s2}^{(i)})$ be a Stackelberg strategy for the ith subgame $\forall i = 1, \ldots, N$. Then, there exists $T^{(i)}$ such that $u_{1s2}^{(i)} = T^{(i)} u_{2s2}^{(i)}$ and, for $u_2^{(i)} \in U_2^{(i)}$,

$$J_1^{(i)}(T^{(i)} u_2^{(i)}, u_2^{(i)}) \leqslant J_1^{(i)}(u_1^{(i)}, u_2^{(i)}), \qquad \forall u_1^{(i)} \in U_1^{(i)}, \qquad \forall i = 1, \ldots, N, \tag{4}$$

$$J_2^{(i)}(T^{(i)} u_{2s2}^{(i)}, u_{2s2}^{(i)}) \leqslant J_2^{(i)}(T^{(i)} u_2^{(i)}, u_2^{(i)}), \qquad \forall u_2^{(i)} \in U_2^{(i)}, \qquad \forall i = 1, \ldots, N. \tag{5}$$

Let $Tu_2 = \{T^{(i)} u_2^{(i)}\}$. By summing (4) and (5) for $i = 1, \ldots, N$, we obtain

$$J_1(Tu_2, u_2) \leqslant J_1(u_1, u_2), \qquad u_2 \in U_2, \qquad \forall u_1 \in U_1, \tag{6}$$

$$J_2(Tu_{2s2}, u_{2s2}) \leqslant J_2(Tu_2, u_2), \qquad \forall u_2 \in U_2. \tag{7}$$

$(u_{1s2} = \{u_{1s2}^{(i)}\}, u_{2s2} = \{u_{2s2}^{(i)}\})$ is therefore a Stackelberg strategy for the composite game.

(b) Let (u_{1s2}, u_{2s2}) be a Stackelberg solution for the composite game. Then, there exists T such that $u_{1s2} = Tu_{2s2}$ and (5)–(7) are satisfied. Fix $u_2 = \{u_2^{(i)}\} \in U_2$, and let $u_1 = Tu_2$ or $u_1^{(i)} = T^{(i)} u_2^{(i)}$. Now, select $u_1 \in U_1$ such that $u_1 = \{T^{(1)} u_2^{(1)}, \ldots, T^{(i-1)} u_2^{(i-1)}, u_1^{(i)}, T^{(i+1)} u_2^{(i+1)}, \ldots, T^{(N)} u_2^{(N)}\}$, where $u_1^{(i)} \in U_1^{(i)}$; then (6) reduces to

$$J_1^{(i)}(T^{(i)} u_2^{(i)}, u_2^{(i)}) \leqslant J_1^{(i)}(u_1^{(i)}, u_2^{(i)}), \qquad \forall u_1^{(i)} \in U_1^{(i)}, \qquad i = 1, \ldots, N. \tag{8}$$

Similarly, if

$$u_2 = \{u_{2s2}^{(1)}, \ldots, u_{2s2}^{(i-1)}, u_2^{(i)}, u_{2s2}^{(i+1)}, \ldots, u_{2s2}^{(N)}\}, \qquad u_2^{(i)} \in U_2^{(i)},$$

then (7) reduces to

$$J_2^{(i)}(T^{(i)} u_{2s2}^{(i)}, u_{2s2}^{(i)}) \leqslant J_2^{(i)}(T^{(i)} u_2^{(i)}, u_2^{(i)}), \qquad \forall u_2^{(i)} \in U_2^{(i)}, \qquad i = 1, \ldots, N. \tag{9}$$

Therefore, $(u_{1s2}^{(i)}, u_{2s2}^{(i)})$ is a Stackelberg strategy for the ith subgame.

The significance of this proposition lies in the fact that, if a set of N games are played consecutively such that the outcome of every game does not affect the outcome of the following games, the players need only calculate the Stackelberg strategies for every subgame in order to obtain the Stackelberg strategies for the composite games. Note that this property does not hold for the noninferior solutions as shown in Ref. 3.

3. Static Games

Static games are games that do not evolve over time. In this section, a class of static games in which the cost functions $J_1(u_1, u_2)$ and $J_2(u_1, u_2)$ are real-valued continuous functions defined over a subset or all of the Euclidean space $R^{m_1} \times R^{m_2}$, where m_1 and m_2 are positive integers, will be considered. Unlike matrix games, the Stackelberg solution in static games need not always exist. In these games, the Nash solution may exist, but the Stackelberg solution may not exist (and *vice versa*), as demonstrated by the following examples.

Example 3.1. Let the cost functions of the two players be

$$J_1(u_1, u_2) = -u_1 u_2 + \tfrac{1}{2}u_1^2 + u_1,$$
$$J_2(u_1, u_2) = -(u_1^2 + 1)\, u_2 + \tfrac{1}{2}(u_2^2 - u_1^2) - 2u_1,$$

where $u_1 \in R^1$ and $u_2 \in R^1$. We have, respectively,

$$\partial J_1/\partial u_1 = -u_2 + u_1 + 1, \qquad \partial^2 J_1/\partial u_1^2 = 1$$
$$\partial J_2/\partial u_2 = -(u_1^2 + 1) + u_2, \qquad \partial^2 J_2/\partial u_2^2 = 1.$$

The rational reaction sets D_1 and D_2 are therefore the line $u_1 = u_2 - 1$ and the curve $u_2 = u_1^2 + 1$. The solution of these equations ($u_1 = 0, u_2 = 1$) and ($u_1 = 1, u_2 = 2$) are the Nash strategies for this game. On the other hand, the Stackelberg strategy with Player 2 as leader is obtained by minimizing J_2 subject to the constraint $u_1 = u_2 - 1$. This reduces to minimizing the function

$$J_2 = -u_2^3 + 2u_2^2 - 3u_2 - 1.5,$$

which has no minimum with respect to u_2, thus implying that a Stackelberg strategy does not exist.

In order to guarantee the existence of Stackelberg strategies, one generally requires compactness of the spaces U_1 and U_2. The following proposition gives sufficient conditions for the existence of the Stackelberg strategies in static games.

Proposition 3.1. If U_1 and U_2 are compact sets, $U_1 \subset R^{m_1}$ and $U_2 \subset R^{m_2}$, and if J_1 and J_2 are real-valued continuous functions on $U_1 \times U_2$, then Stackelberg strategies with either player as leader exist.

Proof. The existence of a Stackelberg strategy with Player 2 as leader will be proved. The proof for the case where Player 1 is the leader is analogous. Since D_1 is a subset of the compact set $U_1 \times U_2$, we need only show that it is closed. Let (u_1^0, u_2^0) be a point in $\bar{D}_1$, the closure of D_1, and let (u_1^n, u_2^n) be a sequence of points in D_1 converging to (u_1^0, u_2^0). We will show that $(u_1^0, u_2^0) \in D_1$. Suppose that $(u_1^0, u_2^0) \notin D_1$, then $\exists (u_1^*, u_2^0) \in D_1$ such that $J_1(u_1^0, u_2^0) > J_1(u_1^*, u_2^0)$. Let $\mathscr{E} = J_1(u_1^0, u_2^0) - J_1(u_1^*, u_2^0)$. Since J_1 is continuous, $\exists \delta_1$ and $\delta_2 > 0$ such that

$$| J_1(u_1, u_2) - J_1(u_1^0, u_2^0)| < \mathscr{E}/3, \qquad \forall (u_1, u_2) \in A,$$

where

$$A = \{(u_1, u_2) \in U_1 \times U_2 : |(u_1, u_2) - (u_1^0, u_2^0)| < \delta_1\},$$

and

$$| J_1(u_1, u_2) - J_1(u_1^*, u_2^0)| < \mathscr{E}/3, \qquad \forall (u_1, u_2) \in B,$$

where

$$B = \{(u_1, u_2) \in U_1 \times U_2 : |(u_1, u_2) - (u_1^*, u_2^0)| < \delta_2\},$$

and $A \cap B = \varnothing$, where $|\cdot|$ denotes the Euclidean norm. Since $(u_1^n, u_2^n) \to (u_1^0, u_2^0)$, $\exists N_1$ such that $(u_1^n, u_2^n) \in A$, $\forall n > N_1$, and also $\exists$ a sequence $u_1^{*n} \to u_1^*$ and N_2 such that $(u_1^{*n}, u_2^n) \in B$, $\forall n > N_2$. Now, pick $N = \max\{N_1, N_2\}$; then, $| J_1(u_1^n, u_2^n) - J_1(u_1^0, u_2^0)| < \mathscr{E}/3$ and $| J_1(u_1^{*n}, u_2^n) - J_1(u_1^*, u_2^0)| < \mathscr{E}/3$, $\forall n > N$. This means that $J_1(u_1^n, u_2^n) > J_1(u_1^{*n}, u_2^n)$, $\forall n > N$. This is a contradiction, since (u_1^n, u_2^n) is a sequence in D_1. Hence, $(u_1^0, u_2^0) \in D_1$, and D_1 is closed. By the continuity of J_2, $\exists (u_{1s2}, u_{2s2}) \in D_1$ such that (1) is satisfied.

When the Stackelberg strategies happen to be in the interior of $U_1 \times U_2$, or when $U_1 \times U_2$ is the whole Euclidean space, necessary conditions for the existence of a solution can be derived easily. If $U_1 = R^{m_1}$, $U_2 = R^{m_2}$ and $J_1(u_1, u_2)$, $J_2(u_1, u_2)$ are twice differentiable on $U_1 \times U_2$, then, if a Stackelberg solution (u_{1s2}, u_{2s2}) with Player 2 as leader exists, it must satisfy the following set of equations:

(i) $$\nabla_{u_1} J_1(u_{1s2}, u_{2s2}) = 0, \tag{10}$$

(ii) $$\nabla_{u_1} J_2(u_{1s2}, u_{2s2}) + J_{1u_1u_1}(u_{1s2}, u_{2s2})\lambda = 0, \tag{11}$$

(iii) $$\nabla_{u_2} J_2(u_{1s2}, u_{2s2}) + J_{1u_2u_1}(u_{1s2}, u_{2s2})\lambda = 0, \tag{12}$$

where λ is an m_1-dimensional Lagrange multiplier. The notation $\nabla_{u_1} J_1$ denotes the gradient vector of J_1 with respect to u_1 and $J_{1u_1u_1}$ and $J_{1u_2u_1}$ represent the $m_1 \times m_1$ and $m_2 \times m_1$ matrices of second partial derivatives whose ijth elements are $\partial^2 J_1/\partial u_1^{(j)} \partial u_1^{(i)}$ and $\partial^2 J_1/\partial u_2^{(j)} \partial u_1^{(i)}$, respectively.

A simple example of planar games illustrating some of the basic properties of the solutions is presented below.

Example 3.2. The static minimization game considered in Refs. 6–8 is reproduced in Fig. 4. The cost functions J_1 and J_2, defined on $R^1 \times R^1$, are assumed to be convex and twice differentiable with respect to u_1 and u_2 and having contour lines as shown in Fig. 4. The rational reaction sets D_1 and D_2 are obtained by joining the points of tangency between the contour lines and the lines of constant u_2 and u_1, respectively. It is clear that J_2 achieves its minimum over D_1 at the point S_2 whose coordinates (u_{1s2}, u_{2s2}) are the Stackelberg strategies when Player 2 is the leader. In other words, if Player 2 is to select his strategy first, he has no better choice than u_{2s2} as long as Player 1 reacts according to the curve D_1. Similarly, point S_1 is the Stackelberg solution when Player 1 is the leader and point N is the Nash solution. An interesting feature of these strategies, illustrated in this example, is that the leader is not necessarily always the only player that benefits. In fact, in this example, both Stackelberg solutions give lower costs for both players than the Nash solution. Thus, by playing Stackelberg (that is, by agreeing that one player will lead and the other will follow), the players will be playing an enforceable solution from which both can benefit over the Nash solution. Furthermore, the Stackelberg solution has great

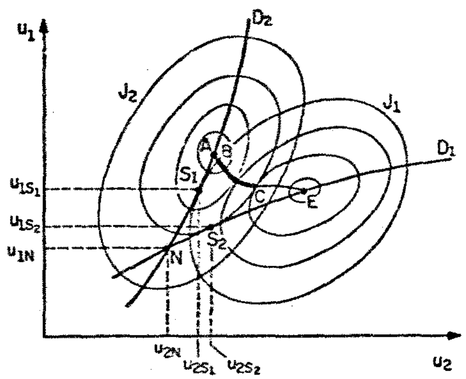

Fig. 4. A game with Nash and Stackelberg solutions.

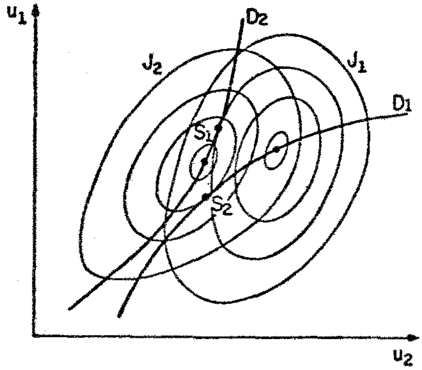

Fig. 5. A game without Nash but with Stackelberg solutions.

impact on the two players in the process of making negotiation.[5] Finally, a slight modification of this game shown in Fig. 5 illustrates a situation where Stackelberg strategies exist while a Nash strategy does not exist. In this case, the Stackelberg strategies are potential substitutes for the Nash strategies.

4. Dynamic Games

Dynamic games are games that evolve over time. Their description is usually done in terms of a dynamic equation that describes the evolution of the state of the game in response to control variables selected by the players from sets of allowable controls. Linear-quadratic games are generally represented by a linear state equation and quadratic cost functions. In this section, linear-quadratic games defined over real Hilbert spaces are treated. This formulation includes several dynamic games, such as continuous-time, discrete-time, etc., that are of interest to control engineers.

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[5] The authors would like to thank one of the reviewers for bringing this fact to their attention. If $P \subset U_1 \times U_2$ is the noninferior set, then a negotiation set N_e can be defined as follows:

$$N_e = \{(u_1, u_2) \in P;\ J_i(u_1, u_2) < \underline{J_i(u_{1si}, u_{2si}) < J_i(u_{1N}, u_{2N})},\ i = 1, 2\},$$

where (u_{1si}, u_{2si}) is the Stackelberg solution when player i is the leader. The underlined relation is always true if the Stackelberg and Nash strategies exist. If any of them do not exist, the corresponding part of the inequality can be ignored. In Fig. 4, P is the curve AE, and N_e is the curve BC.

Let H, H_1, H_2 be real Hilbert spaces, and let the state equation of the game be of the form

$$x = \phi x_0 + L_1 u_1 + L_2 u_2, \tag{13}$$

where the state variable x and the initial state x_0 are in H, and where the control variables u_1 and u_2 of Players 1 and 2 are selected from H_1 and H_2, respectively. $\phi: H \to H$, $L_1: H_1 \to H$, and $L_2: H_2 \to H$ are bounded linear transformations. The cost functionals that the players seek to minimize are of the form

$$J_1(u_1, u_2) = \tfrac{1}{2}(\langle x, Q_1 x\rangle + \langle u_1, R_{11} u_1\rangle + \langle u_2, R_{12} u_2\rangle), \tag{14}$$

$$J_2(u_1, u_2) = \tfrac{1}{2}(\langle x, Q_2 x\rangle + \langle u_1, R_{21} u_1\rangle + \langle u_2, R_{22} u_2\rangle), \tag{15}$$

where Q_1 and Q_2 are bounded linear self-adjoint (BLSA) operators on H, R_{11} and R_{21} are BLSA operators on H_1, and R_{12} and R_{22} are BLSA operators on H_2. The inner products in (14)–(15) are taken over the underlying spaces. Necessary and sufficient conditions for the existence of open-loop and closed-loop Nash controls for this game have been obtained in Refs. 9–10. In the following analysis, necessary and sufficient conditions for the existence of an open-loop Stackelberg control pair $(u_{1s2}, u_{2s2}) \in H_1 \times H_2$ are obtained. Because of difficulties encountered with resulting nonlinear equations, closed-loop Stackelberg controls will not be considered here.

If $u_2 \in H_2$ is fixed, Player 1 can calculate his corresponding optimal strategy by minimizing $J_1(u_1, u_2)$. This minimization, when repeated for all $u_2 \in H_2$, will lead to a description of the rational reaction set D_1. When (13) is substituted into (14), $J_1(u_1, u_2)$ becomes[6]

$$J_1(u_1, u_2) = \tfrac{1}{2}(\langle u_1, (R_{11} + L_1{}^* Q_1 L_1) u_1\rangle + 2\langle u_1, L_1{}^* Q_1(\phi x_0 + L_2 u_2)\rangle + J_{10}), \tag{16}$$

where

$$J_{10} = (\langle \phi x_0 + L_2 u_2, Q_1(\phi x_0 + L_2 u_2)\rangle + \langle u_2, R_{12} u_2\rangle).$$

A necessary condition for u_1 to minimize (16) is obtained by setting

$$[dJ_1(u_1 + \alpha h_1, u_2)/d\alpha]_{\alpha=0} = 0,$$

where $h_1 \in H_1$ and α is a real number. This gives

$$(R_{11} + L_1{}^* Q_1 L_1) u_1 + L_1{}^* Q_1(\phi x_0 + L_2 u_2) = 0.$$

[6] $L_1{}^*$ denotes the adjoint of the operator L_1. For $i = 1, 2$, if $L_i: H_i \to H$, then $L_i{}^*: H \to H_i$ and is defined by $\langle x, L_i u_i\rangle = \langle L_i{}^* x, u_i\rangle$, $x \in H$, $u_i \in H_i$.

A sufficient condition (Ref. 11) for u_1 to minimize J_1 is that the operator

$$S_1 = R_{11} + L_1{}^*Q_1L_1$$

be strongly positive. That is, it must satisfy

$$\alpha \| u_1 \|^2 \leqslant \langle S_1u_1, u_1 \rangle \leqslant \beta \| u_1 \|^2, \qquad 0 < \alpha \leqslant \beta, \qquad \forall u_1 \in H_1, \tag{17}$$

where α and β are real numbers defined by

$$\alpha = \inf_{\|u_1\|=1} \langle S_1u_1, u_1 \rangle \qquad \text{and} \qquad \beta = \sup_{\|u_1\|=1} \langle S_1u_1, u_1 \rangle.$$

If (17) holds, S_1^{-1} exists and

$$u_1 = -S_1^{-1}L_1{}^*Q_1(\phi x_0 + L_2u_2). \tag{18}$$

This relationship defines the mapping T, and the collection of pairs (u_1, u_2) such that (18) is satisfied constitutes the rational reaction set D_1.

Now, for u_1 as in (18), Player 2 (the leader) can find his optimal control by solving an optimization problem for $\tilde{J}_2(u_2) = J_2(u_1, u_2)$ when (18) and (13) are substituted for u_1 and x. Following the same procedure as above, by setting

$$[d\tilde{J}_2(u_2 + \alpha h_2)/d\alpha]_{\alpha=0} = 0,$$

$h_2 \in H_2$ and α a real number, necessary and sufficient conditions for a minimizing u_{2s2} are obtained by

$$S_2u_{2s2} + (L_2{}^*M_1{}^*Q_2M_1\phi + L_2{}^*Q_1L_1S_1^{-1}R_{21}S_1^{-1}L_1{}^*Q_1\phi)\, x_0 = 0, \tag{19}$$

where

$$S_2 = L_2{}^*M_1{}^*Q_2M_1L_2 + L_2{}^*Q_1L_1S_1^{-1}R_{21}S_1^{-1}L_1{}^*Q_1L_2 + R_{22}, \tag{20}$$

$$M_1 = I - L_1S_1^{-1}L_1{}^*Q_1, \tag{21}$$

I being the identity operator on H. If S_2 is strongly positive, S_2^{-1} exists and

$$u_{2s2} = -S_2^{-1}(L_2{}^*M_1{}^*Q_2M_1\phi + L_2{}^*Q_1L_1S_1^{-1}R_{21}S_1^{-1}L_1{}^*Q_1\phi)\, x_0. \tag{22}$$

u_{1s2} is then obtained by substituting (22) into (18). These results are summarized in the following proposition:

Proposition 4.1. If the operators S_1 and S_2 are strongly positive, then the game defined by (13)–(15) has a unique pair (u_{1s2}, u_{2s2}) of open-loop Stackelberg controls satisfying the relations

$$u_{1s2} = -S_1^{-1}L_1{}^*Q_1(\phi - L_2S_2^{-1}L_2{}^*(M_1{}^*Q_2M_1\phi + Q_1L_1S_1^{-1}R_{21}S_1^{-1}L_1{}^*Q_1\phi))\,x_0, \tag{23}$$

$$u_{2s2} = -S_2^{-1}L_2{}^*(M_1{}^*Q_2M_1\phi + Q_1L_1S_1^{-1}R_{21}S_1^{-1}L_1{}^*Q_1\phi)\,x_0. \tag{24}$$

By properly selecting the various operators in (14)–(15), S_1 and S_2 can be made strongly positive in order to guarantee the existence of a solution. One such possible selection is given in the following proposition.

Proposition 4.2. If R_{11} and R_{22} are strongly positive, and Q_1, Q_2, and R_{21} are positive semidefinite,[7] then an open-loop Stackelberg solution (u_{1s1}, u_{2s2}) exists.

Proof. If R_{11} is strongly positive and $Q_1 \geqslant 0$, then clearly S_1 is strongly positive. Similarly, if R_{22} is strongly positive, $Q_2 \geqslant 0$, and $R_{21} \geqslant 0$, then S_2 is strongly positive. By Proposition 4.1, a Stackelberg solution exists and satisfies (23)–(24).

Note that the existence of a Stackelberg solution with Player 2 as leader does not generally imply the existence of a Stackelberg solution with Player 1 as leader. The form in which the Stackelberg strategies (23)–(24) are obtained is most convenient for solving games defined over finite-dimensional Euclidean spaces. However, when infinite-dimensional spaces are considered, an alternative form that does not require inverting the operators S_1 and S_2 is preferable. One such representation, helpful in the numerical computation of the strategies as functions of time only, will be to express the open-loop controls (23)–(24) as a function of x, rather than x_0, and then solve for x as a function of x_0 separately from the state equation. After some algebraic manipulations, (23)–(24) reduce to

$$u_{1s2} = -R_{11}^{-1}L_1{}^*Q_1x, \tag{25}$$

$$u_{2s2} = -R_{22}^{-1}L_2{}^*(Q_2 - Q_1P)x, \tag{26}$$

where the operator P and the state x satisfy the relations

$$P - L_1R_{11}^{-1}L_1{}^*(Q_2 - Q_1P) + L_1R_{11}^{-1}R_{21}R_{11}^{-1}L_1{}^*Q_1 = 0, \tag{27}$$

$$(I + L_1R_{11}^{-1}L_1{}^*Q_1 + L_2R_{22}^{-1}L_2{}^*(Q_2 - Q_1P))x = \phi x_0. \tag{28}$$

The state vector x is obtained in terms of x_0 from (28) and then substituted

[7] If H is a Hilbert space and R is a BLSA operator on H, then R is positive definite (semidefinite), denoted by $R > 0$ $(\geqslant 0)$ if $\langle h, Rh\rangle > 0$ $(\geqslant 0)$, $\forall h \neq 0 \in H$.

in (25)–(26). It is important to note that (25)–(26) are only used for generating the open-loop controls and are not intended for closed-loop control implementation. As a special case of the above analysis, linear-quadratic differential games will be considered in the following section.

5. Linear-Quadratic Differential Games

Recently, linear-quadratic differential games have received considerable interest in the differential games literature (Refs. 8–13). These games have significant importance in studying the local behavior of corresponding nonlinear differential games. The dynamics of the game considered here are assumed to obey the linear differential equation

$$\dot{x} = Ax + B_1 u_1 + B_2 u_2, \qquad x(t_0) = x_0, \tag{29}$$

and the performance criteria are of the form

$$J_1(u_1, u_2) = \tfrac{1}{2} x_f' K_{1f} x_f + \tfrac{1}{2} \int_{t_0}^{t_f} (x' Q_1 x + u_1' R_{11} u_1 + u_2' R_{12} u_2)\, dt, \tag{30}$$

$$J_2(u_1, u_2) = \tfrac{1}{2} x_f' K_{2f} x_f + \tfrac{1}{2} \int_{t_0}^{t_f} (x' Q_2 x + u_1' R_{21} u_1 + u_2' R_{22} u_2)\, dt. \tag{31}$$

In these equations, the initial time t_0 and the final time t_f are finite and fixed, the state x is an n-dimensional vector of continuous functions defined on $[t_0, t_f]$ with $x_f = x(t_f)$, and the controls u_1 and u_2 are square (Lebesque)-integrable m_1-dimensional and m_2-dimensional vector functions defined on $[t_0, t_f]$. The various matrices in (29)–(31) are of proper dimensions and with elements continuous functions on $[t_0, t_f]$. In order to guarantee the existence of an open-loop Stackelberg solution (Propositions 4.1 or 4.2), the matrices in (30)–(31) are assumed to be symmetric and to satisfy the conditions $K_{1f} \geqslant 0$, $K_{2f} \geqslant 0$, $Q_1(t) \geqslant 0$, $Q_2(t) \geqslant 0$, $R_{11}(t) > 0$, $R_{22}(t) > 0$, $R_{21}(t) \geqslant 0$. These are only sufficient conditions and, in games where these conditions are not satisfied (for example, zero-sum or almost zero-sum games), it must be insured that S_1 and S_2 are strongly positive. We note that the matrices $R_{11}(t)$ and $R_{22}(t)$, being positive definite, are in fact strongly positive.[8] The players are seeking an

[8] Since $R_{11}(t) > 0$, $\forall t \in [t_0, t_f]$, then there exists a $\lambda > 0$ such that, $\forall t \in [t_0, t_f]$, $u_1'(t)\, R_{11}(t)\, u_1(t) \geqslant \lambda u_1'(t)\, u_1(t)$, $\forall u_1 \in R^{m_1}$. Therefore,

$$\int_{t_0}^{t_f} u_1'(t)\, R_{11}(t)\, u_1(t)\, dt \geqslant \lambda \int_{t_0}^{t_f} u_1'(t)\, u_1(t)\, dt,$$

which implies that $\langle u_1, R_{11} u_1 \rangle \geqslant \lambda \| u_1 \|^2$. Similar results hold for $R_{22}(t)$.

open-loop Stackelberg solution. That is, the leader is seeking a strategy $u_{2s2}(t)$, a function of time only, that he announces before the game starts. The follower will then calculate his strategy $u_{1s2}(t)$ also as a function of time only.

In order to formulate this problem into the Hilbert-space structure of (13)–(15), the solution of (29) is first obtained. Using the variation of parameters formula, we have

$$x(t) = \phi(t, t_0)\, x_0 + \int_{t_0}^{t} \phi(t, \tau)\, B_1(\tau)\, u_1(\tau)\, d\tau + \int_{t_0}^{t} \phi(t, \tau)\, B_2(\tau)\, u_2(\tau)\, d\tau, \quad (32)$$

where $\phi(t, t_0)$ satisfies the relations

$$\dot{\phi}(t, t_0) = A\phi(t, t_0), \qquad \phi(t, t) = I. \quad (33)$$

Let H, H_1, H_2 be the following spaces:

$$H = \mathscr{L}_2^{n}[t_0, t_f] \times R^n, \qquad H_1 = \mathscr{L}_2^{m_1}[t_0, t_f], \qquad H_2 = \mathscr{L}_2^{m_2}[t_0, t_f],$$

where $\mathscr{L}_2^{j}[t_0, t_f]$ is the set of all j-dimensional real-valued square-integrable functions $v(t)$ satisfying the inequality

$$\int_{t_0}^{t_f} v'(t)\, v(t)\, dt < \infty$$

and accompanied with the inner product

$$\langle v_1(t), v_2(t)\rangle = \int_{t_0}^{t_f} v_1'(t)\, v_2(t)\, dt. \quad (34)$$

Let

$$\tilde{x} = \begin{bmatrix} x(t) \\ x_f \end{bmatrix} \in H, \qquad \tilde{x}_0 = \begin{bmatrix} x_0 \\ x_0 \end{bmatrix} \in H, \qquad u_1 \in H_1 \qquad \text{and} \qquad u_2 \in H_2 .$$

Equation (32), when evaluated at t and t_f, can be written in the form

$$\tilde{x} = \phi\tilde{x}_0 + L_1 u_1 + L_2 u_2, \quad (35)$$

where

$$\phi = \begin{bmatrix} \phi(t, t_0) & 0 \\ 0 & \phi(t_f, t_0) \end{bmatrix} \quad (36)$$

$$L_i u_i = \begin{bmatrix} \int_{t_0}^{t} \phi(t, \tau)\, B_i(\tau)\, u_i(\tau)\, d\tau \\ \int_{t_0}^{t_f} \phi(t_f, \tau)\, B_i(\tau)\, u_i(\tau)\, d\tau \end{bmatrix}, \qquad i = 1, 2. \quad (37)$$

The performance criteria (30)–(31) reduce to[9]

$$J_1(u_1, u_2) = \tfrac{1}{2}(\langle \tilde{x}, \tilde{Q}_1\tilde{x}\rangle + \langle u_1, R_{11}u_1\rangle + \langle u_2, R_{12}u_2\rangle), \tag{38}$$

$$J_2(u_1, u_2) = \tfrac{1}{2}(\langle \tilde{x}, \tilde{Q}_2\tilde{x}\rangle + \langle u_1, R_{21}u_1\rangle + \langle u_2, R_{22}u_2\rangle), \tag{39}$$

where

$$\tilde{Q}_i\tilde{x} = \begin{bmatrix} Q_i(t)\,x(t) \\ K_{if}x_f \end{bmatrix}, \qquad i = 1, 2. \tag{40}$$

The next step is to determine the adjoints of L_1 and L_2. Consider the inner product of L_iu_i with an arbitrary vector

$$\tilde{w} = \begin{bmatrix} w \\ w_f \end{bmatrix} \in H,$$

that is,

$$\langle w, L_iu_i\rangle = \int_{t_0}^{t_f} w'(t) \int_{t_0}^{t} \phi(t, \tau)\, B_i(\tau)\, u_i(\tau)\, d\tau\, dt + w_f' \int_{t_0}^{t_f} \phi(t_f, \tau)\, B_i(\tau)\, u_i(\tau)\, d\tau.$$

By interchanging the order of integration, one easily obtains the relation

$$\langle w, L_iu_i\rangle = \int_{t_0}^{t_f} u_i'(\tau)\, B_i'(\tau) \int_{\tau}^{t_f} \phi'(t, \tau)\, w(t)\, dt\, d\tau + \int_{t_0}^{t_f} u_i'(\tau)\, B_i'(\tau)\, \phi'(t_f, \tau)\, w_f\, d\tau,$$

from which we conclude that $L_i^*\colon H \to H_i$ is defined by

$$L_i^*w = B_i'(t) \int_{t}^{t_f} \phi'(\sigma, t)\, w(\sigma)\, d\sigma + B_i'(t)\phi'(t_f, t)\, w_f, \qquad i = 1, 2. \tag{41}$$

Using these results, and omitting a few algebraic manipulations, (25)–(26) reduce to

$$u_{1s2} = -R_{11}^{-1}B_1'K_1x, \tag{42}$$

$$u_{2s2} = -R_{22}^{-1}B_2'K_2x, \tag{43}$$

[9] The inner product on H is defined by

$$\langle \tilde{x}_1, \tilde{x}_2\rangle = \int_{t_0}^{t_f} x_1'x_2\, dt + x_{1f}'x_{2f}.$$

where x, $K_1(t)$, $K_2(t)$ satisfy the relations

$$\dot{x} = (A - B_1 R_{11}^{-1} B_1' K_1 - B_2 R_{22}^{-1} B_2' K_2) x, \qquad x(t_0) = x_0, \tag{44}$$

$$K_1(t)\, x(t) = \int_t^{t_f} \phi'(\sigma, t)\, Q_1(\sigma)\, x(\sigma)\, d\sigma + \phi'(t_f, t)\, K_{1f} x_f, \tag{45}$$

$$K_2(t)\, x(t) = \int_t^{t_f} \phi'(\sigma, t)[Q_2(\sigma) - Q_1(\sigma)\, P(\sigma)]\, x(\sigma)\, d\sigma + \phi'(t_f, t)[K_{2f} - K_{1f} P(t_f)]\, x_f, \tag{46}$$

where $P(t)$ is obtained from (27) and satisfies the relation

$$P(t)\, x(t) = \int_{t_0}^{t} \phi(t, \tau)\, B_1(\tau)\, R_{11}^{-1}(\tau)[B_1'(\tau)\, K_2(\tau) - R_{21}(\tau)\, R_{11}^{-1}(\tau)\, B_1'(\tau)\, K_1(\tau)]\, x(\tau)\, d\tau. \tag{47}$$

Differentiating (45)–(47) with respect to t, we see that they reduce to

$$\dot{K}_1 = -A'K_1 - K_1 A - Q_1 + K_1 B_1 R_{11}^{-1} B_1' K_1 + K_1 B_2 R_{22}^{-1} B_2' K_2, \qquad K_1(t_f) = K_{1f}, \tag{48}$$

$$\dot{K}_2 = -A'K_2 - K_2 A - Q_2 + Q_1 P + K_2 B_1 R_{11}^{-1} B_1' K_1 + K_2 B_2 R_{22}^{-1} B_2' K_2, \qquad K_2(t_f) = K_{2f} - K_{1f} P(t_f), \tag{49}$$

$$\dot{P} = AP - PA + PB_1 R_{11}^{-1} B_1' K_1 + PB_2 R_{22}^{-1} B_2' K_2 - B_1 R_{11}^{-1} R_{21} R_{11}^{-1} B_1' K_1 + B_1 R_{11}^{-1} B_1' K_2, \qquad P(t_0) = 0, \tag{50}$$

and the open-loop Stackelberg controls are

$$u_{1s2} = -R_{11}^{-1} B_1' K_1 \xi(t, t_0)\, x_0, \tag{51}$$

$$u_{2s2} = -R_{22}^{-1} B_2' K_2 \xi(t, t_0)\, x_0, \tag{52}$$

where

$$\dot{\xi}(t, t_0) = (A - B_1 R_{11}^{-1} B_1' K_1 - B_2 R_{22}^{-1} B_2' K_2)\, \xi(t, t_0), \qquad \xi(t, t) = I. \tag{53}$$

Equations (48)–(49) are identical to the Riccati equations obtained in the corresponding open-loop Nash solution (Ref. 8), except for the terms containing P in (49). These terms account for the fact that the leader is now minimizing his cost on the rational reaction set [Eq. (18)] of the follower. Equation (50), however, is not of the Riccati type, and its solution must be done forward in time in contrast to (48)–(49), whose solution is obtained backward in time.

This two-point boundary-value problem is generally not easy to solve. It is possible, however, to obtain its solution from the solution of a single-point boundary-value problem, as follows. If there exists a $2n \times 2n$ matrix $F(t)$ satisfying

$$\dot{F} = -\tilde{A}'F - F\tilde{A} - \tilde{Q} + F\tilde{B}F, \qquad F(t_f) = F_f, \tag{54}$$

where

$$\tilde{A} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \quad \tilde{Q} = \begin{bmatrix} Q_1 & 0 \\ Q_2 & -Q_1 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B_1R_{11}^{-1}B_1' & B_2R_{22}^{-1}B_2' \\ B_1R_{11}^{-1}R_{21}R_{11}^{-1}B_1' & -B_1R_{11}^{-1}B_1' \end{bmatrix},$$

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}, \quad F_f = \begin{bmatrix} K_{1f} & 0 \\ K_{2f} & -K_{1f} \end{bmatrix},$$

and if (50) has a solution when K_1 and K_2 are of the form

$$\begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & -F_{11} \end{bmatrix} \begin{bmatrix} I \\ P \end{bmatrix}, \tag{55}$$

then the open-loop Stackelberg strategies (51)–(52) can be written as

$$u_{1s2} = -R_{11}^{-1}B_1'(F_{11} + F_{12}P)\,\xi(t, t_0)\,x_0, \tag{56}$$

$$u_{2s2} = -R_{22}^{-1}B_2'(F_{21} - F_{11}P)\,\xi(t, t_0)\,x_0. \tag{57}$$

In (56)–(57), $\xi(t, t_0)$ is the solution of (53) with K_1 and K_2 obtained from (55). It is shown in the appendix that the construction of K_1, K_2, P, as obtained above, satisfies (48)–(50).

It will now be shown that, in the special case of zero-sum games and identical goal games, the Stackelberg solution reduces to the familiar saddle-point and cooperative solutions.

(i) *Zero-Sum Games with Saddle Point.* In zero-sum games, $J_1 = -J_2$. That is, $R_{11} = -R_{21} = R_1$, $R_{22} = -R_{12} = R_2$, $Q_1 = -Q_2 = Q$, $K_{1f} = -K_{2f} = K_f$. The sufficient conditions of Proposition 4.2 are not satisfied. However, assuming that these matrices are selected in such a way that S_1 and S_2 are strongly positive, then a Stackelberg solution exists; and, upon substituting in (48)–(50), we conclude that $P(t) \equiv 0$, $\forall t \in [t_0, t_f]$ and $K_1(t) = -K_2(t) = K(t)$ and satisfying

$$\dot{K} = -A'K - KA - Q + K(B_1R_1^{-1}B_1' - B_2R_2^{-1}B_2')K, \qquad K(t_f) = K_f,$$

in agreement with the saddle-point solution (Ref. 12).

(ii) *Identical Goal Games.* If the two players are cooperating in minimizing the same performance function $J_1 = J_2$, the game is called an identical goal game. This problem can be formulated as an optimal control problem, and its solution obtained in terms of the Riccati equation of the regulator theory. When $R_{11} = R_{21} = R_1$, $R_{12} = R_{22} = R_2$, $Q_1 = Q_2 = Q$, $K_{1f} = K_{2f} = K_f$ are substituted in (48)–(50), it is easily concluded that $P(t) \equiv 0$, $\forall t \in [t_0, t_f]$, and $K_1(t) = K_2(t) = K(t)$ and satisfying

$$\dot{K} = -A'K - KA - Q + K(B_1R_1^{-1}B_1' + B_2R_2^{-1}B_2')K, \qquad K(t_f) = K_f,$$

in agreement with the Riccati equation of regulator theory.

Except for those two special cases, the Stackelberg solution is generally different from the Nash solution. In what follows, a simple pursuit–evasion problem will be considered.

Example 5.1. Consider the nonzero-sum velocity-controlled pursuit–evasion game studied in Refs. 2 and 13. The dynamics of the game are described by the equations

$$\dot{x} = u_1 - u_2, \qquad x(t_0) = x_0,$$

where u_1 and u_2 are the velocities of the pursuer and evader, respectively, and x is their relative position. The performance criteria are

$$J_1 = \tfrac{1}{2}x_f^2 + (1/2c_p)\int_0^1 u_1^2\,dt, \qquad J_2 = -\tfrac{1}{2}x_f^2 + (1/2c_e)\int_0^1 u_2^2\,dt,$$

where

$$c_p > 0, \qquad c_e > 0, \qquad c_pc_e = 1, \qquad c_p/c_e = \omega^2.$$

Assume that Player 2 decides to evade before Player 1 decides to pursue. Naturally, in this case, his best choice will be to announce a Stackelberg strategy u_{2s2} with himself as leader.

Applying (48)–(53), we obtain the open-loop Stackelberg solution with the evader as leader as

$$u_{1s2} = [-c_p/(c_p - \sigma c_e + 1)]\,x_0, \qquad u_{2s2} = [-\sigma c_e/(c_p - \sigma c_e + 1)]\,x_0,$$

where

$$\sigma = 1/(1 + c_p).$$

Sufficient conditions for the existence of a solution are obtained from Propositions 4.1. With the conditions that $c_p > 0$, $c_e > 0$, $c_pc_e = 1$, the operators S_1 and S_2 will be strongly positive if $\sigma^2 c_e < 1$. These conditions imply that $c_e < 2.116$. That is, if $c_e < 2.116$, an open-loop

Stackelberg solution with the evader as leader exists. On the other hand (Ref. 13), the open-loop Nash solution exists if $c_e < 1$, it does not exist if $c_e > 1$, and nothing can be said about its existence if $c_e = 1$. Therefore, in this example, the existence of a Stackelberg strategy with Player 2 as leader is guaranteed over a wider range of parameters.

The performance functions as calculated when the Stackelberg strategy is used are

$$J_1(u_{1s2}, u_{2s2}) = J_{1s2} = \tfrac{1}{2}[(1 + c_p)/(1 - \sigma c_e + c_p)^2]\, x_0^2,$$
$$J_2(u_{1s2}, u_{2s2}) = J_{2s2} = \tfrac{1}{2}[(\sigma^2 c_e - 1)/(1 - \sigma c_e + c_p)^2]\, x_0^2.$$

For the sake of comparison, the open-loop Nash solution for this problem (Ref. 13) is shown below as follows:

$$u_{1N} = [-c_p/(1 + c_p - c_e)]\, x_0,$$
$$u_{2N} = [-c_e/(1 + c_p - c_e)]\, x_0,$$
$$J_1(u_{1N}, u_{2N}) = J_{1N} = \tfrac{1}{2}[(1 + c_p)/(1 - c_e + c_p)^2]\, x_0^2,$$
$$J_2(u_{1N}, u_{2N}) = J_{2N} = \tfrac{1}{2}[(c_e - 1)/(1 - c_e + c_p)^2]\, x_0^2.$$

For the range where both the Stackelberg and the Nash solutions exist (i.e., $c_e < 1$), a comparison of the above quantities will yield $J_{1s2} < J_{1N}$ and $J_{2s2} < J_{2N}$. That is, not only the leader will benefit by using a Stackelberg solution but also the follower will benefit as well. Thus, in this special case, the Stackelberg solution can be looked at as an enforceable negotiated solution that is preferred by both players over the Nash solution.

6. Conclusions

A class of nonzero-sum games in which the strategies are announced sequentially has been investigated, and it was shown that, if the players' sole objective is to minimize their cost functions, the Stackelberg strategy is the most natural way of defining optimality. Games where one player does not know the other's cost function while the other player knows both cost functions and games where one player is faster than the other in computing his strategy are best modeled and solved within this solution concept. In this strategy, the roles of the players, whether leader or follower, must be properly defined and, when compared to the Nash solution, it was shown that it is advantageous to the leader. Conditions for the existence of the Stackelberg strategies have been obtained. It was shown that, in general, one cannot conclude the existence of a Stackelberg solution from the existence of a Nash solution nor *vice versa*. Several examples were considered in order to illustrate the properties of this solution concept. In dynamic games, an abstract formulation in Hilbert spaces has been considered, and necessary and sufficient conditions for the existence of an open-loop Stackelberg solution were obtained. Linear-quadratic differential games were treated as a special case, and the solution was expressed in terms of Riccati-like differential equations. A simple pursuit–evasion problem was solved, and the results were compared to the Nash solution.

7. Appendix

Write Eqs. (48)–(50) in the following form, using the notation in Ref. 14:

$$\dot{K} = -\tilde{A}'K - K\tilde{A} + KNK + K\bar{N}\bar{K} - Q + Q_0\tilde{P}, \qquad K(t_f) = K_f, \tag{58}$$

$$\dot{\tilde{P}} = \tilde{A}'\tilde{P} - \tilde{P}\tilde{A} + \tilde{P}NK + \tilde{P}\bar{N}\bar{K} + RK + \bar{R}\bar{K}, \qquad P(t_0) = 0, \tag{59}$$

where all matrices in (58)–(59) are $2n \times 2n$ and where

$$K = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}, \quad \tilde{P} = \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix}, \quad N = \begin{bmatrix} B_1 R_{11}^{-1} B_1' & 0 \\ 0 & B_2 R_{22}^{-1} B_2' \end{bmatrix},$$

$$Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}, \quad Q_0 = \begin{bmatrix} 0 & 0 \\ 0 & Q_1 \end{bmatrix},$$

$$R = \begin{bmatrix} -B_1 R_{11}^{-1} R_{21} R_{11}^{-1} B_1' & 0 \\ 0 & B_1 R_{11}^{-1} B_1' \end{bmatrix}, \quad K_f = \begin{bmatrix} K_{1f} & 0 \\ 0 & K_{2f} \end{bmatrix},$$

and the operation $\bar{K}$ is as follows:

$$\bar{K} = \begin{bmatrix} K_2 & 0 \\ 0 & K_1 \end{bmatrix}.$$

Let K_1 and K_2 be related to P by (55). In the above notation, this is written as

$$K = F_1 + F_2\tilde{P},$$

where

$$F_1 = \begin{bmatrix} F_{11} & 0 \\ 0 & F_{21} \end{bmatrix}, \quad F_2 = \begin{bmatrix} F_{12} & 0 \\ 0 & F_{22} \end{bmatrix}. \tag{60}$$

Upon differentiating (60) and substituting in it (58)–(59), and after several steps involving algebraic manipulation, (54) is obtained. Furthermore, from the symmetry of (54), it is clear that $F_{22} = -F_{11}$. If (54) has a solution, then K_1 and K_2 can be written as functions of P as in (55) or (60). P and ξ are then obtained by solving (50) and (53) forward in time.

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