Games with Randomly Disturbed Payoffs: A New Rationale for Mixed-Strategy Equilibrium Points

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Abstract: Equilibrium points in mixed strategies seem to be unstable, because any player can deviate without penalty from his equilibrium strategy even if he expects all other players to stick to theirs. This paper proposes a model under which most mixed-strategy equilibrium points have full stability. It is argued that for any game Γ the players' uncertainty about the other players' exact payoffs can be modeled as a disturbed game Γ^*, i.e., as a game with small random fluctuations in the payoffs. Any equilibrium point in Γ, whether it is in pure or in mixed strategies, can "almost always" be obtained as a limit of a pure-strategy equilibrium point in the corresponding disturbed game Γ^* when all disturbances go to zero. Accordingly, mixed-strategy equilibrium points are stable – even though the players may make no deliberate effort to use their pure strategies with the probability weights prescribed by their mixed equilibrium strategies – because the random fluctuations in their payoffs will *make* them use their pure strategies approximately with the prescribed probabilities.

1. Introduction

On the face of it, equilibrium points in mixed strategies are unstable because any player can deviate without penalty from his equilibrium strategy even if all other players stick to theirs. (He can shift to any pure strategy to which his mixed equilibrium strategy assigns a positive probability; he can also shift to any arbitrary probability mixture of these pure strategies.) This instability seems to pose a serious problem because many games have only mixed-strategy equilibrium points.

However, as we shall see, the stability of these equilibrium points will appear in a different light if we take due account of the uncertainty in which every player finds himself about the precise payoffs that other players associate with alternative strategy combinations. Classical game theory assumes that in any game Γ every

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player has precise knowledge of the payoff function of every other player (as well as of his own). But it is more realistic to assume that – even if each player i does have exact knowledge of his own payoff function U_i – he can have at best only somewhat inexact information about the other players' payoff functions U_j, $j \neq i$. This assumption, of course, formally transforms the game into a game with incomplete information. However, it can be shown [HARSANYI, 1967 – 68] that such games can also be modeled, more conveniently, as games with complete information involving appropriate random variables (chance moves), where the players' ignorance about any aspect of the game situation is represented as ignorance about the actual values of these random variables.

Accordingly, we shall propose the following model, to be called a disturbed game $\Gamma^* = \Gamma^*(\varepsilon)$. The payoff function U_i of every player i is subject to small random disturbances within a given range $[-\varepsilon, +\varepsilon]$, due to small stochastic fluctuations in his subjective and objective conditions (e.g., in his mood, taste, resources, social situation, etc.). The probability laws governing these disturbances are known to all players, but their precise effects on any given player's payoff function U_i at the time of his strategy choice are known only to this player i himself. As we shall see, under our assumptions (see below), all equilibrium points of any disturbed game $\Gamma^*(\varepsilon)$ will be in pure strategies. Moreover, any mixed-strategy equilibrium point s in an ordinary game Γ can "almost always" be obtained as the limit of a pure-strategy equilibrium point $s(\varepsilon)$ in the corresponding disturbed game $\Gamma^*(\varepsilon)$ when all random disturbances go to zero (i.e., when the parameter ε goes to zero). Therefore, such a mixed-strategy equilibrium point s can be interpreted as a somewhat unprecise description of this pure-strategy equilibrium point $s(\varepsilon)$ of the disturbed game $\Gamma^*(\varepsilon)$, and can be regarded as having the same stability as $s(\varepsilon)$ has.

More specifically, our results imply the following – somewhat surprising – conclusion. The players may make no deliberate effort to use their pure strategies with the probability weights prescribed by their mixed equilibrium strategies $s_1, \ldots, s_n$. Nevertheless, as a result of the random fluctuations in their payoffs, they will in fact use their pure strategies approximately with the prescribed probabilities.

2. Definitions and Notations

Let Γ be a finite n-person noncooperative game. The k-th pure strategy of player i will be called a_i^k while the set of all his K_i pure strategies will be called A_i. Let

$$K = \prod_i K_i \,. \tag{1}$$

We shall assume that the K possible n-tuples of pure strategies will be numbered consecutively (e.g., in a lexicographical order) as $a^1, \ldots, a^m, \ldots, a^K$ – but see Convention (92) below). Let

$$a^m = (a_1^{k_1}, \ldots, a_i^{k_i}, \ldots, a_n^{k_n}). \tag{2}$$

Then we shall write

$$a^m(i) = a_i^{k_i} \tag{3}$$

to denote the pure strategy used by player i in the strategy n-tuple a^m. The set of all possible pure-strategy n-tuples a^m will be called A. We can write $A = A_1 \times \cdots \times A_n$.

Any mixed strategy s_i of player i will be of the form

$$s_i = \sum_k p_i^k a_i^k, \tag{4}$$

where p_i^k is the probability that this mixed strategy s_i assigns to the pure strategy a_i^k. Of course,

$$p_i^k \geqq 0 \quad \text{for} \quad k = 1, \ldots, K_i \quad \text{and} \quad \sum_k p_i^k = 1. \tag{5}$$

The set of all mixed strategies available to player i will be called S_i. The set of all possible n-tuples $s = (s_1, \ldots, s_n)$ will be called S. We have $S = S_1 \times \cdots \times S_n$. We shall write $s = (s_i, \bar{s}_i)$, where $\bar{s}_i = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$ is the strategy $(n-1)$-tuple representing the strategies of all $(n-1)$ players other than player i.

The set $C(s_i)$ of pure strategies to which s_i assigns positive probabilities is called the *carrier* of s_i. The set $C^*(s_i)$ of all mixed strategies t_i that distribute all probability over the pure strategies in $C(s_i)$ is called the *extended carrier* of s_i. Thus $C^*(s_i)$ is the convex hull of $C(s_i)$.

A mixed strategy s_i whose carrier $C(s_i)$ contains only one pure strategy a_i^k will be identified with this latter so that we shall write $s_i = a_i^k$. On the other hand, if $C(s_i)$ contains two or more pure strategies, then s_i will be called a *proper* mixed strategy.

Suppose that the i-th component of the pure-strategy n-tuple a^m is $a^m(i) = a_i^k$, and that the mixed strategy s_i of player i assigns the probability p_i^k to this pure strategy a_i^k. Then we shall write

$$q_i^m(s_i) = p_i^k. \tag{6}$$

Of course, if $s_i = a_i^k$ is a pure strategy, then we have

$$q_i^m(a_i^k) = 1 \qquad \text{when} \quad a^m(i) = a_i^k, \tag{7}$$

whereas

$$q_i^m(a_i^k) = 0 \qquad \text{when} \quad a^m(i) \neq a_i^k. \tag{8}$$

If the n players use a pure-strategy n-tuple a^m, then player i $(i = 1, \ldots, n)$ will obtain the payoff

$$U_i(a^m) = v_i^m, \tag{9}$$

while if they use a mixed-strategy n-tuple $s = (s_1, \ldots, s_n)$, then his payoff will be

$$U_i(s) = \sum_m [\prod_i q_i^m(s_i)] v_i^m . \tag{10}$$

A given strategy s_i of player i is a *best reply* to some strategy combination $\bar{s}_i$ used by the other $(n-1)$ players if

$$U_i(s_i, \bar{s}_i) \geqq U_i(t_i, \bar{s}_i) \quad \text{for all} \quad t_i \in S_i . \tag{11}$$

A given strategy n-tuple $s = (s_1, \ldots, s_n)$ is an *equilibrium point* [NASH, 1951] if every component s_i of s is a best reply to the corresponding strategy combination $\bar{s}_i$ of the $(n-1)$ other players.

An equilibrium point s is called *strong*[3]) if all n components s_i of s satisfy (11) with the strong inequality sign $>$ for all $t_i \neq s_i$. That is, s is a strong equilibrium point if every player's equilibrium strategy s_i is his *only* best reply to the other players' strategy combination $\bar{s}_i$. An equilibrium point is called *weak* if it is not strong. Any equilibrium point s is always weak if at least one player's equilibrium strategy s_i is a proper mixed strategy because for this player i every strategy t_i in the extended carrier $C^*(s_i)$ of his mixed equilibrium strategy s_i is a best reply to $\bar{s}_i$, and there are infinitely many such strategies t_i in $C^*(s_i)$.

An equilibrium point s is *quasi-strong* if no player i has best replies to $\bar{s}_i$ other than the strategies t_i belonging to the extended carrier $C^*(s_i)$ of his equilibrium strategy s_i. Though an equilibrium point in proper mixed strategies is never strong, it may be (and mostly is) quasi-strong. An equilibrium point that is not even quasi-strong is called *extra-weak.* If an equilibrium point in pure strategies is weak then it is always extra-weak.[4])

3. Disturbed Games

In a disturbed game, $\Gamma^* = \Gamma^*(\varepsilon)$, the payoff of any player i for any given n-tuple a^m of pure strategies can be written as

$$U_i(a^m) = V_i(a^m) + \varepsilon \xi_i^m = v_i^m + \varepsilon \xi_i^m , \quad i = 1, \ldots, n; \quad m = 1, \ldots, K , \tag{12}$$

where $V_i(a^m) = v_i^m$ and ε are constants with $\varepsilon > 0$, whereas ξ_i^m is a random variable. As V_i is the nonrandom part of player i's payoff function U_i, V_i will be called player i's *basic payoff function.* We shall write

$$\xi_i = (\xi_i^1, \ldots, \xi_i^K), \quad i = 1, \ldots, n . \tag{13}$$

Thus ξ_i is a random vector consisting of all random variables ξ_i^m occurring in player i's payoff function U_i.

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[3]) I am using the term "strong equilibrium point" in a sense different from AUMANN'S [1960a, p. 363].

[4]) It can be shown that games possessing extra-weak equilibrium points, whether in pure or in mixed strategies, are very "exceptional": they correspond to a closed set of measure zero in the space of all games of a given size.

We shall assume that the random vectors ξ_i associated with different players i are statistically independent. The probability distribution of each random vector ξ_i will be written as

$$F_i = F_i(\xi_i) = F_i(\xi_i^1, \ldots, \xi_i^K). \tag{14}$$

We shall assume that the nonrandom basic payoff functions $V_1, \ldots, V_n$, the parameter ε, and the probability distributions $F_1, \ldots, F_n$ are known to all n players. But the actual values of the variables $\xi_i^1, \ldots, \xi_i^K$ are known only to player i himself ($i = 1, \ldots, n$).

We shall make the following assumptions about the probability distributions F_i:

The range of each random variable $\xi_i^m (i = 1, \ldots, n; m = 1, \ldots, K)$ is a subset (or possibly the whole) of the closed interval $I = [-1, +1]$. (15)

All probability distributions F_i are absolutely continuous. Moreover, (16)

For each F_i, the corresponding probability density function (17)

$$f_i(\xi_i) = f_i(\xi_i^1, \ldots, \xi_i^K) = \frac{\partial^K F_i}{\partial \xi_i^1, \ldots, \partial \xi_i^K}$$

is continuously differentiable for all possible values of ξ_i.

Suppose that player i uses the pure strategy a_i^k while the other $(n-1)$ players use the mixed-strategy combination $\bar{s}_i = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$. Then, in view of (10) and (12), player i will receive the payoff

$$U_i(a_i^k, \bar{s}_i) = \sum_m [q_i^m(a_i^k) \prod_{j \neq i} q_j^m(s_j)] v_i^m + \varepsilon \sum_m [q_i^m(a_i^k) \prod_{j \neq i} q_j^m(s_j)] \xi_i^m. \tag{18}$$

For convenience, we shall write

$$V_i(a_i^k, \bar{s}_i) = \sum_m [q_i^m(a_i^k) \prod_{j \neq i} q_j^m(s_j)] v_i^m \tag{19}$$

and

$$\zeta_i^k = \sum_m [q_i^m(a_i^k) \prod_{j \neq i} q_j^m(s_j)] \xi_i^m, \tag{20}$$

so that (18) can now be written as

$$U_i(a_i^k, \bar{s}_i) = V_i(a_i^k, \bar{s}_i) + \varepsilon \zeta_i^k. \tag{21}$$

The quantity ζ_i^k is obviously a random variable whose probability distribution depends on the strategy combination $\bar{s}_i$ used by the other $(n-1)$ players. Corresponding to the K_i pure strategies $a_i^1, \ldots, a_i^{K_i}$ available to any given player i, Eq. (20) defines K_i random variables $\zeta_i^1, \ldots, \zeta_i^{K_i}$ for him. Let ζ_i be the random vector

$$\zeta_i = (\zeta_i^1, \ldots, \zeta_i^{K_i}), \quad i = 1, \ldots, n. \tag{22}$$

Clearly, for each player $i (i = 1, \ldots, n)$, the probability distribution $G_i = G_i(\zeta_i | \bar{s}_i)$ of this random vector will once more depend on $\bar{s}_i$. More particularly, let $\Omega_i(\bar{s}_i)$ be the set of all points $\xi_i = (\xi_i^1, \ldots, \xi_i^K)$ satisfying the K_i inequalities of the form

$$\sum_m [q_i^m(a_i^k) \prod_{j \neq i} q_j^m(s_j)] \xi_i^m \leqq \zeta_i^k, \quad \text{for} \quad k = 1, \ldots, K_i. \tag{23}$$

Then we can define

$$G_i(\zeta_i|\bar{s}_i) = G_i(\zeta_i^1, \ldots, \zeta_i^{K_i}|\bar{s}_i) = \int_{\xi_i \in \Omega_i(\bar{s}_i)} \mathrm{d}\xi_i^1 \cdots \int \mathrm{d}\xi_i^K f_i(\xi_i)\,. \tag{24}$$

For any random variable or random vector, we define its *range space* as the set of all possible values it can take in game Γ^*. Thus, we define the range spaces $\Xi_i^m = \{\xi_i^m\}$, $Z_i^k = \{\zeta_i^k\}$, $\Xi_i = \{\xi_i\}$, and $Z_i = \{\zeta_i\}$, with $i = 1, \ldots, n$; $m = 1, \ldots, K$; $k = 1, \ldots, K_i$. In view of (15), we have

$$\Xi_i^m \subseteq I = [-1, +1] \quad \text{for all } i \text{ and } m\,, \tag{25}$$

which, by (20), in turn implies

$$Z_i^k \subseteq I = [-1, +1] \quad \text{for all } i \text{ and } k\,. \tag{26}$$

Moreover, $\Xi_i = \Xi_i^1 \times \cdots \times \Xi_i^K$ and $Z_i = Z_i^1 \times \cdots \times Z_i^{K_i}$.

4. Strategies and Equilibrium Points in Disturbed Games

The disturbed game $\Gamma^*(\varepsilon)$ as defined in Section 3 is not in normal form. One reason for this is that our definitions have not made explicit the dependence of each player's strategy choice on his random payoff-disturbance vector ξ_i. We shall now define *normalized* strategies (i.e., strategies appropriate to the normal form of $\Gamma^*(\varepsilon)$).[5]) In contrast, the pure strategies a_i^k and the mixed strategies s_i introduced in Section 2 will be called *ordinary* strategies. When we speak of a pure or mixed strategy without further specification we shall always mean an ordinary pure or mixed strategy a_i^k or s_i.

A *normalized pure strategy*, or *n-pure strategy*, a_i^* of any player i ($i = 1, \ldots, n$) will be defined as a measurable function from the range space Ξ_i of player i's random vector ξ_i to the set A_i of his ordinary pure strategies a_i^k. For any specific value of the random vector ξ_i, this n-pure strategy a_i^* will select some specific ordinary pure strategy

$$a_i^k = a_i^*(\xi_i) \tag{27}$$

as the strategy to be used by player i whenever ξ_i takes this particular value. Let

$$X_i^k = X_i^k(a_i^*) = \{\xi_i | \xi_i \in \Xi_i \quad \text{and} \quad a_i(\xi_i) = a_i^k\}\,. \tag{28}$$

Thus, X_i^k is the set of all possible ξ_i values that will make player i choose the ordinary pure strategy a_i^k if he follows the n-pure strategy a_i^*. Clearly, any n-pure strategy a_i^* can also be defined as a partitioning of the range space Ξ_i into K_i disjoint measurable subsets $X_i^1, \ldots, X_i^{K_i}$. (Of these K_i sets, as many as $(K_i - 1)$ sets may be empty.) The set of all n-pure strategies a_i^* will be called A_i^*.

An n-pure strategy a_i^* is called a *constant* n-pure strategy and will be denoted as $a_i^* = [a_i^k]$ if it assigns the *same* ordinary pure strategy $a_i^k = a_i^*(\xi_i)$ to all possible ξ_i values, $\xi_i \in \Xi_i$.

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[5]) However, for convenience, we shall keep the payoff functions U_i in their present form and shall not introduce normalized payoff functions (which would have to be defined as the expected values of the U_i's). That is, we shall use the semi-normal form of $\Gamma^*(\varepsilon)$ [HARSANYI, 1967–68, p. 182].

Let $D(a_i^*, a_i^{*\prime})$ be the set of all ξ_i values where the two n-pure strategies a_i^* and $a_i^{*\prime}$ disagree, i.e., where $a_i^*(\xi_i) \neq a_i^{*\prime}(\xi_i)$. Two n-pure strategies a_i^* and $a_i^{*\prime}$ will be called essentially identical if $D(a_i^*, a_i^{*\prime})$ is a set of measure zero. In view of (16), if two n-pure strategies are essentially identical, then with probability one they will prescribe choosing the same ordinary pure strategies. Two n-tuples of n-pure strategies, $a^* = (a_1^*, \dots, a_n^*)$ and $a^{*\prime} = (a_1^{*\prime}, \dots, a_n^{*\prime})$, are called *essentially identical* if their corresponding components a_i^* and $a_i^{*\prime}$ are essentially identical (for $i = 1, \dots, n$). If two n-pure strategies, or two n-tuples of such strategies, are *not* essentially identical, then they are called *essentially different.*

A *normalized mixed strategy*, or *n-mixed strategy*, s_i^* of any player i is a probability distribution over the set A_i^* of his n-pure strategies. As A_i^* is a function space, proper definition of such probability distributions poses certain technical difficulties, which can be overcome in various ways [AUMANN, 1960b, 1961]. We shall not go into these problems because we are introducing n-mixed strategies only in order to show that there is no need for them in disturbed games $\Gamma^*(\varepsilon)$. We shall simply assume that any n-mixed strategy s_i^* to be discussed has been defined in a mathematically satisfactory way.

If an n-mixed strategy s_i^* is a probability mixture of two or more essentially identical strategies $a_i^*, a_i^{*\prime}, \dots$, then s_i^* will be called an *essentially pure* normalized strategy, and will be regarded as being essentially identical with each one of these n-pure strategies $a_i^*, a_i^{*\prime}, \dots$.

Lemma 1:

If player i follows the n-pure strategy a_i^* then he will use any ordinary pure strategy a_i^k of his with the probability

$$p_i^k = \pi_i^k(a_i^*) = \int\limits_{\xi_i \in X_i^k(a_i^*)} \mathrm{d}\xi_i^1 \dots \int \mathrm{d}\xi_i^K f_i(\xi_i), \quad k = 1, \dots, K_i, \tag{29}$$

where, for each particular value of k, $X_i^k(a_i^*)$ is the set defined by (28). Therefore, his behavior will be such as if he followed the ordinary mixed strategy

$$s_i = \pi(a_i^*) = \sum_k p_i^k a_i^k = \sum_k \pi_i^k(a_i^*) a_i^k . \tag{30}$$

This strategy $s_i = \pi(a_i^*)$ will be called the mixed strategy *induced* by the n-pure strategy a_i^*.

Proof:

In view of (27), if player i follows the n-pure strategy a_i^*, then the ordinary pure strategy he will be using will become a function of the random vector ξ_i. He will use a_i^k whenever $\xi_i \in X_i^k(a_i^*)$. But the probability of this event is given by (29).

Lemma 1 directly implies:

Lemma 2:

Let s_i^* be an n-mixed strategy of player i, representing a probability mixture of two or more n-pure strategies $a_i^*, a_i^{*\prime}, \dots$. Let $s_i = \pi(a_i^*)$, $s_i' = \pi(a_i^{*\prime}), \dots$ be the ordinary mixed strategies induced by these component strategies $a_i^*, a_i^{*\prime}, \dots$. Let

t_i be the ordinary mixed strategy representing a probability mixture of these induced mixed strategies $s_i, s_i', \ldots$ such that assigns the same probability weights (or the same probability densities) to $s_i, s_i', \ldots$ as s_i^* itself assigns to the corresponding n-pure strategies $a_i^*, a_i^{*\prime}, \ldots$. Then, whenever player i follows this n-mixed strategy s_i^*, he will be using all his ordinary pure strategies a_i^k with probabilities p_i^k corresponding to the ordinary mixed strategy t_i.

This strategy t_i will be called the ordinary mixed strategy *induced* by s_i^*, and we shall write $t_i = \pi(s_i^*)$.

Let $a^* = (a_1^*, \ldots, a_n^*)$ and let $s_i = \pi(a_i^*)$ be the mixed strategy induced by a_i^*, for $i = 1, \ldots, n$. Let $s = (s_1, \ldots, s_n)$. Then we shall write $s = \pi(a^*)$. Moreover, let $\overline{a_i^*} = (a_1^*, \ldots, a_{i-1}^*, a_{i+1}^*, \ldots, a_n^*)$ and $\bar{s}_i = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$. Then we shall write $\bar{s}_i = \pi(\overline{a_i^*})$.

Likewise, let $s^* = (s_1^*, \ldots, s_n^*)$ and let $t_i = \pi(s_i^*)$ for $i = 1, \ldots, n$. Let $t = (t_1, \ldots, t_n)$. Then we shall write $t = \pi(s^*)$. Finally, let $\overline{s_i^*} = (s_1^*, \ldots, s_{i-1}^*, s_{i+1}^*, \ldots, s_n^*)$ and $\bar{t}_i = (t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)$. Then we shall write $\bar{t}_i = \pi(\overline{s_i^*})$.

Lemma 3:

Suppose that player i uses the n-pure strategy a_i^*, and therefore chooses the ordinary pure strategy $a_i^k = a_i^*(\xi_i)$ at the point ξ_i. If at the same time the other $(n - 1)$ players use the n-pure-strategy combination $\overline{a_i^*}$ then player i will obtain the (expected) payoff

$$U_i(a_i^*, \overline{a_i^*}) = U_i(a_i^k, \bar{s}_i) = V_i(a_i^k, \bar{s}_i) + \varepsilon \zeta_i^k(\xi_i, \bar{s}_i)\,, \tag{31}$$

where

$$\bar{s}_i = \pi(\overline{a_i^*}) \tag{32}$$

and where $\zeta_i^k = \zeta_i^k(\xi_i, \bar{s}_i)$ is the random variable defined by (20). On the other hand, if the other $(n - 1)$ players use the n-mixed-strategy combination $\overline{s_i^*}$ then he will obtain the payoff

$$U_i(a_i^*, \overline{s_i^*}) = U_i(a_i^k, \bar{t}_i) = V_i(a_i^k, \bar{t}_i) + \varepsilon \zeta_i^k(\xi_i, \bar{t}_i)\,, \tag{33}$$

where

$$\bar{t}_i = \pi(\overline{s_i^*})\,. \tag{34}$$

Proof:

By Lemmas 1 and 2, each player $j \neq i$ will use his ordinary pure strategies with probabilities corresponding to the mixed strategy $s_j = \pi(a_j^*)$, or to the mixed strategy $t_j = \pi(s_j^*)$. On the other hand, player i himself will use the ordinary pure strategy $a_i^k = a_i^*(\xi_i)$. Therefore, in view of (21), Eqs. (31) and (33) follow.

A given n-pure strategy a_i^* of player i will be called a *(uniformly) best reply* to some combination $\overline{a_i^*}$ of n-pure strategies (or to some combination $\overline{s_i^*}$ of n-mixed strategies) used by the other $(n - 1)$ players if, for *every* possible value of the random vector ξ_i, this strategy a_i^* maximizes player i's payoff U_i as specified by (31) (or by (33)) when the strategy combination $\overline{a_i^*}$ (or $\overline{s_i^*}$) is kept constant. In view of (31) and (33), in this case we shall also say that a_i^* is a best reply to the combination $\bar{s}_i = \pi(\overline{a_i^*})$ (or to the combination $\bar{t}_i = \pi(\overline{s_i^*})$) of ordinary mixed strategies.

Equilibrium points in a disturbed game $\Gamma^*(\varepsilon)$ will be defined in terms of this concept of uniformly best replies.

Theorem 1:

The best reply of any player i to some combination $\overline{a_i^*}$ of n-pure strategies, or to some combination $\overline{s_i^*}$ of n-mixed strategies, used by the other $(n-1)$ players is always an *essentially unique* n-pure strategy a_i^*. (That is, if *both* a_i^* and $a_i^{*\prime}$ are best replies to $\overline{a_i^*}$ or to $\overline{s_i^*}$ then a_i^* and $a_i^{*\prime}$ must be essentially identical.)

Proof:

We shall discuss only the case where the other $(n-1)$ players use some combination $\overline{a_i^*}$ of n-pure strategies. The proof is basically the same in the case where they use some combination $\overline{s_i^*}$ of n-mixed strategies.

In view of (31), for any specific value of the random vector ξ_i, a given ordinary pure strategy a_i^k will maximize player i's payoff U_i if and only if

$$V_i(a_i^k,\overline{s_i}) - V_i(a_i^{k'},\overline{s_i}) \geqq \varepsilon\cdot(\zeta_i^{k'} - \zeta_i^k), \quad \text{for all} \quad k' \neq k. \tag{35}$$

This maximizing strategy a_i^k will be unique only if

$$V_i(a_i^k,\overline{s_i}) - V_i(a_i^{k'},\overline{s_i}) > \varepsilon\cdot(\zeta_i^{k'} - \zeta_i^k), \quad \text{for all} \quad k' \neq k. \tag{36}$$

On the other hand, two different strategies a_i^k and $a_i^{k'}$ can *both* yield a maximal payoff only if

$$V_i(a_i^k,\overline{s_i}) - V_i(a_i^{k'},\overline{s_i}) = \varepsilon\cdot(\zeta_i^{k'} - \zeta_i^k). \tag{37}$$

But in view of (20), for any given $\overline{s_i}$, the variables ζ_i^k and $\zeta_i^{k'}$ are convex combinations of the variables $\xi_i^1,\dots,\xi_i^K$. Consequently, for any given choice of $\overline{s_i}$, and of a_i^k and $a_i^{k'}$, the locus $L^{kk'}$ of all points $\xi_i = (\xi_i^1,\dots,\xi_i^K)$ satisfying (20) and (37) is a $(K-1)$-dimensional hyperplane. Let $\hat{L}^{kk'}$ be the intersection of $L^{kk'}$ with the K-dimensional range space Ξ_i of the random vector ξ_i. Clearly, $\hat{L}^{kk'}$ will be a set of measure zero in this range space Ξ_i. Let L^* be the union of the $K_i(K_i-1)/2$ sets $\hat{L}^{kk'}$ defined for all possible pairs (k,k') with $k,k' = 1,\dots,K_i$ and with $k' \neq k$. Obviously, L^* will be again a set of measure zero in Ξ_i.

Let $Y_i^k = Y_i^k(\overline{s_i})$ be the set of all points $\xi_i = (\xi_i^1,\dots,\xi_i^K)$ satisfying (20) and (36) for a given strategy a_i^k of player i. Y_i^k will be a relatively open polyhedral subset of Ξ_i, bounded by the various hyperplanes $L^{kk'}$, $L^{kk''}$, etc. (For some values of k, Y_i^k may be empty.)

Any n-pure strategy a_i^* can be a best reply to $\overline{a_i^*}$ only if it makes player i choose his ordinary pure strategy in such a way that

$$a_i^*(\xi_i) = a_i^k, \quad \text{for all} \quad \xi_i \in Y_i^k(\overline{s_i}), \qquad \text{for all} \quad k = 1,\dots,K. \tag{38}$$

This means that if two n-pure strategies a_i^* and $a_i^{*\prime}$ are both best replies to $\overline{a_i^*}$ then they must agree over all sets $Y_i^k (k = 1,\dots,K_i)$ and can disagree at most only over the set L^* of measure zero (or over some subset of L^*). Consequently, a_i^* and $a_i^{*\prime}$ will be essentially identical.

Corollary to Theorem 1:

An n-mixed strategy s_i^* can be a best reply only if it is an *essentially pure* strategy (i.e., it must be a probability mixture of essentially identical n-pure strategies).

This corollary and Lemma 3 in turn imply:

Theorem 2:

In a disturbed game $\Gamma^*(\varepsilon)$ every equilibrium point must be in pure – or at least in essentially pure – normalized strategies. Moreover, for every equilibrium point $s^* = (s_1^*, \ldots, s_n^*)$ partly or wholly in mixed strategies (as we have seen, the latter must be "essentially pure" mixed strategies), there exists an essentially identical equilibrium point $a^* = (a_1^*, \ldots, a_n^*)$ wholly in pure strategies, yielding every player i the same payoff $U_i(a^*) = U_i(s^*)$.

Theorem 2 and the Corollary to Theorem 1 can be regarded as extensions of the results obtained by BELLMAN and BLACKWELL [1949] and by BELLMAN [1952]: if the game itself already contains enough "randomness" (random variables) then the players themselves need not – indeed, should not – introduce any additional randomness by a use of mixed strategies.

Theorem 1 also implies:

Theorem 3:

Every equilibrium point $a^* = (a_1^*, \ldots, a_n^*)$ of a disturbed game is *essentially strong*, in the sense that no player i can counter the strategy combination $\overline{a_i^*}$ of the other $(n - 1)$ players by a best reply essentially different from his equilibrium strategy a_i^*.

Note:

If a given player i does shift from his equilibrium strategy a_i^* to another best-reply strategy $a_i^{*\prime}$ essentially identical to a_i^*, then this will not destabilize the situation. This is so because, by Lemma 3, this shift will leave his own and all other players' payoffs unchanged, and the resulting new strategy n-tuple $a^{*\prime} = (a_1^*, \ldots, a_{i-1}^*, a_i^{*\prime}, a_{i+1}^*, \ldots, a_n^*)$ will again be an equilibrium point. Therefore this shift will not give the other players any incentive to change their own strategies.

Theorem 4:

Every disturbed game $\Gamma^*(\varepsilon)$ has at least one equilibrium point.

Proof:

Let $s = (s_1, \ldots, s_n)$ be an n-tuple of ordinary mixed strategies. For each player i, let a_i^* be his best reply to $\overline{s_i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$ in the sense that, for every possible value of the random vector ξ_i, the ordinary pure strategy $a_i^k = (a_i^*(\xi_i)$ prescribed by a_i^* will satisfy (35) with respect to all $a_i^{k'} \neq a_i^k$. Let a^* be the n-tuple $a^* = (a_1^*, \ldots, a_n^*)$. Finally, let $s' = (s_1', \ldots, s_n')$ be the n-tuple of ordinary mixed strategies induced by a^*, i.e., let $s' = \pi(a^*)$.

Now consider the mapping M: $s \to s'$. Clearly, M is a mapping of the set S of all possible n-tuples of ordinary mixed strategies into itself. This set S is a finite-dimensional convex and compact set.

Moreover, the mapping M is single-valued. Even though in general there will be many n-pure strategies a_i^* that are best replies to a given $(n-1)$-tuple $\bar{s}_i$, all of these strategies a_i^* will be essentially identical (see the proof of Theorem 1). Therefore, all of them will induce exactly the same ordinary mixed strategy $s_i' = \pi(a_i^*)$.

Finally, the mapping M is continuous. For any given n-tuple s, the n-tuple $s' = M(s)$ is defined by conditions (29) and (30). But in (29) the integrand is the continuous function f_i whereas the region of integration, the set $X_i^k(a_i^*)$, can be written as

$$X_i^k(a_i^*) = Y_i^k(\bar{s}_i) \cup D_i^k(\bar{s}_i), \tag{39}$$

where Y_i^k is defined by conditions (20) and (36), whereas D_i^k is a set of measure zero, and therefore does not affect the result of the integration in (29). On the other hand, the set Y_i^k itself depends continuously on the n-tuple s in terms of the Hausdorff metric. This continuous dependence of Y_i^k on s and the continuity of f_i establish the continuity of the mapping M.

Consequently, by Brouwer's Fixed-point Theorem [see, e.g., LEFSCHETZ, 1949, p. 117], the mapping M will have at least one fixed point where $s' = M(s) = s$. Yet, any n-tuple a^* corresponding to such a fixed point $s' = s$ will be an equilibrium point in game $\Gamma^*(\varepsilon)$. First of all, by the definition of M, each component a_i^* of a^* will have the property that $a_i^k = a_i^*(\xi_i)$ will satisfy (35) with respect to all $a_i^{k'} \neq a_i^k$ for all values of ξ_i if we set $\bar{s}_i = \bar{s}_i' = \pi(\bar{a}_i^*)$. But, by Lemma 3, this very property establishes the fact that a_i^* is a best reply to $\bar{a}_i^*$. As this is true for all players i, a^* will be an equilibrium point in $\Gamma^*(\varepsilon)$.

Let $a^* = (a_1^*, \ldots, a_n^*)$ be an equilibrium point in some disturbed game $\Gamma^*(\varepsilon)$, and let $s = (s_1, \ldots, s_n = \pi(a^*)$ be the n-tuple of ordinary mixed strategies induced by a^*. Then s will be called the *s-representation* of the equilibrium point a^*. We shall also say that s is an *s-equilibrium point* in game $\Gamma^*(\varepsilon)$.

5. Equilibrium Points in Ordinary Games as Limits of *s*-Equilibrium Points in Disturbed Games

Let $\{\Gamma^*(\varepsilon)\}$ be a one-parameter family of disturbed games, each game $\Gamma^*(\varepsilon)$ being characterized by a different value of the parameter ε, but all of them having the same n-tuple of basic payoff functions $V_1, \ldots, V_n$ and the same n-tuple of probability distributions $F_1, \ldots, F_n$. If we set $\varepsilon = 0$ then, in view of (12), we obtain an ordinary (undisturbed) game $\Gamma = \Gamma^*(0)$ with the payoff functions $U_1 = V_1, \ldots, U_n = V_n$.

Theorem 5:

Let $\{s(\varepsilon)\}$ be a one-parameter family of n-tuples of ordinary mixed strategies such that, for all ε values in some open interval $(0, \varepsilon^*)$ with $\varepsilon^* > 0$, each n-tuple $s(\varepsilon) = (s_1(\varepsilon), \ldots, s_n(\varepsilon))$ is an s-equilibrium point in the corresponding game $\Gamma^*(\varepsilon)$. Suppose that

$$\lim_{\varepsilon \to 0} s(\varepsilon) = s = (s_1, \ldots, s_n). \tag{40}$$

Then s will be an equilibrium point in game $\Gamma = \Gamma^*(0)$.

Proof:

In view of (40), for any sufficiently small ε we must have

$$C(s_i) \subseteq C(s_i(\varepsilon)), \quad \text{for} \quad i = 1, \ldots, n; \tag{41}$$

where $C(s_i)$ and $C(s_i(\varepsilon))$ are the carriers of strategies s_i and $s_i(\varepsilon)$, respectively. Since $s(\varepsilon)$ is an s-equilibrium point in $\Gamma^*(\varepsilon)$, any ordinary pure strategy a_i^k in the carrier $C(s_i(\varepsilon))$ must satisfy requirement (34) with respect to all other pure strategies $a_i^{k'}$ of player i over some nonempty set Y_i^k of possible ξ_i values. When ε goes to zero, requirement (35) will take the form

$$V_i(a_i^k, \bar{s}_i) - V_i(a_i^{k'}, \bar{s}_i) \geqq 0, \quad \text{for all} \quad k' \neq k. \tag{42}$$

In view of (41), this requirement will be satisfied for all pure strategies a_i^k in $C(s_i)$. But this is precisely the condition that must be satisfied if s is to be an equilibrium point in game $\Gamma = \Gamma^*(0)$. This establishes the theorem.

Let s be an equilibrium point in game $\Gamma = \Gamma^*(0)$. We shall say that s is *approachable* by some s-equilibrium point $s(\varepsilon)$ of a disturbed game $\Gamma^*(\varepsilon)$ if there is some s-equilibrium point $s(\varepsilon)$ (or, more exactly, if there is some family $\{s(\varepsilon)\}$ of s-equilibrium points) satisfying (40) with respect to this equilibrium point s.

Theorems 4 and 5 imply that every ordinary game $\Gamma = \Gamma^*(0)$ has at least *one* approachable equilibrium point s. We shall now show that every *strong* equilibrium point of any game Γ is always approachable, and that a *quasi-strong* equilibrium point is "almost always" approachable (in the sense that this property can fail only for a small class of games, corresponding to a closed set of measure zero in the space of all games of a given size).

Theorem 6:

Let $a = (a_1^{k_1}, \ldots, a_n^{k_n})$ be a *strong* equilibrium point[6]) in game $\Gamma = \Gamma^*(0)$. Let $a^* = (a_1^*, \ldots, a_n^*)$ be an n-tuple of *constant* n-pure strategies such that $a_i^* = [a_i^{k_i}]$ for $i = 1, \ldots, n$. Then, for any small enough ε, a^* will be an equilibrium point in game $\Gamma^*(\varepsilon)$.

Proof:

Let $\bar{a}_i$ be the $(n-1)$-tuple we obtain if we omit the i-th component, the strategy $a_i^{k_i}$, from the n-tuple a. Since a is a strong equilibrium point in game Γ, we must have

$$V_i(a_i^{k_i}, \bar{a}_i) - V_i(a_i^{k'}, \bar{a}_i) > 0, \quad \text{for all} \quad k' \neq k_i, \quad \text{and for} \quad i = 1, \ldots, n. \tag{43}$$

Consequently, for any small enough ε, we can write

$$V_i(a_i^{k_i}, \bar{a}_i) - V_i(a_i^{k'}, \bar{a}_i) \geqq \varepsilon \cdot (\zeta_i^{k_i} - \zeta_i^{k'}) \tag{44}$$

[6]) A strong equilibrium point is always in pure strategies.

for all possible values of $\zeta_i^{k_i}$ and $\zeta_i^{k'}$. Hence, $a_i^{k_i}$ will satisfy (35), and therefore $a_i^{k_i}$ will be a best reply to the strategy combination $\overline{a}_i$ used by the other $(n-1)$ players. As this will be true for all players i, this means that a will be an equilibrium point in any game $\Gamma^*(\varepsilon)$ with a small enough ε.

Consider the constant n-pure strategy $a_i^* = [a_i^{k_i}]$. Clearly, if player i follows this constant strategy then he will choose the corresponding ordinary pure strategy $a_i^{k_i}$ with probability one. Consequently, the "mixed" strategy induced by this n-pure strategy $a_i^* = [a_i^{k_i}]$ is simply the pure strategy $a_i^{k_i}$ itself, so that we can write $\pi(a_i^*) = a_i^{k_i}$. Accordingly, the s-equilibrium point corresponding to the equilibrium point a^* mentioned in Theorem 6 will be simply the n-tuple $a = \pi(a^*)$ of ordinary pure strategies.

Lemma 4:

Let $s = (s_1, \ldots, s_n)$ be an n-tuple of ordinary mixed strategies, and suppose that the probabilities p_i^k characterizing these mixed strategies s_i satisfy the equation

$$p_i^k = \underset{\zeta_i \in \Psi_i^k}{\int d\zeta_i^1 \cdots \int} d\zeta_i^{K_i} g_i(\zeta_i | \overline{s}_i), \quad \text{for} \quad i = 1, \ldots, n; \quad k = 1, \ldots, K_i; \tag{45}$$

where

$$g_i(\zeta_i | \overline{s}_i) = g_i(\zeta_i^1, \ldots, \zeta_i^{K_i} | \overline{s}_i) = \frac{\partial^{K_i} G_i(\zeta_i | \overline{s}_i)}{\partial \zeta_i^1, \ldots, \partial \zeta_i^{K_i}} \tag{46}$$

is the probability density function corresponding to the probability distribution $G_i(\zeta_i | \overline{s}_i)$ defined by (24); whereas, Ψ_i^k is the set of all points $\zeta_i = (\zeta_i^1, \ldots, \zeta_i^{K_i})$ satisfying Condition (36) for a given positive value of ε, and for a given pure strategy a_i^k, with respect to all other strategies $a_i^{k'}$, $k' \neq k$, of the same player i.

Then s will be an s-equilibrium point in game $\Gamma^*(\varepsilon)$.

Proof:

For each player i, let a_i^* be his best reply to $\overline{s}_i = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$, in the sense specified in the proof of Theorem 4. Let $a^* = (a_1^*, \ldots, a_n^*)$. We shall show that a^* is an equilibrium point in game $\Gamma^*(\varepsilon)$, and that s is the corresponding s-equilibrium point. As a_i^* is a best reply to $\overline{s}_i$, it will select any given pure strategy a_i^k whenever a_i^k satisfies Condition (36). Therefore a_i^* will choose each pure strategy a_i^k with the probability p_i^k defined by Eq. (45). (To be sure, a_i^* may make player i choose this pure strategy a_i^k also at some points ξ_i where (36) is not satisfied, though the weaker Condition (35) is. But the total probability mass associated with such points ξ_i will always be zero.) Consequently, for each player i, a_i^* will induce the mixed strategy s_i corresponding to the probabilities p_i^k, so that $s_i = \pi(a_i^*)$. Therefore, the strategy n-tuple s will be a fixed point $s = M(s)$ of the mapping M defined in the proof of Theorem 4, which implies, as we have seen, that a^* is an equilibrium point in $\Gamma^*(\varepsilon)$ whereas $s = \pi(a^*)$ is the corresponding s-equilibrium point.

Let $s = (s_1, \ldots, s_n)$ be an n-tuple of ordinary mixed strategies. Suppose that the carriers $C(s_1), \ldots, C(s_n)$ of the strategies $s_1, \ldots, s_n$ contain $\gamma_1, \ldots, \gamma_n$ pure strategies, respectively. Then, without loss of generality, we can introduce the following notational convention:

The pure strategies a_i^k of each player i are numbered in such a way that the carrier $C(s_i)$ of his mixed strategy s_i will contain his *first* γ_i pure strategies $a_i^1, \ldots, a_i^{\gamma_i}$. (47)

Clearly, the $(\gamma_i - 1)$ probabilities $p_i^1, \ldots, p_i^{\gamma_i - 1}$ fully characterize s_i since

$$p_i^{\gamma_i} = 1 - \sum_{k=1}^{\gamma_i - 1} p_i^k$$

and

$$p_i^k = 0, \quad \text{for} \quad k = \gamma_i + 1, \ldots, K_i.$$

Let p_i denote the vector

$$p_i = (p_i^1, \ldots, p_i^{\gamma_i - 1}). \tag{48}$$

Let $\overline{p_i}$ denote the composite vector

$$\overline{p_i} = (p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n), \tag{49}$$

where $p_1, \ldots, p_n$ are the vectors defined by (48). In order to make the dependence of the functions V_i and g_i on the probabilities p_i^k more explicit, we shall now write

$$V_i(a_i^k, \overline{s_i}) = V_i(a_i^k, \overline{p_i}) \tag{50}$$

and

$$g_i(\zeta_i | \overline{s_i}) = g_i(\zeta_i | \overline{p_i}). \tag{51}$$

For each player i $(i = 1, \ldots, n)$, we introduce the random variables

$$\delta_i^{kk'} = \zeta_i^{k'} - \zeta_i^k, \quad \text{for} \quad k, k' = 1, \ldots, K_i; \ k' \neq k; \tag{52}$$

as well as the random vectors

$$\delta_i^k = (\delta_i^{k1}, \delta_i^{k2}, \ldots, \delta_i^{k(k-1)}, \delta_i^{k(k+1)}, \delta_i^{k(k+2)}, \ldots, \delta_i^{kK_i}), \quad \text{for} \quad k = 1, \ldots, \gamma_i - 1. \tag{53}$$

For convenience, we shall also write

$$\delta_i^{kk} = \zeta_i^k - \zeta_i^k = 0. \tag{54}$$

Lemma 5:

Let $s = (s_1, \ldots, s_n)$ be an n-tuple of ordinary mixed strategies. Suppose that for each player i $(i = 1, \ldots, n)$ the probabilities p_i^k characterizing his mixed strategy s_i, together with some appropriately chosen quantities y_i^k and $z_i^{kk'}$, satisfy the following conditions with respect to a specific positive value of ε:

$$\varepsilon y_i^k = V_i(a_i^k, \overline{p_i}) - V_i(a_i^{k+1}, \overline{p_i}), \quad \text{for} \quad k = 1, \ldots, \gamma_i - 1; \tag{55}$$

$$z_i^{kk'} = \sum_{j=k}^{k'-1} y_i^j, \quad \text{for} \quad k = 1, \ldots, \gamma_i - 1; \quad k' = k + 1, \ldots, \gamma_i; \tag{56}$$

$$z_i^{kk'} = - \sum_{j=k'}^{k-1} y_i^j, \quad \text{for} \quad k = 2, \ldots, \gamma_i; \quad k' = 1, \ldots, k - 1; \tag{57}$$

$$p_i^k = H_i^k(z_i^{k1}, \ldots, z_i^{k(k-1)}, z_i^{k(k+1)}, \ldots, z_i^{k\gamma_i}; \overline{p_i}) = \tag{58}$$

$$= \int_{-\infty}^{\sigma_1} \mathrm{d}\,\delta_i^{k1} \ldots \int_{-\infty}^{\sigma_{k-1}} \mathrm{d}\,\delta_i^{k(k-1)} \int_{-\infty}^{\sigma_{k+1}} \mathrm{d}\,\delta_i^{k(k+1)} \ldots \int_{-\infty}^{\sigma_{K_i}} \mathrm{d}\,\delta_i^{kK_i} \int_{-\infty}^{+\infty} \mathrm{d}\zeta_i^k h_i^k(\delta_i^k;\zeta_i^k),$$

where

$$\sigma_j = z_i^{kj}, \quad \text{for} \quad j = 1,\ldots,k-1,k+1,\ldots,\gamma_i; \tag{59}$$

but

$$\sigma_j = +\infty, \quad \text{for} \quad j = \gamma_i + 1,\ldots,K_i; \tag{60}$$

whereas

$$\begin{aligned} h_i^k(\delta_i^k;\zeta_i^k) &= h_i^k(\delta_i^{k1},\ldots,\delta_i^{k(k-1)},\delta_i^{k(k+1)},\ldots,\delta_i^{kK_i};\zeta_i^k) = \\ &= g_i(\zeta_i^k + \delta_i^{k1},\ldots,\zeta_i^k + \delta_i^{kj},\ldots,\zeta_i^k + \delta_i^{kK_i}|\bar{p}_i); \end{aligned} \tag{61}$$

and finally

$$V_i(a_i^k,\bar{p}_i) - V_i(a_i^{k'},\bar{p}_i) > \varepsilon\cdot(\zeta_i^{k'} - \zeta_i^k) \tag{62}$$

for all possible values of $\zeta_i^{k'}$ and $\zeta_i^k(\zeta_i^{k'} \in Z_i^{k'}$ and $\zeta_i^k \in Z_i^k)$; and for $k = 1,\ldots,\gamma_i$; $k' = \gamma_i + 1,\ldots,K_i$.

Then s will be an s-equilibrium point in game $\Gamma^*(\varepsilon)$. Moreover, the quantities $z_i^{kk'}$ will satisfy

$$-2 < z_i^{kk'} < +2. \tag{63}$$

Proof:

As $\varepsilon > 0$, (55), (56), and (57) imply that

$$z_i^{kk'} = [V_i(a_i^k,\bar{p}_i) - V_i(a_i^{k'},\bar{p}_i)]/\varepsilon. \tag{64}$$

Hence, (58) can be obtained from (45) by a change of variables. In view of (36), (52), and (64), for each variable $\delta_i^{kk'}$ with $k,k' = 1,\ldots,\gamma_i$ and with $k' \neq k$, the upper limit of integration is $\sigma_{k'} = z_i^{kk'}$. Moreover, since in this case both a_i^k and $a_i^{k'}$ are in $C(s_i)$, each upper limit $z_i^{kk'}$ will have the property that the two inequalities $\delta_i^{kk'} < z_i^{kk'}$ and $\delta_i^{kk'} > z_i^{kk'}$ are *both* satisfied with positive probabilities. Therefore, each quantity $z_i^{kk'}$ must lie in the interior of the range of the corresponding random variable $\delta_i^{kk'}$. By (26) and (52), this range is a subset of $[-2, +2]$, which implies (63).

In contrast, for the variables $\delta_i^{kk'}$ with $k \leqq \gamma_i < k'$, the upper limit of integration can be taken to be $\sigma_{k'} = +\infty$. This is so because now the inequality $\delta_i^{kk'} < \sigma_{k'}$ must be satisfied with probability one, since in this case, $a_i^k \in C(s_i)$, whereas $a_i^{k'} \notin C(s_i)$. This completes the proof.

Lemma 6:

Equation (58) can also be written as

$$p_i^k = E_i^k(y_i^1,\ldots,y_i^{\gamma_i-1};\bar{p}_i), \quad \text{for} \quad i = 1,\ldots,n; \quad k = 1,\ldots,\gamma_i - 1. \tag{65}$$

These functions E_i^k are continuously differentiable.

Proof:

(56) and (57) make the quantities $z_i^{kk'}$ functions of the variables y_i^k. Continuous differentiability follows from (17) and from Leibnitz's Rule [Olmsted, 1959, p. 417].

Lemma 7:

Let $s = (s_1, \ldots, s_n)$ be an n-tuple of ordinary mixed strategies, characterized by the probabilities p_i^k. Then, we can always find some quantities y_i^k and $z_i^{kk'}$ which, together with these probabilities p_i^k, will satisfy (56) to (61).

Proof:

First, we shall show that we can find quantities y_i^k satisfying (65) for $i = 1, \ldots, n$ and for $k = 1, \ldots, \gamma_i - 1$. Let I^* be the closed interval $I^* = [-2, +2]$. For each player i, we define a set $Y_i^* = I_1 \times \cdots \times I_{\gamma_i}$ where $I_1 = \cdots = I_{\gamma_i} = I^*$. For each vector $y_i^* = (y_i^1, \ldots, y_i^{\gamma_i})$ in Y_i^* we define a vector $\omega_i = (\omega_i^1, \ldots, \omega_i^{\gamma_i})$ as follows. Let

$$\pi_i^k = E_i^k(y_i^1, \ldots, y_i^{\gamma_i - 1}), \quad \text{for} \quad k = 1, \ldots, \gamma_i - 1; \tag{66}$$

$$\pi_i^{\gamma_i} = 1 - \sum_{k=1}^{\gamma_i - 1} \pi_i^k, \tag{67}$$

and

$$p_i^{\gamma_i} = 1 - \sum_{k=1}^{\gamma_i - 1} p_i^k. \tag{68}$$

Finally, we set

$$\omega_i^k = y_i^k[1 - \max(0, \pi_i^k - p_i^k)], \quad \text{for} \quad k = 1, \ldots, \gamma_i. \tag{69}$$

For each player i, Eqs. (66) to (69) establish a mapping $M_i : y_i^* \to \omega_i$. Clearly M_i is a mapping of the finite-dimensional convex and compact set Y_i^* into itself. Moreover, by Lemma 6, M_i is continuously differentiable and therefore continuous. Consequently, by Brouwer's Fixed-Point Theorem, M_i must have a fixed point where

$$\omega_i = M_i(y_i^*) = y_i^*. \tag{70}$$

For each player i, the first $(\gamma_i - 1)$ components $y_i^1, \ldots, y_i^{\gamma_i - 1}$ of any vector y_i^* satisfying (70) will satisfy (65). Moreover, if we substitute these variables y_i^k in (56) and (57), then we shall obtain quantities $z_i^{kk'}$ satisfying (58). This completes the proof.

Any vector y_i^* satisfying (70) must lie in the interier of Y_i^* [cf. the proof of Lemma 5, regarding Condition (63)]. Thus we can write

$$-2 < y_i^k < +2, \quad \text{for } i = 1, \ldots, n; \quad k = 1, \ldots, \gamma_i - 1. \tag{71}$$

(This inequality is, of course, true also for $k = \gamma_i$, but now we do not need this fact.)

We now define

$$\gamma^* = \sum_{i=1}^{n} (\gamma_i - 1) = \sum_{i=1}^{n} \gamma_i - n, \tag{72}$$

and

$$y_i = (y_i^1, \ldots, y_i^{\gamma_i - 1}), \quad \text{for} \quad i = 1, \ldots, n. \tag{73}$$

Let p and y denote the composite vectors

$$p = (p_1, \ldots, p_n) \tag{74}$$

and

$$y = (y_1, \ldots, y_n), \tag{75}$$

where $p_1, \ldots, p_n$ and $y_1, \ldots, y_n$ are the vectors defined by (48) and (73). Clearly, p has γ^* components of form p_i^k, and y has γ^* components of form y_i^k.

Let P be the set of all vectors p satisfying

$$p_i^k > 0, \quad \text{for} \quad i = 1, \ldots, n; \quad \text{and for } k = 1, \ldots, \gamma_i - 1; \tag{76}$$

and

$$\sum_{k=1}^{\gamma_i - 1} p_i^k < 1, \quad \text{for} \quad i = 1, \ldots, n. \tag{77}$$

Let Y be the set of all vectors y satisfying (71).

In vector notation, we can now write Eq. (65) as

$$p_i = E_i(y_i; \overline{p}_i), \quad \text{for} \quad i = 1, \ldots, n. \tag{78}$$

Lemma 8:

There exists a mapping $\tau: p \to y$ from set P to set Y with the following properties:
(a) For every p in P, the vectors p and $y = \tau(p)$ together satisfy (65).
(b) The set P^0 of all points p in P where τ fails to be continuously differentiable is closed and has measure zero as a subset of P.

Proof:

Let J be the Jacobian

$$J = \frac{\partial(E_1^1, \ldots, E_1^{\gamma_1 - 1}; \ldots; E_i^1, \ldots, E^{\gamma_i - 1}; \ldots; E_n^1, \ldots, E_n^{\gamma_n - 1})}{\partial(y_1^1, \ldots, y^{\gamma_1 - 1}; \ldots; y_i^1, \ldots, y_i^{\gamma_i - 1}; \ldots; y_n^1, \ldots, y_n^{\gamma_n - 1})}. \tag{79}$$

It is easy to verify that

$$J = \prod_{i=1}^{n} J_i, \tag{80}$$

where J_i is the smaller Jacobian

$$J_i = \frac{\partial(E_i^1, \ldots, E_i^{\gamma_i - 1})}{\partial(y_i^1, \ldots, y_i^{\gamma_i - 1})}, \quad \text{for} \quad i = 1, \ldots, n. \tag{81}$$

By Lemma 7, a mapping τ always exists. By Lemma 6 and the Implicit Function Theorem, τ can be chosen so as to be continuously differentiable at every point $(p, \tau(p))$, where the Jacobian J is nonzero; i.e., where all n Jacobians $J_1, \ldots, J_n$ are nonzero. We shall now show that this condition will be met everywhere except on a closed set P^0 of measure zero.

Let 0_i be the set of all points $(y_i, \overline{p}_i)$ where the Jacobian $J_i = J_i(y_i, \overline{p}_i)$ vanishes. Since J_i depends continuously on $(y_i, \overline{p}_i)$, and since both y_i and $\overline{p}_i$ are bounded, 0_i is a compact set.

Now, consider the mapping $\tau_i^*: (y_i, \overline{p}_i) \to (p_i, \overline{p}_i) = p$, where $p_i = E_i(y_i; \overline{p}_i)$. By Lemma 6, this mapping τ_i^* is continuously differentiable. Therefore, by Sard's Theorem [SARD, 1942], the set $P_i^0 = \tau_i^*(0_i)$ of all points p associated with zeroes of J_i is a set of measure zero in P. Moreover, as P_i^0 is the image of 0_i under the continuous mapping τ_i^*, P_i^0 itself is also a compact set, and therefore closed.

As each set P_i^0 is a closed set of measure zero, their union, the set

$$P^0 = \bigcup_{i=1}^{n} P_i^0 \tag{82}$$

will have the same properties. But continuous differentiability of τ can fail only on this set P^0. This establishes the lemma.

We now introduce the (nK)-vector

$$v = (v_1^1, \ldots, v_1^K; \ldots; v_i^1, \ldots, v_i^K; \ldots; v_n^1, \ldots, v_n^K). \tag{83}$$

Let us write Eq. (55) in the form

$$\begin{aligned} \varphi_i^k(p,y,v,\varepsilon) &= \varepsilon y_i^k - V_i(a_i^k,\bar{p}_i) + V_i(a_i^{k+1},\bar{p}_i) = \\ &= \varepsilon y_i^k - \sum_{m=1}^{K} \left[q_i^m(a_i^k) \prod_{j \neq i} q_j^m(s_j) \right] v_i^m \\ &+ \sum_{m=1}^{K} \left[q_i^m(a_i^{k+1}) \prod_{j \neq i} q_j^m(s_j) \right] v_i^m = 0, \quad \text{for} \quad i = 1, \ldots, n; \\ & \qquad k = 1, \ldots, \gamma_i - 1. \end{aligned} \tag{84}$$

In view of Lemma 8, this can also be written as

$$\varphi_i^k(p,\tau(p),v,\varepsilon) = \psi_i^k(p,v,\varepsilon) = 0. \tag{85}$$

Lemma 9:

Let $s = (s_1, \ldots, s_n)$ be a *quasi-strong* equilibrium point, characterized by the probability vector p, in game $\Gamma = \Gamma^*(0)$. Suppose that

(i) The mapping τ from P to Y is continuously differentiable on some neighborhood N of p; and that

(ii) The Jacobian

$$J^* = \frac{\partial(\psi_1^1, \ldots, \psi_1^{\gamma_1-1}; \ldots; \psi_i^1, \ldots, \psi_i^{\gamma_i-1}; \ldots; \psi_n^1, \ldots, \psi_n^{\gamma_n-1})}{\partial(p_1^1, \ldots, p_1^{\gamma_1-1}; \ldots; p_i^1, \ldots, p_i^{\gamma_i-1}; \ldots; p_n^1, \ldots, p_n^{\gamma_n-1})} \neq 0 \tag{86}$$

at the point p if we set $\varepsilon = 0$.

Then, there exists a family $\{s(\varepsilon)\}$ of n-tuples of ordinary strategies such that, for any sufficiently small positive ε, $s(\varepsilon)$ is an s-equilibrium point in game $\Gamma^*(\varepsilon)$, and such that $s(\varepsilon)$ and s satisfy (40).

Proof:

First, we propose to show that p satisfies (55) if we set $\varepsilon = 0$. The left-hand side of (55) now will be zero, and the same will be true about the right-hand side since s is an equilibrium point in Γ. Therefore, in view of (86) and the Implicit Function Theorem, for any small enough positive ε, there exists a vector $p(\varepsilon)$ satisfying (55) in conjunction with the vector $y(\varepsilon) = \tau(p(\varepsilon))$. This vector $p(\varepsilon)$ will depend continuously on ε. In view of Lemmas 7 and 8, these vectors $p(\varepsilon)$ and $y(\varepsilon)$, together with some appropriately chosen quantities $z_i^{kk'}(\varepsilon)$, will also satisfy Conditions (56) to (61). Finally, as s is a quasi-strong equilibrium point in Γ, we have

$$V_i(a_i^k,\bar{p}_i) - V_i(a_i^{k'},\bar{p}_i) > 0, \quad \text{for} \quad k = 1, \ldots, \gamma_i; \; k' = \gamma_i + 1, \ldots, K_i. \tag{87}$$

This is so because $a_i^k \in C(s_i)$ whereas $a_i^{k'} \notin C(s_i)$. Consequently, for any small enough positive ε, the strategy n-tuple $s(\varepsilon)$ defined by $p(\varepsilon)$ will satisfy (62). Thus, by Lemma 5, $s(\varepsilon)$ will be an s-equilibrium point in $\Gamma^*(\varepsilon)$. Finally, (40) follows from the continuous dependence of $s(\varepsilon)$ on ε.

We now define

$$m(i,1) = 1, \quad \text{for} \quad i = 1, \ldots, n; \tag{88}$$

$$m(1,k) = k, \quad \text{for} \quad k = 2, \ldots, \gamma_1 - 1; \tag{89}$$

and

$$m(i,k) = 1 + \sum_{j=1}^{i-1} (\gamma_j - 2) + k = \sum_{j=1}^{i-1} \gamma_j - 2i + k + 3, \tag{90}$$

$$\text{for} \quad i = 2, \ldots, n; \; k = 2, \ldots, \gamma_i - 1.$$

Thus, $m(i,k)$ ranges over all positive integers from $m(i,1) = 1$ to

$$m(n, \gamma_n - 1) = \sum_{j=1}^{n} \gamma_j - 2n + 3 = \gamma^* - n + 3 = \gamma^{**}. \tag{91}$$

In addition to (47), we now introduce the following notational convention, which again involves no loss of generality:

> The first γ^{**} pure-strategy n-tuples $a^1, \ldots, a^{\gamma^{**}}$ are numbered in such a way that for any $m^* = m(i,k)$ with $i = 1, \ldots, n$ and with $k = 1, \ldots, \gamma_i - 1$, we have
> $$a^{m^*} = (a_1^1, \ldots, a_{i-1}^1, a_i^k, a_{i+1}^1, \ldots, a_n^1). \tag{92}$$

We now define

$$v(i,k) = v_i^{m^*} = v_i^{m(i,k)} = V_i(a_1^1, \ldots, a_{i-1}^1, a_i^k, a_{i+1}^1, \ldots, a_n^1), \tag{93}$$

$$\text{for} \quad i = 1, \ldots, n; \; k = 1, \ldots, \gamma_i - 1.$$

Let v^* be the vector formed of those γ^* components of vector v which can be written in form (93). Let v^{**} be the vector formed of the remaining $(nK - \gamma^*)$ components of v. Thus

$$v = (v^*, v^{**}). \tag{94}$$

Let $\mathscr{I} = \mathscr{I}(n; K_1, \ldots, K_n)$ be the set of all ordinary (undisturbed) n-person games in which players $1, \ldots, n$ have exactly $K_1, \ldots, K_n$ strategies, respectively. Thus $\mathscr{I}$ is the set of all games of a given size. Each game Γ in $\mathscr{I}$ can be characterized by its vector $v = v(\Gamma)$ of possible payoffs for pure-strategy combinations. As v is an (nK)-vector, where $K = \prod_i K_i$, $\mathscr{I}$ can be regarded as an (nK)-dimensional Euclidean space, and we can write $\mathscr{I} = \{v\}$. Let $\bar{\mathscr{I}}$ be the set of all games Γ in $\mathscr{I}$ for which a given statement $\mathscr{S}$ is false. We shall say that $\mathscr{S}$ is true for *almost all* games Γ if, for every possible set $\mathscr{I} = \mathscr{I}(n; K_1, \ldots, K_n)$, this set $\bar{\mathscr{I}}$ is closed and has measure zero in $\mathscr{I}$. [As to the requirement of closure for $\bar{\mathscr{I}}$, cf. DEBREU, 1970, p 387.]

Theorem 7:

For *all* probability distributions $F_1, \ldots, F_n$ satisfying Conditions (15) to (17), and for *almost all* games Γ, the following statement is true. Let $s = (s_1, \ldots, s_n)$ be a *quasi-strong* equilibrium point in Γ. Let $\{\Gamma^*(\varepsilon)\}$ be a family of disturbed games characterized by these probability distributions $F_1, \ldots, F_n$ with $\Gamma^*(0) = \Gamma$. Then there exists a family $\{s(\varepsilon)\}$ of n-tuples of ordinary mixed strategies with the property that, for any small enough positive ε, $s(\varepsilon)$ is an s-equilibrium point in the disturbed game $\Gamma^*(\varepsilon)$, with

$$\lim_{\varepsilon \to 0} s(\varepsilon) = s. \tag{95}$$

Proof:

Each equation of form (55) (or, equivalently, of form (84) and (85)) can be written as

$$v^{m^*} = \frac{\varepsilon y_i^k - \sum_{\substack{1 \leqq m \leqq K \\ m \neq m^*}} \left[q_i^m(a_i^k) \prod_{j \neq i} q_j^m(s_j) \right] v_i^m + \sum_{\substack{1 \leqq m \leqq K \\ m \neq m^*}} \left[q_i^m(a_i^{k+1}) \prod_{j \neq i} q_j^m(s_j) \right] v_i^m}{\prod_{j \neq i} q_j^{m^*}(s_j)}, \tag{96}$$

where $m^* = m(i,k)$. Each equation of from (55) can be written in this way because $q_j^{m^*}(s_j) = p_j^1 > 0$ for all $j \neq i$ since $a_i^1 \in C(s_j)$. Moreover, $q_i^{m^*}(a_i^k) = 1$, whereas $q_i^{m^*}(a_i^{k+1}) = 0$ since the i-th component of the strategy combination a^{m^*} is a_i^k. Note that each quantity $v_i^{m^*}$ occurs only in *one* equation of form (96) with a nonzero coefficient. Consequently, the γ^* equations of form (96) define a continuously differentiable mapping ρ: $(p,y) \to v^*$, which is a mapping from $(P \times Y)$ to the γ^*-dimensional Euclidean space $\mathscr{I}^* = \{v^*\}$.

Let P^0 be the set defined by (82), and let $Y^0 = \tau(P^0)$. By the proof of Lemma 8, $(P^0 \times Y^0)$ is a compact set of measure zero in $(P \times Y)$. Therefore, its image $\mathscr{I}^{*0} = \rho(P^0 \times Y^0)$ under the continuously differentiable mapping ρ will again be a compact set with measure zero in $\mathscr{I}^*$. Let $\mathscr{I}^{**} = \{v^{**}\}$ be the set of all possible vectors v^{**}. $\mathscr{I}^{**}$ will be an $(nK - \gamma^*)$-dimensional Euclidean space. We have $\mathscr{I} = \mathscr{I}^* \times \mathscr{I}^{**}$. Let $\mathscr{I}^0 = \mathscr{I}^{*0} \times \mathscr{I}^{**}$. Clearly, $\mathscr{I}^0$ will be a closed set of measure zero in $\mathscr{I}$.

We can use the mapping ρ to define another mapping $\rho^*: (p, v^{**}) \to (v^*, v^{**}) = v$, by setting $v^* = \rho(p, \tau(p))$. This mapping ρ^* is from set $(P \times \mathscr{I}^{**})$ to $\mathscr{I}$, and is continuously differentiable whenever $v \in (\mathscr{I} - \mathscr{I}^0)$ because in this case, τ will be continuously differentiable. Therefore, by Sard's Theorem, the set $\mathscr{I}^{00}$ of all points v in $(\mathscr{I} - \mathscr{I}^0)$ corresponding to zeroes of the Jacobian J^* of (86) will have measure zero in $(\mathscr{I} - \mathscr{I}^0)$ and therefore also in $\mathscr{I}$.

We shall now show that $\mathscr{I}^{00}$ will be a closed set. We set

$$u_i = \min_m v_i^m \quad \text{and} \quad w_i = \max_m v_i^m, \quad \text{for} \quad i = 1, \ldots, n. \tag{97}$$

We define

$$\dot{v}_i^m = \mu(v_i^m) = \frac{v_i^m - u_i}{w_i - u_i}, \quad \text{if} \quad u_i < w_i, \tag{98}$$

and

$$\dot{v}_i^m = \mu(v_i^m) = 0, \quad \text{if} \quad u_i = w_i, \quad \text{for} \quad i = 1, \ldots, n; \; m = 1, \ldots, K. \tag{99}$$

We shall write

$$\dot{v} = \mu(v) = (\dot{v}_1^1, \ldots, \dot{v}_1^K; \ldots; \dot{v}_i^1, \ldots, \dot{v}_i^K; \ldots; \dot{v}_n^1, \ldots, \dot{v}_n^K), \tag{100}$$

and also

$$\dot{v}^* = \mu(v^*) \quad \text{and} \quad \dot{v}^{**} = \mu(v^{**}). \tag{101}$$

We also define

$$\dot{\mathscr{I}} = \{\dot{v}\}, \; \dot{\mathscr{I}}^* = \{\dot{v}^*\}, \quad \text{and} \quad \dot{\mathscr{I}}^{**} = \{\dot{v}^{**}\}. \tag{102}$$

Any component $\dot{v}_i^m$ of any vector $\dot{v}$ in $\dot{\mathscr{I}}$ will satisfy

$$0 \leqq \dot{v}_i^m \leqq 1, \quad \text{for} \quad i = 1, \ldots, n; \; m = 1, \ldots, K. \tag{103}$$

Clearly, $\mu\colon v \to \dot{v}$ is a linear mapping from $\mathscr{I}$ to $\dot{\mathscr{I}}$, which normalizes each player's payoff so as to satisfy Condition (103).

Let $\mathscr{F}$ be the set of all points $(p, \dot{v}^{**})$ where the Jacobian J^* vanishes. As J^* is a continuous function of $(p, \dot{v}^{**})$, $\mathscr{F}$ will be a closed set. Indeed, $\mathscr{F}$ will be a compact set since both $P = \{p\}$ and $\dot{\mathscr{I}}^{**} = \{\dot{v}^{**}\}$ are bounded. Consequently, $\dot{\mathscr{I}}^{00} = \rho^*(\mathscr{F})$ will also be compact since ρ^* is continuous. But $\dot{\mathscr{I}}^{00} = \mu(\mathscr{I}^{00})$, and μ is continuous. Therefore, $\mathscr{I}^{00}$ will be a closed set.

Thus, we can conclude that the assumptions of Lemma 9 are everywhere satisfied on $\mathscr{I}$, except on a closed set $\bar{\mathscr{I}} = \mathscr{I}^0 \cup \mathscr{I}^{00}$ with measure zero in $\mathscr{I}$. This completes the proof.

6. Numerical Examples

Consider the following three games:

Game Γ_1

	a_2^1	a_2^2
a_1^1	(3,4)	(2,2)
a_1^2	(1,1)	(2,1)

Game Γ_2

	a_2^1	a_2^2
a_1^1	(1,5)	(4,1)
a_1^2	(2,1)	(0,3)

Game Γ_3

	a_2^1	a_2^2
a_1^1	(1,1)	(1,1)
a_1^2	(1,1)	(1,1)

In all three games player i's payoff funtion will be called V_i $(i = 1,2)$. For each game Γ_t we shall define a disturbed game $\Gamma_t^*(\varepsilon)$ by assuming that each player's payoff function U_i will be of form (12). For the sake of simplicity, we shall also assume that in each game Γ_t^* the probability functions F_1 and F_2 are such that the random variables $\delta_i^{kk'} = \delta_i^{12}$ defined by (52) are uniformly distributed and that this distribution is independent of the other player's strategy $s_j, j \neq i$. More particularly, δ_1^{12} is uniformly distributed over the interval $[-\alpha, +\alpha]$ whereas δ_2^{12} is uniformly distributed over the interval $[-\beta, +\beta]$, with $\alpha \leqq 1$ and $\beta \leqq 1$. We shall write $\alpha^* = \varepsilon\alpha$ and $\beta^* = \varepsilon\beta$.

Let $s = (s_1, s_2)$ be a pair of proper ordinary mixed strategies with $s_1 = q a_1^1 + (1-q) a_1^2$ and $s_2 = r a_2^1 + (1-r) a_2^2$. In view of (58), s will be an s-equilibrium point in Γ_t^* if we have

$$[r v_1^{11} + (1-r) v_1^{12}] - [r v_1^{21} + (1-r) v_1^{22}] = \alpha^* \left(q - \frac{1}{2}\right) \tag{104}$$

and

$$[q v_2^{11} + (1-q) v_2^{21}] - [q v_2^{21} + (1-q) v_2^{22}] = \beta^* \left(r - \frac{1}{2}\right), \tag{105}$$

where

$$v_i^{km} = V_i(a_1^k, a_2^m), \quad \text{for} \quad i, k, m = 1, 2. \tag{106}$$

More particularly, Γ_1 has two equilibrium points in pure strategies. Of these, $a^1 = (a_1^1, a_2^1)$ is *strong* so that the pair $a^* = ([a_1^1], [a_2^1])$ of constant n-pure strategies is an equilibrium point in $\Gamma_1^*(\varepsilon)$ with a small ε. In contrast, $a^2 = (a_1^2, a_2^2)$ is an *extra weak* equilibrium point. Eqs. (104) and (105) yield:

$$q(\alpha^*, \beta^*) = -\beta^* \left(1 + \frac{\alpha^*}{2}\right) / (4 - \alpha^* \beta^*)$$

and

$$r(\alpha^*, \beta^*) = -\alpha^* \left(1 + \frac{\beta^*}{2}\right) / (4 - \alpha^* \beta^*).$$

These quantities q and r are close to the probabilities $q = 0$ and $r = 0$ characterizing a^2 but they are negative and so cannot be probabilities. Therefore a^2 is not approachable by any s-equilibrium point of game $\Gamma_1^*(\varepsilon)$.

Γ_2 has only one *quasi-strong* equilibrium point in mixed strategies, viz. $s = (s_1, s_2)$ with $s_1 = (\frac{1}{3} a_1^1 + \frac{2}{3} a_1^2)$ and $s_2 = (\frac{4}{5} a_2^1 + \frac{1}{5} a_2^2)$. By (104) and (105), s is approachable by the s-equilibrium point $s(\alpha^*, \beta^*)$, characterized by the probabilities

$$q(\alpha^*, \beta^*) = \frac{1}{3}\left(1 + \frac{3}{20}\beta^* + \frac{1}{20}\alpha^*\beta^*\right) \Big/ \left(1 + \frac{1}{30}\alpha^*\beta^*\right)$$

and

$$r(\alpha^*, \beta^*) = \frac{4}{5}\left(1 - \frac{1}{24}\alpha^* + \frac{1}{48}\alpha^*\beta^*\right) \Big/ \left(1 + \frac{1}{30}\alpha^*\beta^*\right).$$

Clearly, for small values of α^* and β^*, these probabilities will be very close to the probabilities $q = \frac{1}{3}$ and $r = \frac{4}{5}$ characterizing s itself.

Finally, Γ_3 has infinitely many equilibrium points. In fact, *any* possible pair $s = (s_1, s_2)$ of pure or mixed strategies is an equilibrium point. Those equilibrium points where one or both players use pure strategies are *extra weak*. Those where both use proper mixed strategies are *quasi-strong*. However, the Jacobian J^* of (86) vanishes at all these equilibrium points. (Actually, it can be shown that J^* will *always* vanish at all equilibrium points belonging to any infinite connected family of equilibrium points. This is not inconsistent with Theorem 7 since all games containing such families of equilibrium points form a closed set of measure zero in any $\mathscr{I}$.) Eqs. (104) and (105) now yield the probabilities

$$q(\alpha^*, \beta^*) = r(\alpha^*, \beta^*) = \frac{1}{2},$$

which happen to be independent of α^* and β^*. This means that, out of the infinitely many equilibrium points in game Γ_3, only *one* is approachable, and this is the strategy pair $s = (s_1, s_2)$ with $s_i = \frac{1}{2}a_i^1 + \frac{1}{2}a_i^2$ for $i = 1, 2$. But it is easy to verify that, in general, different choices of probability distributions for the random variables δ_1^{12} and δ_2^{12} would have made other equilibrium points of Γ_3 approachable. (Once more, it is a common occurrence that, out of any infinite connected family of equilibrium points, only a finite subset is approachable, and that this subset itself strongly depends on the choice of the probability distributions $F_1, \ldots, F_n$ used in defining the disturbed game $\Gamma^*(\varepsilon)$.)

7. Conclusion

We have found that the players' uncertainty about the exact payoffs that the other players will associate with various strategy combinations can be modeled as a game $\Gamma^*(\varepsilon)$ with randomly disturbed payoffs. Under this model, every ordinary game Γ will have at least one stable (i.e., approachable) equilibrium point. Indeed, all strong and "almost all" quasi-strong equilibrium points will be stable, including "almost all" equilibrium points in mixed strategies. Moreover, under this model, these equilibrium points will be stable, not because the players will make any deliberate effort to use their pure strategies with the probabilities prescribed by their mixed equilibrium strategies, but rather because the random fluctuations in their payoffs will *make* them use their pure strategies approximately with the prescribed probabilities.

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